In contemplating long operating life the question of vacuum deserves consideration. Some observations were made last year on power Thyratrons which had been in operation for two years at constant current (175 amperes, 120 degree conduction). These tubes were tested in the laboratory for ability to operate at high voltage, and the results compared to initial tests on the same tubes. In every case the voltage tolerance was found to be higher than at the start. This shows that vacuum is not impaired by operation under these conditions.

JULY 1, 1939 PHYSICAL REVIEW VOLUME 56

Change of Resistance in a Magnetic Field

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(Received April 18, 1939)

The general theory of the Hall effect and the change of resistance in a magnetic field expresses these quantities in terms of a number of integrals over the surface of the Fermi distribution. The values of these integrals depend upon the form of the electron energy and the relaxation time as functions of the wave vector. If the free electron situation is assumed, the Hall effect has the right order of magnitude, but there is no change of resistance. This can be seen from a qualitative consideration of the effect of the fields on the distribution function. A general form for the functions in other cases can be obtained as an expansion in spherical harmonics with the symmetry of the crystal lattice. The results can then be expressed in terms of the coefficients in this expansion. When only the first two harmonics are retained, the computed change of resistance and Hall effect are close to the observed values. However, contrary to the available observations, the ratio of the transverse to the longitudinal change of resistance shows a minimum value of about four. It seems improbable that this result could be changed in any material way by the inclusion of higher series members, so that if the experimental results are to be taken as reliable, doubt is thrown on the general method of treatment.

§1. INTRODUCTION

CHANGE of resistance in a magnetic field cannot be interpreted in terms of a free electron picture. But since such a change is observed in the alkalies, it is of interest to see if it can be understood as a small departure from the free electron situation. Jones and Zener[1] have given a theory for this effect that seems to give quite satisfactory numerical results in the case of lithium. However, their approximate method of evaluating the integrals made it necessary to apply, later, an estimated correction factor of about six. This paper attempts to give a method of evaluating the integrals that will enable one to get analytically as good an approximation as is desired, provided the surfaces of constant energy are not too irregular. Also it is not assumed that the relaxation time is a function of the energy only. §2 gives a summary of the general theory, in Wilson's[2] notation, and §3 gives a physical picture of the situation, in which particular attention is given to the free electron case. In §4 the energy and the relaxation time are expressed in terms of series of cubically symmetric spherical harmonics. By a proper choice of independent variables, the integrals can all be evaluated in terms of the coefficients of the spherical harmonics. If the first two terms of the series are taken as a satisfactory approximation, expressions are obtained for the conductivity, the Hall coefficient, and the two coefficients, B_t and B_l, of the change of resistance in a magnetic field. In §5 the conclusions that may be drawn from these expressions are discussed. It is found that theory and experiment give radically different values of B_t/B_l. Experimentally this ratio is about unity, while no choice of parameters can give a theoretical value less than 4. The expressions for B_t and B_l show that the variation of the

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[1] H. Jones and C. Zener, Proc. Roy. Soc. **A145**, 268 (1934).

[2] A. H. Wilson, *The Theory of Metals* (Cambridge Univ. Press, 1937), Chapter V.

relaxation time with direction, as well as with energy, is nearly as important in all these effects as the fact that the surfaces of constant energy are not spheres.

§2. General Results

We will specify the state of an electron in the usual manner by the wave vector $\mathbf{k}$ whose Cartesian components are (k_1, k_2, k_3). The energy eigenfunctions of the electrons when unperturbed by external fields can be taken as

$$\psi_{\mathbf{k}}(\mathbf{r}) = \exp(i\mathbf{k}\cdot\mathbf{r})u_{\mathbf{k}}(\mathbf{r}),$$

where $u_{\mathbf{k}}(\mathbf{r})$ has the periodicity of the lattice. The number of electrons per unit volume whose wave vectors lie in the range (dk_1, dk_2, dk_3) is

$$(1/4\pi^3)f(\mathbf{k})dk_1dk_2dk_3.$$

We omit any dependence on $\mathbf{r}$ since we are going to be interested only in the case in which the temperature and composition of the metal are independent of $\mathbf{r}$.

