A Continuum Theory of the Compound Nucleus

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The nucleus is described by an absorption coefficient σ which gives the probability per unit time that an incident particle becomes amalgamated with the nucleus (Eq. (1)). This absorption coefficient appears as an imaginary potential in the Schrödinger equation. It is shown that a gradual decrease of σ at the nuclear boundary is essential for achieving agreement with experiments (§2). This model gives automatically unit sticking probability for fast neutrons, a cross section proportional to $1/v$ for slow neutrons, and no one-particle resonances for particles which have to penetrate a potential barrier (§3). Quantitative calculations are made with σ varying as $e^{-(r-R)/b}$ outside the nucleus. For neutrons of zero orbital momentum, the formation probability of the compound nucleus is found to be $\zeta=1-e^{-2\pi kb}$ where k is the wave number. It is significant that ζ depends on the diffuseness b of the nuclear boundary rather than on the nuclear radius R. On the other hand, the factor 2π ensures that ζ is close to unity already for energies of about 1 Mev (§4). The total cross section in the region of overlapping levels, and the average level width in the region of separated levels are expressed in terms of the formation probability ζ. The relation with the elastic scattering is discussed (§5). The case of slow neutrons is treated in detail. With an average spacing D of 10 volts between levels of the same J, the average neutron width is about $2\times10^{-3}E^{\frac{1}{2}}$ for a neutron energy E, in rough agreement with the meager experimental data. With these assumptions, the neutron width will become larger than the radiation width already for $E\approx10^3$ ev; experiments on the capture of "medium fast" neutrons ($\approx2\times10^5$ ev) can be interpreted roughly on this basis. The elastic potential scattering of slow neutrons is shown to be equivalent to the scattering from a hard sphere whose radius R' is defined by the condition that $\sigma(R')=(\hbar/2mb^2)e^{-2C}$ where C is Euler's constant $0.577\cdots$ (§6). The case of particles which move in a non-nuclear potential V (electrostatic or centrifugal) is treated in §4, 7, 8 for various relations between the energy E of the incident particle and the height $V(R')$ of the potential barrier. If $E-V(R')$ is more than about 1 Mev, the formation probability is close to one, as for a fast neutron (§4). If E is about equal to $V(R')$, ζ is still of the order of unity (§8). For $E<V(R')$, ζ contains the well-known penetrability of the potential barrier, e^{-2G}, aside from other factors which increase slowly with $|E-V(R')|$ (§7). The magnitude of σ inside the nucleus is derived for the case of extremely high energies from the Born approximation and the variation of σ with energy is shown to be slight in this case. Although quantitative conclusions on the case of moderate energies cannot be drawn, it seems likely that σ is at least 20–40 Mev in that case (§9). Finally, it is shown that no appreciable change of results is caused by an attractive or repulsive nuclear potential added to the nuclear absorption potential (§10). In the main part of the paper, it has been assumed that the average interaction between nucleus and particle is zero.

§1. INTRODUCTION

THE treatment of the compound nucleus has in the past been based mostly on dispersion theory.[1–5] This treatment is eminently well suited for the resonance phenomena observed with slow neutrons and protons for which it was originally devised by Breit and Wigner.[1] Moreover, Bohr, Peierls and Placzek[5] have shown that the dispersion theory (in the form of Peierls and Kapur[4]) can also be applied in the region of dense levels, i.e. when the width of the compound levels is greater than their spacing. On the other hand, it is of course *not necessary* to use dispersion theory in the region of overlapping levels because the cross section then no longer depends on the individual energy levels. It is instead permissible to treat the energy levels statistically which means a great simplification of the formalism.[6–8] Bohr, Peierls and Placzek have shown that the statistical and the dispersion method give the same cross section for the formation of the compound nucleus and have investigated which quantities in dispersion theory correspond to the quantities used in the statistical treatment.

In either case, quantitative results can only be obtained when certain assumptions are made

[1] G. Breit and E. Wigner, Phys. Rev. **49**, 519 (1936).
[2] H. A. Bethe and G. Placzek, Phys. Rev. **51**, 450 (1937).
[3] N. Bohr and F. Kalckar, Kgl. Dansk. Vid. Selsk., Math.-fys. Med. **14**, 10 (1937).
[4] P. L. Kapur and R. Peierls, Proc. Roy. Soc. **166**, 277 (1938).
[5] N. Bohr, R. Peierls and G. Placzek, to appear shortly. I am indebted to Dr. Placzek for the opportunity to see the manuscript before publication.
[6] L. Landau, Physik. Zeits. Sowjetunion **11**, 556 (1937).
[7] V. F. Weisskopf, Phys. Rev. **52**, 295 (1937).
[8] V. F. Weisskopf and D. H. Ewing, Phys. Rev. **57**, 472 (1940).

about the physical quantities entering the theory. In the statistical theory it is shown that the total cross section can be expressed in terms of the sticking probability ξ, i.e., the probability that an incident particle hitting the nucleus will stick to it and thus form a compound nucleus. It is usually assumed that this sticking probability is unity for high energy particles and proportional to the velocity for slow particles, but it has not been possible thus far to give a general theory comprising both limiting cases and permitting an interpolation between them, or even to say which energies are to be considered as "high" and "low."

In this paper, we shall attempt to give a theory of the sticking probability. For this purpose, we shall consider the case of dense levels and shall try to develop a *continuum theory* of the compound nucleus, disregarding all effects of the individual levels. This theory will be related to the dispersion theory as the classical theory of a solid—describing it by phenomenological constants like conductivity or elasticity—is related to a quantum theory which would take into account each (electronic and vibrational) quantum state of the crystal.

In our continuum theory, we shall start from the basic idea underlying the theory of the compound nucleus,[9] *viz.* that there is a strong interaction between a nucleus and any nuclear particle incident upon it which leads to an amalgamation of the two. This means that there is a large probability for a "free" nuclear particle inside a nucleus to become absorbed and lose its individuality. We shall therefore *describe a nucleus by a large absorption coefficient for nuclear particles*. For reasons to be explained later (below Eq. (2a)), we shall refer to this absorption coefficient as "absorption potential."

An incident particle can then either be absorbed by the initial nucleus or elastically reflected. Our aim will be to calculate the relative probabilities of these two processes (in other words, the sticking probability) in terms of the absorption potential and of the velocity of the incident particle. The absorption of the particle is, of course, equivalent to the formation of the compound nucleus, and therefore the absorption probability gives the total cross section for all possible reactions. How this cross section is distributed over the various possible reactions is another matter; according to the original hypothesis of the compound nucleus, its disintegration is independent of the manner of its formation, and, if this is true, the relative probabilities of various modes of disintegration can be calculated by statistical arguments from the probabilities of the inverse processes.[8]

[9] N. Bohr, Nature **137**, 344 (1936).

§2. Definition of the Absorption Potential

We have proposed in the preceding section to describe the nucleus by an absorption coefficient. We shall assume that the number of particles absorbed per unit time at a given point in the nucleus is proportional to the density of particles at that point. Then we have the continuity equation

$$\partial\rho/\partial t+\text{div}\,\mathbf{j}=-2\sigma(\mathbf{r})\rho/\hbar, \qquad (1)$$

where $\mathbf{j}$ is the current density, ρ the density of particles and σ the "absorption potential." The function σ is defined so as to have the dimension of an energy; it will in general depend on the position $\mathbf{r}$ in the nucleus.

Equation (1) contains the assumption that the absorption depends only on the density of the particles but not on their kinetic energy, direction of motion, spin, etc. This assumption cannot be justified from first principles. However, it seems reasonable to assume that the dependence on spin and direction is slight, and that the dependence on the energy E is only appreciable if E changes at least by an amount of the order of a nuclear binding energy ($\sim$10 Mev). That the dependence on E is slight for very high energies E, will be shown in §9. Then, for our purposes, σ may be considered as a function of $\mathbf{r}$ only.

If we use the wave-mechanical definition of density ρ and current density $\mathbf{j}$, we can show easily that (1) is equivalent to the Schrödinger equation

$$\nabla^2\psi+(2m/\hbar^2)(E-V+i\sigma)\psi=0, \qquad (2)$$

where E is the energy of the incident particle and V its ordinary (real) potential energy. In order to show that (2) is equivalent to (1), we multiply (2) by ψ^* and subtract from the product its

complex conjugate; remembering then the definitions of density and current density, *viz.*,

$$\rho=\psi^*\psi,\ \mathbf{j}=(\hbar/2im)(\psi^*\ \text{grad}\ \psi-\psi\ \text{grad}\ \psi^*),\quad (2a)$$

we obtain Eq. (1) after a simple transformation. In Eq. (2), σ appears as the imaginary part of the potential. This justifies our use of the term "absorption potential," in preference to the colorless term "absorption coefficient."

The absorption potential σ may be assumed to be a function only of the distance r from the center of the nucleus. Inside the nucleus $(r<R)\sigma$ will be of the order of the nuclear interaction forces, i.e., perhaps 30 Mev (cf. §9). At the nuclear boundary σ will drop to zero. We shall show presently that the results for the sticking probability will depend most sensitively on the way in which σ goes to zero, i.e., whether this happens suddenly or gradually. On the other hand, the results will be practically independent of the absolute magnitude of σ inside the nucleus. (§3, 9).

If our theory is to agree with experimental results, it must give a sticking probability very close to unity for fast neutrons ($\gg$1 Mev). This follows from experiments of Bacher[10] on the scattering of fast neutrons by Cu and Pb. These experiments show that practically all the scattering is inelastic; what little elastic scattering there is, goes mainly in the forward direction[10,11] and is readily explained by shadow diffraction.[5, 11a] Therefore experimentally the nucleus behaves like a blackbody to incident fast neutrons; it absorbs practically all of the incident particles and reflects only very few elastically.

In order to find out how σ must behave in order to give blackbody properties to the nucleus it is convenient to remember the electrodynamic analog. Consider a medium of complex refractive index $N=n+ik$ with a sharp plane boundary. If light falls normally on the boundary from the vacuum, the reflection coefficient is

$$R=\left|\frac{N-1}{N+1}\right|^2=\frac{(n-1)^2+k^2}{(n+1)^2+k^2}. \quad (3)$$

All the light which is not reflected is absorbed if the medium is sufficiently thick; the fraction absorbed is equivalent to the sticking probability in nuclear physics and (for infinite thickness) is given by

$$\zeta=1-R=4n/[(n+1)^2+k^2]. \quad (3a)$$

From this formula it is seen that a large absorption coefficient k is by no means favorable for a large sticking probability; on the contrary, the only case when ζ becomes nearly 1 is for $N=n+ik$ nearly 1, i.e. when the medium is very similar to vacuum. We know, however, that the complex refractive index N is by no means close to unity in the nuclear case; in fact, $|N^2-1|=|-V+i\sigma|/E$ is in general large compared with unity. Therefore, if the nucleus had a sharp boundary, we would obtain a large reflectivity and small sticking probability.

The result is quite different if σ and V change *gradually* at the nuclear boundary. If the variation of $U=E-V+i\sigma$ over one wave-length is small compared with U itself, we can use the WKB method. As is well known, this method is equivalent to classical mechanics; therefore a stream of particles incident upon the nucleus will proceed towards the nucleus without being reflected, until they arrive in the regions where they are absorbed. Therefore a model in which the complex potential $V-i\sigma$ changes gradually at the nuclear boundary will, for fast incident particles (small wave-length), give *nearly unit sticking probability*.

A gradual variation of V and σ is very plausible. Both these quantities are connected with the forces between nuclear particles. Therefore, even if the nucleus had a sharp boundary, V and σ would extend beyond the geometrical boundary over a distance of the order of the range of the nuclear forces, i.e., about 10^{-13} cm. Actually, the boundary of the nucleus itself will not be sharp, because of the zero-point oscillation of the nuclear particles, which constitutes a further reason for a gradual change of V and σ. We shall use the term "diffuseness of the nuclear boundary," b, for the distance over which the absorption potential σ (and also V) fall to, say, $1/e$ of their value inside the nucleus. According to the foregoing, there are two contributions to

[10] R. F. Bacher, Phys. Rev. **55**, 679 (1939); **57**, 352 (1940).
[11] T. Wakatuki and S. Kikuchi, Proc. Phys.-Math. Soc. Japan **21**, 656 (1939).
[11a] G. Placzek and H. A. Bethe, Phys. Rev. **57**, 1075A (1940).

b, probably of about equal importance, one being the diffuseness of the region actually occupied by the particles in the nucleus and the other the range of the nuclear forces.

We thus see that very plausible assumptions lead to unit sticking probability for fast neutrons. The same assumptions will also eliminate one-particle resonances without the artificial introduction of a repulsive potential.[2] As we shall see in §3, our absorption theory will give the "$1/v$ law" for slow neutrons and Weisskopf's formula[8] for charged particle reactions.

§3. General Calculation

Many results can be obtained without special assumptions about the potential V and the absorption potential σ. The potential V will in general contain one part, V_n, which arises from nuclear forces, and another part, V_{ext}, which is not of nuclear origin and contains the Coulomb and the centrifugal force. Since the diffuseness of the nuclear boundary (cf. end of §2) is small, we can for most purposes consider the slowly varying part V_{ext} as constant over the interesting region, i.e. over the region outside the nucleus in which V_n and σ are still appreciable (cf., however, §8). We shall introduce

$$W=E-V_{\text{ext}}(R), \tag{4}$$

i.e., the kinetic energy of the incident particle at the surface of the nucleus (R=nuclear radius) disregarding nuclear forces. Then the radial Schrödinger equation is

$$\frac{d^2u}{dr^2}+\frac{2m}{\hbar^2}(W-V_n+i\sigma)u=0, \tag{5}$$

where u is r times the radial factor of ψ. We shall frequently use the abbreviation

$$\Phi=(2m/\hbar^2)(-W+V_n-i\sigma). \tag{5a}$$

A. High energy

If W is large and positive, the WKB method will be applicable for all values of r. Then

$$u=c\Phi^{-\frac{1}{4}}\exp\left(\int\Phi^{\frac{1}{2}}dr\right), \tag{6}$$

where c is a constant. The square root of Φ has to be chosen in such a way that the wave function (6) *decreases* towards the interior of the nucleus because the nucleus absorbs particles coming from the outside but does not emit any. This means that the real part of $\Phi^{\frac{1}{2}}$ must be positive. Now the imaginary part of Φ (cf. (5a)) is negative; therefore $\Phi^{\frac{1}{2}}$ must lie in the fourth quadrant of the complex plane (real part positive, imaginary part negative). As σ and V_n decrease, Φ becomes real and negative; and, since $\Phi^{\frac{1}{2}}$ stays all the time in the fourth quadrant, it will become purely negative imaginary far outside the nucleus. Therefore (6) goes over automatically into

$$u=c\Phi^{-\frac{1}{4}}\exp\left(-i(2mW)^{\frac{1}{2}}r/\hbar\right) \tag{6a}$$

or, for still larger distances, into

$$u=c\Phi^{-\frac{1}{4}}\exp\left\{-i(2m)^{\frac{1}{2}}\hbar^{-1}\int(E-V_{\text{ext}})^{\frac{1}{2}}dr\right\}, \tag{6b}$$

in other words, into an incident wave without any outgoing part. Therefore, for high energy, the nucleus does not reflect any particles; the sticking probability is

$$\zeta=1. \tag{6c}$$

Any deviation from this result is due to the inaccuracy of the WKB.

B. Small energy

For slow neutrons of zero orbital momentum, we have $V_{\text{ext}}=0$ and $W=E$ small. Then the WKB method is valid only inside the nucleus, i.e., as long as $|V_n-i\sigma|$ is sufficiently large. Let R be the value of r for which the WKB method just ceases to be valid. Then at $r=R$ we have approximately, according to (6):

$$\left(\frac{1}{u}\frac{du}{dr}\right)_{r=R}\approx\Phi^{\frac{1}{2}}(R)\approx\left[\frac{2m}{\hbar^2}(V_n-i\sigma)\right]^{\frac{1}{2}}. \tag{7}$$

The condition for validity of the WKB method is that the change of Φ over one wave-length be small compared to Φ itself,[12] one wave-length being equal to $|\Phi^{-\frac{1}{2}}|$. At the limit of validity, then, $|\Phi^{-\frac{1}{2}}|$ will be approximately equal to the

[12] Actually, the quantity neglected in the WKB is only the second derivative of $\Phi^{-\frac{1}{4}}$, rather than of Φ. Therefore the limit of validity may be given by the point where the wave-length $|\Phi|^{-\frac{1}{2}}$ is equal to $4b$ rather than to b. This would increase the result (11) for ζ by another factor 4. (With the exponential potential of §4, ζ comes out about 2.2 times the value given in (11).)

distance over which Φ changes by its own amount, i.e., of the order of the diffuseness of the nuclear boundary, b. Thus we have

$$(d \log u/dr)_R = (1/b)e^{-\frac{1}{2}i\varphi} \tag{7a}$$

with

$$\tan \varphi = (\sigma/V_n)_R. \tag{7b}$$

Immediately outside R, there will be a small region in which V_n and σ are still important but the effect of this region on $d \log u/dr$ is slight. We can therefore join the wave function at $r=R$ immediately to that of a free particle of energy E, *viz.*

$$u = \sin(kr+\delta) \tag{8}$$

with

$$k = (2mE)^{\frac{1}{2}}/\hbar. \tag{8a}$$

Since k and δ are small, we can write for r of the order R:

$$u = kr+\delta, \tag{8b}$$

$$\left(\frac{1}{u}\frac{du}{dr}\right)_R = \frac{k}{kR+\delta}. \tag{8c}$$

Equating (8c) and (7a), we find

$$\delta = -kR + kbe^{\frac{1}{2}i\varphi}. \tag{9}$$

The term kR gives a potential scattering with a cross section $4\pi R^2$. From the other term we may obtain the sticking probability (besides an additional contribution to the scattering). For this purpose, we re-write (8) in the form

$$2iu = e^{ikr+i\delta} - e^{-ikr-i\delta}. \tag{10}$$

This expression consists of an incident wave and a reflected wave, the ratio of their amplitudes being $e^{2i\delta}$. The reflection coefficient is then

$$P = |e^{4i\delta}| = \mathrm{Re}\,(1+4i\delta), \tag{10a}$$

where terms of order δ^2 have been neglected and Re denotes the real part. The sticking probability is then (cf. (9))

$$\zeta = 1-P = \mathrm{Im}\,(4\delta) = 4kb\sin\tfrac{1}{2}\varphi. \tag{11}$$

According to (7b), $\sin\frac{1}{2}\varphi$ will be of the order unity if σ is comparable to V_n at the limit of validity of the WKB. Therefore ζ is of the order of $4kb$.

This shows that ζ decreases as $k \sim E^{\frac{1}{2}}$ for slow neutrons. ζ has been defined here as the ratio of the number of particles absorbed to the number of incident particles with $l=0$. The latter number is π/k^2 for unit total incident current; therefore the absorption cross section is

$$\sigma = (\pi/k^2)\zeta \approx 4\pi b/k, \tag{12}$$

i.e., the well-known $1/v$ law. Our theory does of course not describe the resonance phenomena (cf. §6).

It must be pointed out that our definition of ζ differs from the usual definition[13] of the sticking probability ξ. The latter was made so as to make ξ close to unity for any case, including slow neutrons and particles going through a potential barrier. Our definition of ζ is very much simpler, but of course we cannot, in general, expect it to be unity. In order to avoid confusion, we propose to call ζ the "formation probability," i.e., the probability of formation of the compound nucleus.

According to (11), the formation probability will become of the order unity when $\lambda\!\!\!^{-} = 1/k \approx 4b$. This solves a question of old standing, *viz.* whether it is sufficient to make the wave-length $\lambda\!\!\!^{-}$ smaller than the nuclear radius R in order to make the formation probability almost unity, or whether it is necessary to make $\lambda\!\!\!^{-}$ smaller than the diffuseness b of the nuclear boundary. We see that b rather than R is important which is very plausible because, for an imaginary nucleus of very large dimensions, the size can obviously not be important for its physical properties. On the other hand, b is multiplied by the numerical factor 4 which arises from the $e^{4i\delta}$ in Eq. (10a) for the reflection coefficient; this means that already for rather large values of $\lambda\!\!\!^{-}$ (rather small energies) the formation probability becomes unity, in agreement with experiment.[10] It was mainly in view of the experimental results that it had been believed[13] that R was the critical value for $\lambda\!\!\!^{-}$ below which $\zeta \approx 1$. A more quantitative discussion will be given in §4.