If there exist in the metal an electric field, $\boldsymbol{E}$, and a magnetic field, $\mathbf{H}$, then the condition that the distribution function, $f(\mathbf{k})$, remains constant in time under the influence of the fields and the collisions of the electrons with the lattice points is given by Boltzmann's equation,

$$-(\epsilon/\hbar)[\boldsymbol{E}+\mathbf{v}_{\mathbf{k}}\times\mathbf{H}/c]\cdot\text{grad.}_{\mathbf{k}}f + (f-f_0)/\tau(\mathbf{k}) = 0, \quad (1)$$

where $-\epsilon$ is the charge on an electron, $\mathbf{v}_{\mathbf{k}}$ is the velocity associated with the state $\mathbf{k}$, and f_0 is the value of f when $\boldsymbol{E}=\mathbf{H}=0$. We have assumed that a relaxation time, $\tau(\mathbf{k})$, can be defined so that we can use this form of Boltzmann's equation. A relaxation time can be used if the temperature is greater than the Debye characteristic temperature, and perhaps under other conditions. Eq. (1) is only valid if $\epsilon H\tau/2mc \ll 1$.

To solve Eq. (1), take the direction of the magnetic field as the z axis and substitute for $\mathbf{v}$ its value in terms of $\mathbf{k}$,

$$\mathbf{v}_{\mathbf{k}} = (1/\hbar)\,\text{grad.}_{\mathbf{k}}\,E(\mathbf{k}), \quad (2)$$

where $E(\mathbf{k})$ is the energy associated with the state $\mathbf{k}$. It should be noted that E refers to the energy of an electron while $\mathbf{E}$ always refers to the electric field.

Let

$$f = f_0 - \Phi(\mathbf{k})\partial f_0/\partial E.$$

The equation which $\Phi(\mathbf{k})$ is to satisfy is

$$\Phi(\mathbf{k})/\tau(\mathbf{k}) + (\epsilon/\hbar)\boldsymbol{E}\cdot\text{grad.}_{\mathbf{k}}\,E - (\epsilon H/\hbar^2 c)\Omega\Phi(\mathbf{k}) = 0, \quad (3)$$

where the product of $\boldsymbol{E}$ and Φ is neglected, and Ω is the operator

$$\Omega = \frac{\partial E}{\partial k_2}\frac{\partial}{\partial k_1} - \frac{\partial E}{\partial k_1}\frac{\partial}{\partial k_2}.$$

This process is equivalent to a development of f as a power series in the components of $\boldsymbol{E}$ and the neglect of all terms of higher than the first degree.

To solve Eq. (3) we write $\Phi(\mathbf{k})$ as a power series in H, and obtain

$$\Phi(\mathbf{k}) = -(\epsilon/\hbar)\{\tau\boldsymbol{E}\cdot\text{grad.}\,E + (\epsilon H/\hbar^2 c)\tau\Omega(\tau\boldsymbol{E}\cdot\text{grad.}\,E) + (\epsilon H/\hbar^2 c)^2\tau\Omega[\tau\Omega(\tau\boldsymbol{E}\cdot\text{grad.}\,E)] + \cdots\}. \quad (4)$$

This is one particular integral of (3) and may or may not be the solution we desire. The general solution of (3) is obtained by adding to (4) the general solution of

$$\Phi(\mathbf{k}) = \tau(\mathbf{k})(\epsilon H/\hbar^2 c)\Omega\Phi(\mathbf{k}) = (\tau\epsilon H/\hbar^2 c)[\text{grad.}_{\mathbf{k}}\,\Phi(\mathbf{k})]\cdot[\text{grad.}_{\mathbf{k}}\,E(\mathbf{k})]\times\mathbf{e}_3,$$

where $\mathbf{e}_3$ is a unit vector in the k_3 direction. Consider the curve of intersection of any surface of constant energy with the plane $k_3=0$. The vector $(\text{grad.}_{\mathbf{k}}\,E)\times\mathbf{e}_3$ will be directed along this curve. In all cases of interest $E(\mathbf{k})$ is such that this vector is directed in the clockwise sense of describing the curve. Since $\tau(\mathbf{k})$ is essentially positive, our equation shows that the component of $\text{grad.}_{\mathbf{k}}\,\Phi$ in the direction of $(\text{grad.}_{\mathbf{k}}\,E)\times\mathbf{e}_3$ is always of the same sign as Φ. Hence if we traverse our curve in the clockwise sense, $|\Phi|$ will increase unless $\Phi=0$. Therefore the only single-valued solution of the homogeneous equation is $\Phi\equiv 0$. Hence (4) is the desired solution.