C. Negative energy

If W is negative (potential barrier), $\Phi^{\frac{1}{2}}$ will become positive real outside the nucleus (cf. case A). This means that the wave function will be much smaller at the nuclear boundary than

[13] H. A. Bethe, Rev. Mod. Phys. **9**, 69 (1937), (cf. p. 96).

farther out so that our theory leads automatically to the ordinary result of small penetration through the potential barrier, and also to the usual expressions for the penetrability. Furthermore, the increase of the wave function outwards is obtained quite generally which shows that there are no one-particle resonances. The only condition is that the WKB method is applicable everywhere near the nuclear boundary. This means that *if* the nuclear potential V_n is attractive and becomes greater than $|W|$ at a certain point r_0, then $\sigma(r_0)$ should be large enough to insure the validity of the WKB near r_0. This will be the case if σ is of the same order as V_n or larger which is to be expected on general grounds.

§4. Quantitative Calculation for Exponential Absorption Coefficient

For the purpose of a more detailed discussion, we shall assume that σ depends exponentially on r, *viz.*

$$\sigma=(\hbar^2/2mb^2)e^{-(r-R)/b}. \tag{13}$$

b is the "diffuseness of the nuclear boundary." The factor $\hbar^2/2mb^2$ is chosen for convenience in the calculations. Eq. (13) contains a definition of the nuclear radius R, *viz.*, R is that value of r for which σ has the value $\hbar^2/2mb^2$. If we take for b simply the range of the nuclear forces (cf. §2), and if we determine this range from the theory of the triton,[14] we obtain $b=0.86\times10^{-13}$ cm, and $\hbar^2/2mb^2=28$ Mev, i.e. rather large. The expression (13) becomes very large inside the nucleus ($r<R$) whereas we should expect σ to reach a constant value of perhaps 30 Mev. This does not affect the results perceptibly because the inside value of σ is large enough to make the WKB valid for the transition from the inside to the boundary region, and therefore the behavior of the wave function in the boundary region depends only on the value of σ in that region and is independent of the behavior of σ inside the nucleus.

We shall further assume that V_n is zero. This assumption is probably a good approximation because, as far as we know, nuclear forces are sometimes repulsive, sometimes attractive according to the relative orientation of the spins of the interacting particles, their relative orbital momentum, their charge, etc. The interaction of a complicated nucleus, containing many particles of different spin, etc., with another particle will be very close to zero, at least on the average over the surface of the nucleus. We shall show in §10 that finite V_n gives essentially the same results as $V_n=0$.

The radial Schrödinger equation becomes now

$$\frac{d^2u}{dr^2}+\left\{k^2+\frac{i}{b^2}e^{-(r-R)/b}\right\}u=0. \tag{14}$$

Here k is the wave number of the incident particle near the nucleus, *viz.* [cf. Eq. (4)]

$$k^2=2mW/\hbar^2=(2m/\hbar^2)(E-V_{\text{ext}}(R)) \tag{14a}$$

$$=\frac{2m}{\hbar^2}\left(E-\frac{Zze^2}{R}\right)-\frac{l(l+1)}{R^2}, \tag{14b}$$

where l is the orbital momentum of the incident particle, ze its charge, Ze the charge and R the radius of the nucleus.

Equation (14) may be solved by introducing the independent variable

$$x=e^{-(r-R)/2b}. \tag{15}$$

Then (14) becomes

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+4\left(\frac{k^2b^2}{x^2}+i\right)u=0, \tag{15a}$$

whose solution is

$$u=cH_{2ikb}^{(1)}(2i^{\frac{1}{2}}x)=cH_{2ikb}^{(1)}(\sqrt{2}(1+i)x). \tag{16}$$

The Hankel function $H^{(1)}$, rather than any other type of Bessel function, must be chosen in order to make u zero rather than infinite for large x, i.e. inside the nucleus. c is a constant.

We want to know the asymptotic behavior of u far outside the nucleus ($r\gg R$). According to (15), this corresponds to *small* values of x. Neglecting all higher powers of z, we have for small z and any p[15]

$$H_p^{(1)}(2z)=\frac{i}{\sin\pi p}\left(e^{-\pi ip}\frac{z^p}{p!}-\frac{z^{-p}}{(-p)!}\right). \tag{16a}$$

[14] W. Rarita and R. D. Present, Phys. Rev. **51**, 788 (1937).

[15] Cf. Jahnke-Emde, *Table of Functions*, second edition, p. 200 and 194.

Therefore, choosing c appropriately, (16) becomes for small x

$$u=x^p-e^{\frac{1}{2}\pi i p}x^{-p}p!/(-p)! \tag{16b}$$

Inserting $p=2ikb$ and x from (15), this gives

$$u=e^{-ik(r-R)}-e^{ik(r-R)}e^{-\pi kb+2i\eta}, \tag{17}$$

where η is the complex phase of $(2ikb)!$, i.e.

$$e^{i\eta}=(2ikb)!/|(2ikb)!|. \tag{17a}$$

Equation (17) contains the main result. The radial wave function u consists of an incident wave of unit amplitude and an outgoing wave of absolute amplitude $e^{-\pi kb}$. This shows that the amplitude of the outgoing wave is always less than that of the incident one, as it must be. The reflection coefficient of the nucleus is

$$R=e^{-2\pi kb}. \tag{17b}$$

The difference of the intensities of incident and outgoing wave represents the particles absorbed by the nucleus; the formation probability of the compound nucleus is therefore

$$\zeta=1-e^{-2\pi kb}. \tag{18}$$

Equation (18) represents the principal result of our theory. It gives the desired expression for the formation probability as a function of the energy of the particle, which is necessary in order to deduce quantitative results for the cross sections of nuclear reactions. The expression (18) has the following properties:

(a) For high energies ($kb\gg 1/2\pi$), the formation probability is practically unity, in agreement with the experiments on fast neutrons. This shows that our assumptions about the absorption potential were appropriate to obtain the desired result.

(b) For low energies ($kb\ll\frac{1}{2}\pi$), the formation probability is

$$\zeta\approx 2\pi kb, \tag{18a}$$

i.e., ζ is proportional to k and therefore to the velocity. This gives the $1/v$ law for the capture cross section [cf. Eq. (27a)].

(c) The critical wave-length is

$$\lambda\!\!\!^{-}_{cr}=1/k_{cr}=2\pi b. \tag{19}$$

For a range of the nuclear forces $b=0.86\times 10^{-13}$ cm, we have $\lambda\!\!\!^{-}_{cr}=5.4\times 10^{-13}$ cm. This means that $\lambda\!\!\!^{-}_{cr}$ is of the order of the radius of medium weight nuclei although it is determined not by the nuclear radius but by the diffuseness of the nuclear boundary. Thus our explicit calculation confirms the results obtained in §3 from the WKB. (It should be noted that $\lambda\!\!\!^{-}$ is the ordinary wave-length divided by 2π, so that the "ordinary" critical wave-length would be $4\pi^2 b$.) The energy of a neutron or proton of wave-length $\lambda\!\!\!^{-}_{cr}$ is $E_{cr}=0.7$ Mev. Therefore neutrons above 1 Mev have a sticking probability close to unity. The sticking probability (18a) for slow neutrons may be written

$$\zeta\approx(E/E_{cr})^{\frac{1}{2}}\quad(E\ll E_{cr}). \tag{19a}$$

According to (14a), k depends on the kinetic energy W of the incident particle at the surface of the nucleus. Only in the case of neutrons with zero orbital momentum, W is identical with the actual energy E and only in this case, Eqs. (18a) to (19a) are directly applicable. In all other cases, W is less than E because of the electrostatic and the centrifugal potential barrier. Moreover, in these other cases V_{ext} is a function of r. Then, among the statements above, only (a) is correct, i.e., for large kinetic energy the sticking probability is still nearly unity. However, if W is small (<1 Mev, say), it is necessary to take into account the variation of V_{ext} over the diffuse nuclear boundary. This will be done in §8 with the result that the formation probability remains of the order unity even for W small, i.e. even when the energy of the incident particle is only just sufficient to go over the top of the barrier.

In the next section we shall discuss the connection between our ζ and the observable cross sections as well as the level widths occurring in dispersion theory. In the following sections we shall consider the cases of slow neutrons (§6), of particles going through a potential barrier (§7), of particles just able to go over the top of a barrier (§8) and of very fast particles (§9). Finally, we shall investigate the influence of a nonvanishing nuclear potential V_n (§10).

§5. Cross Section, Level Width and Formation Probability

From the formulas given in the preceding section, we can compute the formation proba-

bility ζ for a given energy E of the incident particle as a function of the orbital momentum l. The probability ζ_l is defined as the fraction of the incident particles of the given angular momentum which are absorbed by the nucleus. The total number of incident particles of orbital momentum l per second is $(2l+1)\pi\lambda\!\!\!^{-2}$, if the total incident current is unity and $\lambda\!\!\!^{-}$ is the wavelength of the particle at large distance from the nucleus. The total cross section for all possible processes is therefore

$$\sigma_t=\sum_{l=0}^{\infty}\sigma_l=\pi\lambda\!\!\!^{-2}\sum_l(2l+1)\zeta_l. \tag{20}$$

If the fundamental assumption of the theory of the compound nucleus is correct, i.e., if the disintegration of the compound nucleus is independent of the way of its formation, the partial cross section for a certain process A is obtained by the well-known formula

$$\sigma_A=\sigma_t\gamma_A/\gamma, \tag{20a}$$

where γ_A is the partial width for the given process and γ the total width. Both γ_A and γ represent averages over all compound levels near the energy of the incident particle.

Equation (20) does not include the whole elastic scattering, but only the part due to formation of the compound nucleus and reemission of the incident particle without phase relations [cf. Eqs. (20), (20a)]. This part which corresponds to the resonance scattering in the region of separated levels, is small if many other processes (inelastic scattering or disintegrations) can occur. The remaining part of the elastic scattering follows from the asymptotic behavior of the radial wave function u. For a free particle, the radial wave of orbital momentum l behaves asymptotically as

$$e^{-ikr}+(-1)^{l+1}e^{ikr}. \tag{21a}$$

If the actual wave function behaves as

$$e^{-ikr}+(-1)^{l+1}\beta_l e^{ikr}, \tag{21b}$$

the scattered amplitude will be

$$F(\vartheta)=-\tfrac{1}{2}i\lambda\!\!\!^{-}\sum_l(2l+1)(\beta_l-1)P_l(\cos\vartheta). \tag{21}$$

The total elastic cross section is then

$$\sigma_{el}=2\pi\int|F|^2\sin\vartheta d\vartheta =\pi\lambda\!\!\!^{-2}\sum_l(2l+1)|\beta_l-1|^2. \tag{22}$$

From Eq. (17) we have, taking into account the phase shift outside the nucleus[15a] due to V_{ext}

$$\beta=\exp\{\pi il-2l^{\frac{1}{2}}(l+1)^{\frac{1}{2}}\arctan[l^{\frac{1}{2}}(l+1)^{\frac{1}{2}}/kR] -2ikR+2i\eta-\pi kb\} \tag{22a}$$

β is related to the formation probability by

$$\zeta_l=1-|\beta_l|^2. \tag{22b}$$

The cross section for formation of the compound nucleus by incident particles of orbital momentum l is $(2l+1)\pi\lambda\!\!\!^{-2}\zeta_l$. Adding to this the contribution l to the elastic cross section (22), we find for the l part of the total cross section (elastic plus inelastic)

$$s_l=(2l+1)\pi\lambda\!\!\!^{-2}\cdot 2(1-\text{Re}\,\beta_l), \tag{23}$$

Re denoting the real part. Comparing this with (21), we find

$$s_l=4\pi\lambda\!\!\!^{-}\,\text{Im}\,F_l, \tag{24}$$

where F_l is the factor of P_l in (21) and Im denotes the imaginary part. The relation (24) between the total cross section and the imaginary part of the elastically scattered amplitude was found by Bohr, Peierls and Placzek[5] and will be referred to below as the BPP theorem.

According to Eq. (22a), β is very small as long as W is positive and greater than about 1 Mev. For fast neutrons this is the case if $l<R/\lambda\!\!\!^{-}$. On the other hand, we shall show in §7 that β is nearly unity for W negative and $|W|$ greater than about 1 Mev, i.e. for $l>R/\lambda\!\!\!^{-}$. The intermediate region, -1 Mev $<W<1$ Mev, is not very important for high E. Therefore [cf. Eq. (20)] the total cross section for all inelastic processes is about πR^2 and the cross section for elastic nonresonance scattering [cf. Eq. (22)] is also equal to πR^2. This latter scattering represents the diffraction of the geometrical shadow of the nucleus, as has been pointed out by

[15a] This expression was calculated by S. P. Frankel of the University of Rochester to whom I am indebted for its communication.

Bohr, Peierls and Placzek;[5] the "shadow scattering" goes mostly in the forward direction.[5, 11, 11a]

In the case of *separated compound levels*, it is important to know the average width of the levels. This quantity must also be known in the case of overlapping levels when we want to calculate the partial cross sections [cf. Eq. (20a)]. From statistical arguments (cf. reference 8), we can deduce the relation

$$\langle\Gamma_{Al}{}^{J}\rangle_{\text{Av}}=D_J\frac{(2i+1)(2s+1)}{2J+1}\frac{\sigma_{lJ}}{2\pi^2\lambda^2}. \tag{25}$$

Here $\Gamma_{Al}{}^{J}$ represents the partial width for disintegration of a compound level of angular momentum J into a particle P of orbital momentum l and a residual nucleus in the quantum state A; i and s are the spins of residual nucleus and particle P, respectively; σ_{lJ} is the cross section for formation of the compound nucleus in the inverse process, i.e. of a compound nucleus of angular momentum J by an incident particle of orbital momentum l; λ is the wave-length of the particle emitted, and D_J is the average spacing between the levels of angular momentum J of the compound nucleus.

We may write

$$\sigma_{lJ}=\sigma_l\,\alpha_{lJ}, \tag{25a}$$

where σ_l is the total cross section for formation of the compound nucleus by particles of orbital momentum l, and α_{lJ} the probability that the nucleus so formed has angular momentum J. If the nuclear reaction is statistical with respect to angular momentum, i.e. if ζ does not depend on the total angular momentum J (which must, e.g., be true if $\zeta=1$), we have

$$\alpha_{lJ}=g_{lis,J}(2J+1)/(2l+1)(2i+1)(2s+1), \tag{25b}$$

where $g_{lis,J}$ is an integer giving the number of ways in which the resultant J can be obtained by compounding l, i and s vectorially. If one of the three numbers l, i, s is zero, g is unity for all J's which can be obtained at all. If $s=\frac{1}{2}$, as for proton and neutron, g is either one or two, *viz.*

$$\begin{aligned} &g_{li\frac{1}{2},J}=1 \quad \text{for} \quad J=l+i+\tfrac{1}{2} \quad \text{and} \quad |l-i|-\tfrac{1}{2} \\ &g_{li\frac{1}{2},J}=2 \quad \text{for} \quad |l-i|+\tfrac{1}{2}\leqslant J\leqslant l+i-\tfrac{1}{2}. \end{aligned} \tag{25c}$$

For $l=i$, the level $J=|l-i|-\frac{1}{2}$ does, of course, not exist.

Inserting (25a), (25b) and (20) into (25), we find

$$\langle\Gamma_{Al}{}^{J}\rangle_{\text{Av}}=g_{lis,J}D_J\,\zeta/2\pi. \tag{26}$$

This equation contains the well-known result that the partial width of a compound level for a given mode of disintegration is of the order of the spacing between levels if the formation probability ζ is of the order unity.

§6. Slow Neutrons. Relation to the Dispersion Theory

A. Capture

It has been shown in §4 that the formation probability of the compound nucleus for slow neutrons of zero orbital momentum is [cf. Eq. (18a)]

$$\zeta=2\pi b/\lambda. \tag{27}$$

The total cross section exclusive of potential scattering is then [cf. Eq. (20), $l=0$]

$$\sigma_t=2\pi^2\lambda b. \tag{27a}$$

This formula contains the $1/v$ law. It is, of course, only valid when the energy levels of the compound nucleus overlap. The average neutron width $\langle\Gamma_N\rangle_{\text{Av}}$ of the energy levels can be obtained from (26). Since $l=0$, we have $g_{lis,J}=1$ and therefore

$$\langle\Gamma_N\rangle_{\text{Av}}=Db/\lambda. \tag{28}$$

With $b=0.86\times10^{-13}$ cm, this gives

$$\langle\Gamma_N\rangle_{\text{Av}}=1.9\times10^{-4}DE^{\frac{1}{2}}, \tag{28a}$$

if E is the neutron energy in ev. If we assume for D a value of 10 ev which seems of the right order for medium heavy nuclei ($A\approx100$), $\langle\Gamma_N\rangle_{\text{Av}}E^{-\frac{1}{2}}$ comes out about 2×10^{-3} volt$^{\frac{1}{2}}$ which is in fair agreement with the observed values (reference 13, p. 150). Values of Γ_N for individual levels will, of course, differ considerably from the average.

Our result for the neutron width is rather larger than was believed in the beginnings of slow neutron physics. On the other hand, the radiation width Γ_γ of the compound levels seems to be rather smaller than was believed originally. In practically all cases of medium heavy nuclei,

Γ_γ seems to be smaller than 0.1 ev, and in some cases considerably so. If $\langle\Gamma_\gamma\rangle_{Av}=0.05$ ev and $D=10$ ev, the average neutron width (28a) becomes greater than the average radiation width already for neutron energies as low as 1000 ev. For lighter nuclei, let us say $A\approx 50$, the spacing D will be considerably greater while $\langle\Gamma_\gamma\rangle_{Av}$ will be only slightly larger than for $A\approx 100$; then the neutron width will, in general, be larger than the radiation width even for energies of the order of D.

The cross section for neutron capture in the region of separated levels can be calculated from the dispersion formula of Peierls and Kapur.[4] Our theory adds to dispersion theory only the formula (28) for the average neutron width.

We can make more direct predictions from our theory for neutrons of medium energy, such as the photoneutrons produced in deuterium by the γ-rays from ThC'' (neutron energy ~ 225 kev). In this energy region we can use the formulae for the cross section averaged over the resonances. For the radiative capture of the neutrons the averaged cross section is

$$\sigma_c=2\pi^2\lambda\!\!\!^{-2}\langle\Gamma_N\Gamma_\gamma/D\Gamma\rangle_{Av}, \tag{29}$$

where Γ_N is the partial width for disintegration of the compound nucleus into a neutron and a residual nucleus in the ground state, Γ_γ the radiation width and Γ the total width, all quantities being averaged over the compound levels. As we have pointed out above, the neutron width can be expected to be considerably larger than the radiation width for a neutron energy of 225 kev. Therefore the total width Γ is determined by the neutron width, and we may write

$$\Gamma/\Gamma_N=N(E)=\sum_n(E-W_n)^{\frac{1}{2}}/E^{\frac{1}{2}}. \tag{29a}$$

Here n labels the energy levels of the initial (not the compound) nucleus whose excitation energy W_n is less than the kinetic energy E of the incident neutron. The partial width for disintegration into a neutron and a residual nucleus in state n, has been assumed to be simply proportional to the square root of the kinetic energy of the outgoing neutron, in accord with (26), (27) and remembering that $g_{lis,J}=1$ because $l=0$.

Inserting (29a) in (29), we obtain

$$\sigma_c=\frac{4.1\times 10^{-24}}{EN(E)}\frac{\Gamma_\gamma}{D}\text{ cm}^2, \tag{29b}$$

where E is measured in Mev. $N(E)$ will in general be of the order unity because E is rather low. Taking $E=0.22_5$ Mev, $\Gamma_\gamma=0.05$ ev, $D=10$ ev and $N(E)=1$, we get $\sigma_c=9\times 10^{-26}$ cm^2. The cross sections observed by v. Halban and Kowarski[16] vary considerably from one element to another, the largest ones being of the order of magnitude predicted by our theory, *viz.* about $2-5\times 10^{-26}$ cm^2. The cross sections of the lighter elements ($A\approx 60$) are consistently smaller, *viz.* between 3×10^{-28} and 10^{-27} cm^2, which is readily explained by a greater spacing D between the energy levels. However, some of the nuclei of $A\approx 100-150$ also have small cross sections of the order of 10^{-27} cm^2; whether this is due to an unusually large D or to a large $N(E)$, cannot be decided at present.

B. Scattering

Multiplying (17) by e^{-ikR}, we obtain

$$ue^{-ikR}=e^{-ikr}-e^{ikr}e^{-2ikR-\pi kb+2i\eta}. \tag{30}$$

The amplitude of the elastically scattered wave is then (cf. (21); in our case, $k=1/\lambda\!\!\!^{-}$)

$$F=(e^{-2ikR-\pi kb+2i\eta}-1)/2ik. \tag{30a}$$

η is [cf. Eq. (17a)] the complex phase of $(2ikb)!$; since kb is supposed to be small, we may write

$$\eta=2kb\psi(0)=-2kbC, \tag{30b}$$

where ψ is the logarithmic derivative of the factorial function and $C=0.577\cdots$ Euler's constant. Then, again remembering that k is small, (30a) becomes

$$F=-(R+2Cb)+\tfrac{1}{2}i\pi b. \tag{31}$$

The scattered amplitude (31) contains a real and an imaginary part. The former is, in sign and magnitude, the same as for a hard sphere of radius

$$R'=R+2Cb. \tag{31a}$$

[16] H. v. Halban and L. Kowarski, Nature **142**, 392 (1938).

This part is usually called potential scattering. It was originally explained in a very artificial way,[2] using a repulsive potential. Later, it was shown to come out naturally from the theory of Peierls and Kapur[4] but the result was somewhat uncertain because of the contribution of the distant levels (cf. 32c). It is now seen to result equally naturally from the assumption of a strong absorption in the nucleus. Our derivation has the advantage of giving a more definite physical meaning to the nuclear radius. The effective radius, R', is slightly larger than the R defined above. We may define R' directly as that radius for which the absorption potential is

$$\sigma(R')=(\hbar^2/2mb^2)e^{-2C}, \tag{31b}$$

which, for $b=0.86\times10^{-13}$ cm, is about 9 Mev. Mathematically, R' can be considered as the point where the WKB treatment of σ breaks down. The sign of the potential scattering should be the same (negative) for all nuclei.

The scattered amplitude F (31) also has an imaginary part. This is connected with the probability of neutron capture because it derives from the factor $e^{-\pi kb}$ in (30a) whose square is equal to one minus the formation probability ζ [cf. Eqs. (17b), (18)]. The relation between the imaginary part of (31) and the cross section for formation of the compound nucleus is a special case of the theorem of Bohr, Peierls and Placzek [cf. Eq. (24)]. (In our case, the formation cross section is practically equal to the total cross section because it contains the factor $1/v$). In the region of overlapping levels where (31) is valid, its imaginary part is not very important for the total scattering, in comparison with the real part. The total elastic scattering cross section is then

$$\sigma_s=4\pi|F|^2=4\pi(R'^2+\tfrac{1}{4}\pi^2b^2). \tag{31c}$$

We shall now discuss the scattering according to the dispersion formula, and compare it with the results just obtained for overlapping levels. According to Peierls and Kapur [reference 4, Eq. (24)], the scattered amplitude is approximately[17]

$$F=\frac{1}{2ik}\left[e^{-2ikR''}\left(1-i\sum_n\frac{\gamma_{nN}}{E-E_n+\frac{1}{2}i\gamma_n}\right)-1\right], \tag{32}$$

where R'' is the nuclear radius introduced by Peierls and Kapur (their r_0), E_n is the neutron energy which gives exact resonance with the nth compound level, γ_n is the total width of the level, and γ_{nN} its partial width for disintegration into a neutron and a residual nucleus in the ground state. For small k,

$$F=-R''-\tfrac{1}{2}\lambda\sum_n\frac{\gamma_{nN}}{E-E_n+\frac{1}{2}i\gamma_n}. \tag{32a}$$

This expression can be separated into a real and an imaginary part:

$$F=F_r+iF_i, \tag{32b}$$

$$F_r=-R''-\tfrac{1}{2}\lambda\sum_n\frac{\gamma_{nN}(E-E_n)}{(E-E_n)^2+\frac{1}{4}\gamma_n^2}, \tag{32c}$$

$$F_i=\tfrac{1}{4}\lambda\sum_n\frac{\gamma_n\gamma_{n\lambda}}{(E-E_n)^2+\frac{1}{4}\gamma_n^2}. \tag{32d}$$

The exact theory of Peierls and Kapur is still valid when the compound levels overlap. In this case, our theory using an absorption potential is also valid, and a comparison can be made. In the case of overlapping levels the theory of Peierls and Kapur simplifies to (cf. reference 5)

$$F_i=\tfrac{1}{2}\pi\lambda\langle\gamma_N\rangle_{Av}/D, \tag{33}$$

the average being taken over the compound levels near the neutron energy E. (Γ is the value of γ at the resonance energy, cf. reference 2.) With (28), this gives

$$F_i=\tfrac{1}{2}\pi b \tag{33a}$$

in exact agreement with Eq. (31). This agreement is not surprising and means only that both the dispersion theory and our theory satisfy the theorem of Bohr, Peierls and Placzek [Eq. (24)].

The sum in the real part of the Peierls-Kapur scattered amplitude consists of the contribution of the levels close to E and of the distant levels just as (32c). The former can be shown to be zero when the levels overlap.[5] The latter is almost independent of the energy and therefore

[17] Since we shall apply (32) only to the case of low energies, we have put the u_n^2/N_n of Peierls and Kapur equal to $|u_n|^2=\gamma_{nN}$.

represents simply an addition to R''. Since the Peierls-Kapur theory is a consistent treatment of the compound nucleus, the final result, i.e. R'' *plus* the contribution of the distant levels, must be independent of the (arbitrary) choice of R''. Comparing (32c) with (31), (31a) we find that for overlapping levels

$$F_r = -R'. \tag{34}$$

In other words, *the contribution of the distant levels* in (33a) is *zero when R'' is chosen equal to R'*. Thus our theory provides a natural choice for the nuclear radius in the Peierls-Kapur theory. Since the contribution of distant levels is almost independent of E, we can write in the region of separated levels

$$F_r = -R' - \tfrac{1}{2}\lambda\sum_n \frac{\gamma_{nN}(E-E_n)}{(E-E_n)^2+\frac{1}{4}\gamma_n{}^2}, \tag{34a}$$

where the sum is to be extended only over the levels close to E. Taking into account the conservation of angular momentum, we obtain then for the elastic scattering cross section the value given in Eq. (54) of reference 2, with R replaced by R'.

It is interesting to estimate the relative importance of resonance and potential scattering. In the region of overlapping levels we have seen that the potential scattering is considerably larger than the resonance scattering [cf. Eq. (31c)]. (The overlapping of levels will certainly occur for neutron energies above 1 Mev because then the partial neutron width is greater than the spacing; it may already occur at lower energies if there are many low excited levels of the initial nucleus.) In the region of separated levels, the average of the elastic scattering cross section over an energy interval large compared with D, is

$$\langle\sigma_{el}\rangle_{Av} = 4\pi(R'^2+\tfrac{1}{2}\pi\lambda^2\langle\Gamma_N\rangle_{Av}{}^2/\Gamma_{Av}D) \tag{35}$$

$$= 4\pi(R'^2+\tfrac{1}{2}\pi\lambda b\langle\Gamma_N\rangle_{Av}/\Gamma_{Av}). \tag{35a}$$

If the neutron energy E is high enough so that the partial neutron width is greater than the radiation width (i.e. perhaps for $E>1$ kev for medium heavy nuclei), and if, on the other hand, E is smaller than the first excited level of the initial nucleus, the total level width Γ is nearly equal to the partial neutron width Γ_N. Then

$$\langle\sigma_{el}\rangle_{Av} = 4\pi(R'^2+\tfrac{1}{2}\pi\lambda b). \tag{35b}$$

If we take $R'=6\times10^{-13}$ cm (medium heavy nuclei) and $b=0.86\times10^{-13}$ cm, the second term in (35b) (average resonance scattering) becomes greater than the first (potential scattering) for $E<30$ kev. Therefore, according to (35b), we should have an energy region (from about 1 to 30 kev) in which the average elastic scattering cross section is proportional to $1/v$.

For smaller energies (below about 1 kev) we have $\Gamma_\gamma > \Gamma_N$; therefore the elastic cross section, averaged over the neutron energy, becomes independent of E, *viz.*

$$\langle\sigma_{el}\rangle_{Av} = 4\pi(R'^2+\tfrac{1}{2}\pi b^2 D/\Gamma_\gamma). \tag{35c}$$

With our assumed values for b, D and Γ_γ, the second term gives a cross section of 30×10^{-24} cm^2. It is not surprising that the scattering cross section for thermal neutrons is considerably smaller than this amount for practically all nuclei. This is because the main contribution to the average cross section (35c) comes from the immediate neighborhood of the resonances, and the thermal region is in general "between resonances." If we estimate the scattering amplitude (32a) between resonances by assuming $E-E_n=\frac{1}{2}D$ for one level n, and no contribution from other levels, we obtain for the resonance scattering $\approx\lambda\gamma_{nN}/D=b$. Therefore "between resonances" the potential scattering will probably be the more important term, although it must be admitted that our estimate of the resonance scattering is extremely uncertain. It would be very important to investigate experimentally the relative importance of resonance and potential scattering for slow neutrons. This could be done by interference experiments because the potential scattering amplitude should have the same sign (negative) for all nuclei while the resonance scattering may be positive or negative, according to the position of the nearest levels. Moreover, the resonance scattering is partly incoherent.[18,19]

[18] G. C. Wick, Physik. Zeits. **38**, 689 (1937).
[19] O. Halpern and M. H. Johnson, Phys. Rev. **55**, 898 (1939).

§7. Potential Barrier

In this section, we shall treat the case of a particle which must penetrate a potential barrier in order to enter the nucleus. In deriving Eq. (17), it was not assumed that the kinetic energy W at the surface of the nucleus was positive. If W is negative, we may simply put

$$k=i\kappa, \tag{36}$$

where

$$\kappa^2=-2mW/\hbar^2. \tag{36a}$$

Then (17) goes over into

$$u=e^{\kappa(r-R)}-e^{-\kappa(r-R)}\frac{(-\kappa b)!}{(\kappa b)!}e^{-\pi i\kappa b}. \tag{37}$$

Since the kinetic energy $E-V_{\text{ext}}$ [cf. Eq. (4)] varies with r, this expression must be replaced by the WKB expression

$$u(r)=|\Phi|^{-\frac{1}{4}}\exp\left(\int_R^r|\Phi(\rho)|^{\frac{1}{2}}d\rho\right)-|\Phi|^{-\frac{1}{4}}e^{-\pi i\kappa b}\exp\left(-\int_R^r|\Phi(\rho)|^{\frac{1}{2}}d\rho\right)(-\kappa b)!/(\kappa b)!, \tag{37a}$$

where

$$\Phi=(2m/\hbar^2)(E-V_{\text{ext}}). \tag{37b}$$

For large r, Φ becomes positive and (37a) goes over into

$$u=2\Phi^{-\frac{1}{4}}\sin\left(\int_{r_0}^r\Phi^{\frac{1}{2}}d\rho+\tfrac{1}{4}\pi\right)\exp\left(\int_R^{r_0}|\Phi(\rho)|^{\frac{1}{2}}d\rho\right)$$
$$-\Phi^{-\frac{1}{4}}\cos\left(\int_{r_0}^r\Phi^{\frac{1}{2}}d\rho+\tfrac{1}{4}\pi\right)\exp\left(-\int_R^{r_0}|\Phi(\rho)|^{\frac{1}{2}}d\rho\right)e^{-\pi i\kappa b}(-\kappa b)!/(\kappa b)!, \tag{37c}$$

where r_0 is the point at which Φ is zero and ρ an integration variable. The function (37c) can be represented as an incident plus an outgoing wave. The reflection coefficient can be calculated in the usual way as the absolute square of the ratio of the amplitudes of the two waves. The formation probability then is one minus the reflection coefficient and comes out to be

$$\zeta=2\sin\pi\kappa b\ e^{-2G}(-\kappa b)!/(\kappa b)!+0(e^{-4G}) \tag{38}$$

with

$$G=\int_R^{r_0}|\Phi(r)|^{\frac{1}{2}}dr, \tag{38a}$$

the Gamow integral. Using the well-known relation between $(-n)!$ and $n!$ (cf. reference 15, p. 89), we find

$$\zeta=2\pi\kappa be^{-2G}/(\kappa b)!^2 \tag{38b}$$

The formation probability ζ, as here defined, includes the penetrability of the potential barrier (factor e^{-2G}) and differs in this respect from the sticking probability ξ used commonly in the literature (cf. §3B). For small κb, the ζ from (38b) is very nearly $2\pi\kappa be^{-2G}$, i.e. it contains, besides the penetrability factor e^{-2G}, another factor which is analogous to the formation probability for small positive W, *viz.* $2\pi kb$. If we neglect the variation of V_{ext} with r, the formation probability drops from 1 at high energies to 0 at $W=0$, then rises again for negative W as $|W|^{\frac{1}{2}}$, and finally decreases again because of the factor e^{-2G}.

Our formula for ζ is in agreement with the result of Peierls and Kapur,[4] which has also been used by Weisskopf and Ewing,[8] namely that the probability of formation of the compound nucleus is inversely proportional to the square of the irregular solution of the wave equation in the Coulomb field. For $r<r_0$ this

solution behaves as

$$|\Phi|^{-\frac{1}{4}}(r)\exp\left[\int_r^{r_0}|\phi(\rho)|^{\frac{1}{2}}d\rho\right]$$

which makes the formation probability proportional to $|\Phi(R)|^{\frac{1}{2}}e^{-2G}=\kappa e^{-2G}$, in agreement with our result. Previously, it had been supposed that the formation probability was directly proportional to the square of the regular solution, i.e. $\sim(1/\kappa)e^{-2G}$, or else that it contained only[20] the exponential e^{-2G} without any power of κ. However, it must be remembered that the variation of V_{ext} with r invalidates our Eq. (38b) for small κ for which the factor κ would be most important. We shall show in the next section that, because of the variation of V_{ext}, ζ does not actually go to zero for $W=0$ and then rise again to a secondary maximum for negative W, but that it is of the order unity for $W=0$ and decreases monotonically with decreasing W. Therefore the simple exponential e^{-2G} is probably a better approximation to ζ than the complete formula (38b), at least as long as the factor $2\pi\kappa b$ is smaller than unity.

For very high barrier (W large and negative), (38b) decreases faster than the penetrability e^{-2G}, because of the denominator $(\kappa b)!^2$. This result is spurious because it depends essentially on the assumption that V_{ext} is constant everywhere. A more appropriate treatment of the case of large negative W can be made on the basis of the WKB which in this case is a very good approximation.

The WKB gives for all $r<r_0$:

$$u=c(V-E-i\sigma)^{-\frac{1}{4}}\exp\left\{(2m)^{\frac{1}{2}}\hbar^{-1}\int_{r_0}^{r}(V(\rho)-E-i\sigma(\rho))^{\frac{1}{2}}d\rho\right\}, \tag{39}$$

if we write for brevity V instead of V_{ext}. Then the wave function at large distances ($r\gg r_0$, $V\ll E$) is

$$u=2cE^{-\frac{1}{4}}\cos(2mE/\hbar^2)^{\frac{1}{2}}r. \tag{39a}$$

It will be most convenient to normalize to unit incident current, which means $c=(m/2)^{\frac{1}{4}}$. The total absorption per second can in our case be most easily calculated from the definition (1) of σ; we have

$$\zeta=(2/\hbar)\int_0^\infty|u(r)|^2\sigma(r)dr. \tag{40}$$

If we insert (13) for σ and (39) for u, and neglect σ^2 compared with $(V-E)^2$ in u, we obtain

$$\zeta=\frac{\hbar}{(2m)^{\frac{1}{2}}(V-E)^{\frac{1}{2}}b^2}\int_0^\infty dr\exp\left\{-\frac{r-R}{b}+2\frac{(2m)^{\frac{1}{2}}}{\hbar}\int_{r_0}^{r}(V-E)^{\frac{1}{2}}d\rho\right\}. \tag{40a}$$

The exponential in (40a) has a maximum for $r=r_1$ where r_1 is defined by

$$V(r_1)-E=\hbar^2/8mb^2. \tag{40b}$$

In order to justify our neglect of σ compared with $V-E$ in the integrand of (40a), we must require that r_1 be appreciably greater than R, in other words that $W(R)>\hbar^2/8mb^2$. For our value $b=0.86\times10^{-13}$ cm, we have $\hbar^2/8mb^2=7$ Mev for a proton or neutron. We note that the condition $W>\hbar^2/8mb^2$ is equivalent to $\kappa b>\frac{1}{2}$; for smaller values of κb, (38b) can be used, and the $(\kappa b)!$ in that formula can be replaced by unity.

If $W>\hbar^2/8mb^2$, there will be a certain $r_1>R$ for which (40b) is fulfilled. If we expand the exponent in (40a) in powers of $r-r_1$ up to the quadratic terms and integrate, we obtain, remembering (40b)

$$\zeta=\left(\frac{2\pi\hbar^2}{mb^3}\right)^{\frac{1}{2}}\left(\frac{dV}{dr}\right)_{r_1}^{-\frac{1}{2}}\exp\left\{-2G+2\frac{(2m)^{\frac{1}{2}}}{\hbar}\int_R^{r_1}(V-E)^{\frac{1}{2}}dr-\frac{r_1-R}{b}\right\}. \tag{41}$$

[20] E. J. Konopinski and H. A. Bethe, Phys. Rev. **54**, 130 (1938).

It can easily be seen that the exponential in (41) is greater than e^{-2G}. The difference arises from the fact that the main absorption occurs not at the boundary of the nucleus but farther out (at r_1) because the square of the wave function, $|u|^2$, increases faster from R to r_1 than σ decreases, a behavior similar to the Oppenheimer-Phillips process. The effect depends, of course, essentially on the exponential decrease of σ outside the nucleus; if σ decreased more strongly, e.g., as $e^{-(r-R)^2}$, the main absorption would always be in the neighborhood of the nucleus.

Because of its dependence on the model, the deviation of ζ from the simple exponential e^{-2G} should not be taken too seriously; but apart from this, the deviation is never appreciable compared with e^{-2G} itself. If we assume $V-E$ to vary as $1/r$ the additional terms in the exponent in (41) have the value

$$\gamma_1=(R/b)(\nu^{\frac{1}{2}}-1)^2, \tag{41a}$$

where

$$\nu=W/(\hbar^2/8mb^2). \tag{41b}$$

If $V-E$ varies as $1/r^2$, we have instead of (41a)

$$\gamma_2=(R/b)(\tfrac{1}{2}\nu^{\frac{1}{2}}\log\nu-\nu^{\frac{1}{2}}+1). \tag{41c}$$

Both (41a) and (41c) are relatively small. E.g., for $\nu=2$, i.e. $W=14$ Mev, we have $\gamma_1=0.17R/b$ and $\gamma_2=0.077R/b$. Since R/b is only 10 for the heaviest nuclei, the exponential differs from e^{-2G} only by a factor $e^{1.7}$ or $e^{0.77}$, respectively, which is not important compared with the large factor e^{-2G}. It is very unlikely that the additional terms in the exponential in (41) can ever be tested experimentally. Thus far, no experiments at all have been made in which the energy of the incident particle was more than $\hbar^2/8mb^2\approx 7$ Mev below the top of the potential barrier.

Summarizing, we find that (38b) is valid for moderate negative energies, perhaps from $\kappa b=1/2\pi$ to $\kappa b=\frac{1}{2}$ or $W=-0.7$ to -7 Mev. The value of ζ for greater $|W|$ is of no practical importance and depends on the exact behavior of σ as a function of r.

Our formulae for ζ can, of course, be used both to obtain the total cross section for all possible reactions in the case of overlapping levels, according to (20), and also the average width of resonance levels if such exist, according to (26).

The elastic scattering can be obtained from the behavior of the wave function for large r. It must be remembered, of course, that the Coulomb potential in the case of charged particles produces scattering by itself. Let δ be the phase shift due to the Coulomb potential for a given l ($\delta=0$ for neutrons), then the actual scattered amplitude for this l is

$$F_l=-\tfrac{1}{2}i\lambda\{e^{2i\delta}[1-ie^{-\pi i\kappa b}e^{-2G}(-\kappa b)!/(\kappa b)!]-1\}(2l+1)). \tag{42}$$

Terms of order e^{-4G} have been neglected. The differential scattering cross section per unit solid angle is $|\sum_l F_l P_l(\cos\vartheta)|^2$. The term proportional to e^{-2G} in (42) represents the nuclear scattering. For neutrons, this term alone exists, and we have

$$F_l=-(l+\tfrac{1}{2})\lambda e^{-2G}e^{-\pi i\kappa b}(-\kappa b)!/(\kappa b)! \tag{42a}$$

(The potential V_{ext} for neutrons is simply the centrifugal potential, $\hbar^2 l(l+1)/2mr^2$.) The expression (42a) is nearly real for small κb and corresponds then to the potential scattering found for small positive W (§6B). The real part decreases in relative importance as κb increases and reaches zero for $\kappa b=\frac{1}{2}$. Then there remains only the imaginary part which is required by the theorem of Bohr, Peierls and Placzek, and is equal to

$$F_l=\tfrac{1}{2}i(l+\tfrac{1}{2})\lambda\zeta. \tag{42b}$$

For $\kappa b>\frac{1}{2}$, (42a) is no longer valid, for the same reasons as (38). However, the WKB is then very nearly valid; and it can easily be shown that F_l is purely imaginary when terms of the order $\sigma^2/(V-E)^2$ are neglected [cf. above, Eq. (40) ff.]. Therefore (42b) remains valid; the elastic scattering at very large negative kinetic

energies, just as at large positive energies, is only the amount necessary because of the existence of inelastic processes according to the optical theorem.

§8. Spatial Variation of the External Potential

With the exception of the case of neutrons of zero orbital momentum, the extranuclear potential V_{ext} changes (decreases) with r. Therefore, if W is very small, $V_{\text{ext}}-E$ will change by its own amount and more over a very short distance. In this case, it is not justified to consider $V_{\text{ext}}-E$ as constant over the range of action of the nuclear absorption coefficient σ, as has been done in §4–7. It will, however, be sufficient to consider $V_{\text{ext}}-E$ as changing linearly with r. We shall put

$$(2m/\hbar^2)(E-V_{\text{ext}})=k^2+A(r-R'), \tag{43}$$

where A is a positive constant measuring the force on the particle,

$$A=-(2m/\hbar^2)(dV/dr)_{R'}, \tag{43a}$$

and R' is the radius of the hard sphere which is equivalent to the nucleus with regard to the potential scattering of slow neutrons (§6B). It is more convenient to use R' than R for our purpose. $(\hbar^2/2m)k^2$ is the kinetic energy at $r=R'$.

At large distances from the nucleus, σ is negligible and we have

$$d^2u/dr^2+[k^2+A(r-R')]u=0. \tag{44}$$

Here we introduce the abbreviation

$$x=r-R'+k^2/A. \tag{44a}$$

The solution of (44) is well known and has been investigated in detail by Kramers[21] in connection with the WKB method. The regular solution which vanishes for large negative x and is identical with the solution investigated by Kramers, is given by

$$v=\tfrac{1}{3}(\pi x)^{\frac{1}{2}}\{J_{1/3}(\tfrac{2}{3}A^{\frac{1}{2}}x^{\frac{3}{2}})+J_{-1/3}(\tfrac{2}{3}A^{\frac{1}{2}}x^{\frac{3}{2}})\}, \tag{45}$$

where J is the Bessel function. The function v is so normalized as to go over, for large x, into the WKB solution

$$v\rightarrow(Ax)^{-\frac{1}{4}}\cos(\tfrac{2}{3}A^{\frac{1}{2}}x^{\frac{3}{2}}-\tfrac{1}{4}\pi). \tag{45a}$$

For negative x, the analytical continuation of (45) is

$$v=\tfrac{1}{2}(\pi|x|/3)^{\frac{1}{2}}e^{2\pi i/3}H_{1/3}^{(1)}(\tfrac{2}{3}A^{\frac{1}{2}}i|x|^{\frac{3}{2}}). \tag{45b}$$

The irregular solution of (44) which is shifted in phase by $\pi/2$ against v, is the function

$$w=3^{-\frac{1}{2}}(\pi x)^{\frac{1}{2}}\{J_{-1/3}(\tfrac{2}{3}A^{\frac{1}{2}}x^{\frac{3}{2}})-J_{1/3}(\tfrac{2}{3}A^{\frac{1}{2}}x^{\frac{3}{2}})\}, \tag{46}$$

whose asymptotic behavior for large x is

$$w\rightarrow(Ax)^{-\frac{1}{4}}\cos(\tfrac{2}{3}A^{\frac{1}{2}}x^{\frac{3}{2}}+\tfrac{1}{4}\pi). \tag{46a}$$

The general solution may be written

$$u=v+\alpha w. \tag{47}$$

Then the ratio of the amplitude of the outgoing to that of the incident wave is

$$\beta=(1+i\alpha)/(1-i\alpha), \tag{47a}$$

the reflection coefficient is $R=|\beta|^2$ and the scattered amplitude is proportional to $\beta-1$. Therefore we need only determine α. Inserting (45), (46), and omitting a constant factor, we may write (47) in the form

$$u\sim x^{\frac{1}{2}}[(1+\alpha\sqrt{3})J_{-1/3}(\tfrac{2}{3}A^{\frac{1}{2}}x^{\frac{3}{2}})+(1-\alpha\sqrt{3})J_{1/3}(\tfrac{2}{3}A^{\frac{1}{2}}x^{\frac{3}{2}})]. \tag{48}$$

We shall be interested in the behavior of u for $r\approx R'$, and small k^2, i.e., for small x [cf. Eq. (44a)].

[21] H. A. Kramers, Zeits. f. Physik **39**, 828 (1926).

In this case we have

$$x^{\frac{1}{2}}J_{-1/3}(\tfrac{2}{3}A^{\frac{1}{2}}x^{\frac{3}{2}})=3^{\frac{1}{3}}A^{-\frac{1}{6}}(-\tfrac{1}{3})!^{-1}(1-\tfrac{1}{6}Ax^3+\cdots), \tag{48a}$$

$$x^{\frac{1}{2}}J_{1/3}(\tfrac{2}{3}A^{\frac{1}{2}}x^{\frac{3}{2}})=3^{-\frac{1}{3}}A^{\frac{1}{6}}(\tfrac{1}{3})!^{-1}x(1-\tfrac{1}{12}Ax^3+\cdots). \tag{48b}$$

Inserting in (48) and leaving out a constant factor, we find

$$u\sim 1-\tfrac{1}{6}Ax^3+\cdots+\frac{1-\alpha\sqrt{3}}{1+\alpha\sqrt{3}}\frac{3^{\frac{1}{3}}}{2}\frac{(\frac{2}{3})!}{(\frac{1}{3})!}A^{\frac{1}{3}}x(1-\tfrac{1}{12}Ax^3+\cdots). \tag{49}$$

It will be convenient to use the abbreviation

$$\kappa=2\cdot 3^{-\frac{1}{3}}\cdot(\tfrac{1}{3})!/(\tfrac{2}{3})!=1.372. \tag{49a}$$

Kramers has given a table of the regular solution v for small values of $\xi=A^{\frac{1}{3}}x$. By numerical calculation, it can be shown that this solution differs by only $4\frac{1}{2}$ percent from the WKB value at $\xi=+1$, and by 8 percent at $\xi=-1$. It will therefore be necessary to use the more complicated theory of this section only if $|\xi|<1$ at the boundary of the nucleus ($r=R'$). This condition is equivalent to

$$|k^2|<A^{\frac{2}{3}} \tag{50}$$

or

$$|E-V_{\text{ext}}(R')|<(\hbar^2/2m)^{\frac{1}{3}}(dV/dr)^{\frac{2}{3}}. \tag{50a}$$

For $|\xi|<1$, the expressions given in (48a, b) for the J's are accurate to $\frac{1}{2}$ and $\frac{1}{5}$ percent, respectively.

Our problem is now to solve the Schrödinger equation including σ, i.e.

$$\frac{d^2u}{dr^2}+(Ax+\frac{i}{b^2}e^{-(r-R)/b})u=0. \tag{51}$$

This can, of course, not be done exactly. However, we can use the fact that k^2 is small to get a good approximation. In the region where σ is important, Ax will be relatively small and it will be sufficient to take it into account in first approximation. The solution obtained in this way will, for large r, go over automatically into the solution of the equation without σ, neglecting terms in A^2 in the latter. As pointed out in the last paragraph, this is justified for all cases in which the variation of V_{ext} with r is at all important.

Equation (51) with $A=0$ has the solution (cf. §4)

$$u_0=2H_0^{(1)}(2i^{\frac{1}{2}}e^{-(r-R)/2b}), \tag{52}$$

the factor 2 being introduced for convenience in the following. For large r, (52) behaves as

$$u_0\rightarrow-(2i/\pi)(r-R')/b. \tag{52a}$$

With the usual method, we obtain between u and u_0 the relation

$$u_0du/dr-udu_0/dr=\int_{-\infty}^{r}[A(\rho-R')+k^2]u(\rho)u_0(\rho)d\rho. \tag{53}$$

Here it has been assumed that u as well as u_0 decreases towards small r inside the nucleus. In first approximation, we may put $u=u_0$ in the integral. Then the integration is elementary for values of r large enough so that (52a) is valid. Only the integration constant cannot be determined in an elementary way; it must be a linear function of A and k^2. The coefficient of k^2 can be determined by analytical integration, that of A was found by numerical integration. Neglecting terms of order $e^{-(r-R')/b}$ and using the abbreviation

$$y=(r-R')/b, \tag{53a}$$

we obtain the result:

$$u=1-2iy/\pi+k^2b^2(\lambda y-\tfrac{1}{2}y^2+(i/3\pi)y^3)+Ab^3(\mu y-\tfrac{1}{6}y^3+(i/6\pi)y^4) \tag{54}$$

with

$$\lambda=\frac{32}{3\pi^2}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\cdots\right)-\frac{\pi}{6}i=1.298-0.523i, \tag{54a}$$

$$\mu=1.339-2.034i. \tag{54b}$$

We may now compare (49) with (54). Remembering that [cf. Eqs. (44a), (53a)]

$$x=by+k^2/A \tag{55}$$

and introducing the abbreviations

$$a=A^{\frac{1}{3}}b, \tag{55a}$$

$$v=k^2/A^{\frac{1}{3}} \tag{55b}$$

we obtain

$$\frac{1-\alpha\sqrt{3}}{1+\alpha\sqrt{3}}=\kappa\frac{-2i/\pi a+\mu a^2+\lambda av+\frac{1}{2}v^2+iv^3/3\pi a}{1+2iv/\pi a-\mu a^2v-\lambda av^2-\frac{1}{3}v^3-iv^4/6\pi a}. \tag{56}$$

Let us denote numerator and denominator of the right-hand side by ν and ρ, respectively; then the reflected amplitude β defined in (47a) is

$$\beta=\frac{\rho(\sqrt{3}+i)+\nu(\sqrt{3}-i)}{\rho(\sqrt{3}-i)+\nu(\sqrt{3}+i)}. \tag{56a}$$

Of the two constants v and a occurring in (56), v lies between the limits $+1$ and -1, because v is simply the value of $A^{\frac{1}{3}}x$ for $r=R'$, and because the WKB is valid when the absolute value of $A^{\frac{1}{3}}x$ is greater than 1 [cf. Eq. (50)]. The constant a is always positive and given by [cf. Eqs. (43a), (55a)]

$$a^3=2mb^3\hbar^{-2}|dV/dr|_{R'}. \tag{57}$$

We have

$$V(R')=Ze^2/R'+\hbar^2l(l+1)/2mR'^2, \tag{57a}$$

$$V/R'\leqslant|dV/dr|\leqslant 2V/R'. \tag{57b}$$

Since we are interested in the case of small kinetic energy $V-E$, we may put

$$|dV/dr|=\eta E/R', \tag{57c}$$

where $\eta=2$ for neutrons, $\eta=1$ for charged particles with $l=0$, and η between 1 and 2 for charged particles with $l\neq 0$. Therefore (57) becomes

$$a=(b/\lambda)^{\frac{1}{3}}(\eta b/R')^{\frac{1}{3}}. \tag{58}$$

For atomic weights between 10 and 240, the last factor is between 0.8 and 0.45. The first factor, for $b=0.86\times10^{-13}$, is equal to $(E/28\text{ Mev})^{\frac{1}{3}}$, i.e., 0.7 for 10 Mev and 0.33 for 1 Mev. Therefore a is in general less than one, and may vary between about 0.15 and 0.5.

For $v=0$, i.e. when the energy of the incident particle is exactly equal to the height of the potential barrier, the numerator and denominator of (56) reduce to

$$\nu=-2i\kappa/\pi a+\kappa\mu a^2;\quad \rho=1. \tag{51}$$

In Table I we give the value of β and of the formation probability, $\zeta=1-|\beta|^2$, for a few values of a in the important region. It is seen that ζ is of the order unity and changes only slightly with a, in the expected direction, i.e., increasing with increasing a. This shows that, due to the variation of V with r, the point $v=0$ ceases to be a singular point and is characterized by a fairly large formation probability, as has been anticipated in §7. From the value of β, the elastic scattering can be found [cf. Eq. (22)].

For $v\neq 0$, Eq. (56) is more complicated. However, it can easily be seen that $|\beta|$ decreases and

TABLE I. *Changing potential, $W=0$.*

a	β	ζ
small	$\frac{1}{2}-\frac{1}{2}i\sqrt{3}-1.146a(\sqrt{3}-i)$	$3.97a$
0.2	$0.369-0.570\,i$	0.539
0.3	$0.350-0.467\,i$	0.659
0.4	$0.356-0.404\,i$	0.710
0.5	$0.382-0.369\,i$	0.718

therefore ζ increases with increasing v, as it should be. Computations for $v\neq 0$ are in progress at the University of Rochester.

§9. VERY HIGH ENERGY

For very high energy, the formalism of the compound nucleus breaks down. Instead, the Born approximation is applicable. Because of its simplicity, this method permits a comparison between our theory and general nuclear theory. Neglecting the correlations between the various particles in the nucleus, the total cross section for all processes is from general nuclear theory:

$$\sigma_t=(A/2\pi\hbar^2v^2)\times\int_0^\infty qdq\left|\int V_n(\mathbf{r})\exp\left[i\mathbf{q}\cdot\mathbf{r}\right]d\tau\right|^2, \quad (60)$$

where $V_n(\mathbf{r})$ is the nuclear interaction potential as a function of the distance r between the interacting particles, $\hbar\mathbf{q}$ is the momentum change and v the initial velocity of the incident particle and A is the number of particles in the nucleus.

In (60), we have assumed that the interaction of very fast nuclear particles can still be described by a potential. In the following, we shall even assume that this potential V_n is the same as for the usual energies of a few Mev. There is no justification for such an assumption. However, we are not interested in the actual behavior of high energy nuclear particles but rather in the mathematical problem of the behavior of fast particles which have the interaction V_n. From the solution of this problem which can be found because of its mathematical simplicity, we can draw conclusions about the problem which forms the subject of this paper, *viz.* the behavior of particles of moderate energy having an interaction V_n with the nucleus.

The integral in (60) reduces to a quantity closely related to the binding energy of the deuteron, e.g., for an exponential potential

$$V_n(r)=Be^{-r/b}. \quad (60a)$$

(60) reduces to

$$\sigma_t=(16\pi/3)A(Bb^2/\hbar v)^2. \quad (60b)$$

According to the theory of the deuteron, with $b=0.86\times10^{-13}$ cm,

$$Bb^2=2.23\hbar^2/m. \quad (60c)$$

Considering that for the singlet potential Bb^2 is only about $1.31\hbar^2/m$, we obtain

$$\sigma_t=4\pi\lambda\!\!\!^{-2}A\cdot 5.5, \quad (61)$$

where $\lambda\!\!\!^{-}$ is the wave-length of the incident particle. This approximation is applicable when $\sigma_t\ll\pi R^2$. Putting (cf. reference 8)

$$R=r_0A^{\frac{1}{3}},\quad r_0\approx 1.3_5\cdot10^{-13}\text{ cm}, \quad (61a)$$

we obtain for the critical wave-length, at which $\sigma_t=\pi R^2$:

$$\lambda\!\!\!^{-}_c=r_0/(2\cdot5.5^{\frac{1}{2}}A^{\frac{1}{6}})=2.8_5\cdot10^{-14}A^{-\frac{1}{6}}\text{ cm}. \quad (61b)$$

For $A=100$, this is 1.3×10^{-14} cm and corresponds to a neutron or proton of 1200 Mev energy. The Born approximation is restricted to energies higher than this.

From our theory of the absorption potential, [cf. Eq. (1)] we have in the case of the Born approximation

$$\sigma_t=\frac{2\sigma_0}{\hbar v}\cdot\frac{4\pi}{3}R^3, \quad (62)$$

where σ_0 is the absorption potential inside the nucleus. Comparing (62) with (61), (61a), we find that in the region of the Born approximation we must have

$$\sigma_0=\frac{3}{2}\frac{\hbar^2}{mr_0^3}\cdot5.5\lambda\!\!\!^{-}\approx140\lambda\!\!\!^{-}\text{ Mev}, \quad (62a)$$

if $\lambda\!\!\!^{-}$ is measured in 10^{-13} cm. This shows (1) that σ_0 is proportional to $1/v$ and thus changes only slowly with the energy of the incident particle, (2) that σ_0 depends on $|V_n|^2$ [cf. Eq. (60)] rather than on the average of V_n, (3) that σ_0 is very large. Point (1) has been assumed when σ was introduced in §2 and is essential in order to give validity to the dependence of the formation probability ζ on the energy as derived in this paper. Point (2) is important in order to

justify the neglect of the average nuclear potential V_n compared to the absorption potential σ (cf. the beginning of §4); the attractive and repulsive interactions existing between particles of different relative spin and charge will average out in V_n but will all give a positive contribution to σ. Finally (3) the magnitude of σ_0 may not be quite as large as would follow from (62a) for moderate energies ($\lambda\!\!\!^{-}=1-5\times10^{-13}$ cm, $E=20-1$ Mev) because (62a) has only been proved for $\lambda\!\!\!^{-}<\lambda\!\!\!^{-}_c$ [cf. Eq. (61b)]. For $\lambda\!\!\!^{-}_c\approx1.3\times10^{-14}$ cm, we have $\sigma_0=18$ Mev, and it seems safe to assume that σ_0 is greater than this value for moderate energies.

In our theory, it is quite easy to treat the transition from moderate to high energies. It is only necessary to consider the finite size of the nucleus and to make the wave function u equal to zero at $r=0$. Since we can certainly apply the WKB for these energies, we obtain

$$u(r)=\exp\left(-i(2m)^{\frac{1}{2}}\hbar^{-1}\int_0^r(E-V+i\sigma)^{\frac{1}{2}}d\rho\right)$$
$$-\exp\left(i(2m)^{\frac{1}{2}}\hbar^{-1}\int_0^r(E-V+i\sigma)^{\frac{1}{2}}d\rho\right). \quad (63)$$

Assuming σ small compared with E everywhere, and neglecting σ^2, we obtain for large r:

$$u(r)=\exp\left(-ikr-i\delta+\frac{m^{\frac{1}{2}}}{\sqrt{2}\hbar}\int_0^r\frac{\sigma(\rho)}{(E-V(\rho))^{\frac{1}{2}}}d\rho\right)$$
$$-\exp\left(ikr+i\delta-\frac{m^{\frac{1}{2}}}{\sqrt{2}\hbar}\int_0^r\frac{\sigma(\rho)}{(E-V(\rho))^{\frac{1}{2}}}d\rho\right), \quad (63a)$$

where δ is the phase shift. Taking $\sigma=\sigma_0$ for $r\leqslant R$ and $\sigma=0$ for $r>R$, and taking for V the pure centrifugal potential $\hbar^2l(l+1)/2mr^2$, we find

$$\zeta=1-\exp\left(-\frac{4\sigma_0}{\hbar v}(R^2-l^2\lambda\!\!\!^{-2})^{\frac{1}{2}}\right). \quad (63b)$$

Summing over all l from 0 to $R/\lambda\!\!\!^{-}$, the total cross section is then

$$\sigma_t=\pi R^2\left\{1-2\frac{1-e^{-x}(1+x)}{x^2}\right\} \quad (64)$$

with

$$x=4\sigma_0R/\hbar v. \quad (64a)$$

§10. Influence of Nuclear Potential

Although we have given arguments for the assumption that the average nuclear potential V_n is small compared with the absorption potential σ (beginning of §4), we shall investigate the influence on our results of a nonvanishing V_n, attractive or repulsive. We shall assume that V_n is proportional to σ, *viz.*

$$V_n=\alpha\sigma. \quad (65)$$

Then the solution (16) of the Schrödinger equation is to be replaced by

$$u=cH_{2ikb}^{(1)}(2(i-\alpha)^{\frac{1}{2}}x). \quad (65a)$$

For small argument x, this becomes [cf. Eq. (16b)]

$$u=x^p-e^{\pi ip}(i-\alpha)^{-p}x^{-p}p!/(-p)!, \quad (65b)$$

so that (17) is replaced by

$$u=e^{-ik(r-R)}-e^{ik(r-R)}$$
$$\times e^{-(\pi+2\text{ arc tan }\alpha)kb+2i\eta-ikb\log(1+\alpha^2)} \quad (66)$$

and the formation probability (18) by

$$\zeta=1-e^{-2(\pi+2\text{ arc tan }\alpha)kb}. \quad (66a)$$

The main change is the appearance of $\pi+2$ arc tan α, instead of π, in the exponent. For an attractive potential, $\alpha<0$ and therefore the sticking probability is *decreased*. On the other hand, a repulsive nuclear potential will serve to increase ζ. The change of ζ will be only slight if α is small, i.e. if the nuclear potential is smaller than the absorption potential. Only for $|\alpha|\gg1$ and α negative, will the formation probability be very much less than for $\alpha=0$. Then we have practically only an attractive potential, and we consequently get more elastic scattering and less absorption, A large repulsive potential, on the other hand, increases the sticking probability for slow neutrons by a factor of two. Neither case is of practical importance.

The elastic scattering is modified by the term $-ikb\log(1+\alpha^2)$ in the exponential in (65) which corresponds to an increase of the radius of the effective hard sphere by $\frac{1}{2}b\log(1+\alpha^2)$, both for repulsive and attractive potential.

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