The reason that we can terminate the series in the components of $\boldsymbol{E}$ with the linear terms while we must include at least the quadratic term in the series in H is that the magnetic force is very much larger than the electrostatic for electrons whose representative points are at the surface of the Fermi distribution. If the current density in silver is 10 amp. cm^{-2}, and if the magnetic field is 1000 gauss, the magnetic force is 10^8 times as great as the electric force.

We can get expressions for the conductivity, the Hall coefficient, and the change of resistance in a magnetic field by substituting the value of Φ found above in the equation giving the current density,

$$\mathbf{J} = -(\epsilon/4\pi^3)\int \mathbf{v}_{\mathbf{k}} f(\mathbf{k})dV$$
$$= (\epsilon/4\pi^3\hbar)\int \operatorname{grad}_{\cdot\mathbf{k}} E(\partial f_0/\partial E)\Phi dV,$$

where dV is an element of volume in $\mathbf{k}$-space.

To do this we introduce the following abbreviations:

$$I_1 = -\int \tau(\partial f_0/\partial E)(\partial E/\partial k_1)^2 dV,$$

$$I_2 = -\int \tau(\partial f_0/\partial E)(\partial E/\partial k_2)^2 dV,$$

$$I_3 = -\int \tau(\partial f_0/\partial E)(\partial E/\partial k_3)^2 dV,$$

$$I_4 = I_2 I_6 - I_7^2,$$

$$I_5 = \int \tau\frac{\partial f_0}{\partial E}\frac{\partial E}{\partial k_3}\Omega\left\{\tau\Omega\left(\tau\frac{\partial E}{\partial k_3}\right)\right\}dV$$
$$= -\int \tau\frac{\partial f_0}{\partial E}\left\{\Omega\left(\tau\frac{\partial E}{\partial k_3}\right)\right\}^2 dV, \quad (5)$$

$$I_6 = \int \tau\frac{\partial f_0}{\partial E}\frac{\partial E}{\partial k_1}\Omega\left\{\tau\Omega\left(\tau\frac{\partial E}{\partial k_1}\right)\right\}dV$$
$$= -\int \tau\frac{\partial f_0}{\partial E}\left\{\Omega\left(\tau\frac{\partial E}{\partial k_1}\right)\right\}^2 dV,$$

$$I_7 = \int \tau\frac{\partial f_0}{\partial E}\frac{\partial E}{\partial k_1}\Omega\left(\tau\frac{\partial E}{\partial k_2}\right)dV$$
$$= -\int \tau\frac{\partial f_0}{\partial E}\frac{\partial E}{\partial k_2}\Omega\left(\tau\frac{\partial E}{\partial k_1}\right)dV.$$

The second form of each integral is obtained by integrating the first form by parts.

For those metals in which $E(\mathbf{k})$ and $\tau(\mathbf{k})$ are even functions of k_1, of k_2, and of k_3, we find that the electrical conductivity in the absence of a magnetic field is

$$\sigma = J_x/E_x = (\epsilon^2/\pi h^2)I_1, \quad (6)$$

that the Hall coefficient is

$$R = E_y/HJ_x = E_y/H\sigma E_x = -(4\pi^3/c\epsilon)I_7/I_1I_2, \quad (7)$$

and that the coefficients of the change of resistance in magnetic fields that are perpendicular and parallel, respectively, to the direction of the current are

$$B_t = (\sigma_0-\sigma)/\sigma H^2 = (\epsilon/\hbar^2 c)^2 I_4/I_1I_2 \quad (8)$$

and

$$B_l = (\sigma_0-\sigma)/\sigma H^2 = (\epsilon/\hbar^2 c)^2 I_5/I_3. \quad (9)$$

By Schwarz's inequality the change of resistance is always an increase. These formulas are equivalent to those given by Wilson.[2]

§3. Physical Picture

A careful study of the above equations enables one to get a physical picture of what is going on. In the absence of external forces, the distribution of representative points in $\mathbf{k}$-space is given by the Fermi function, $f_0(\mathbf{k})$. The density of distribution depends on the energy only. The application of a force, $\mathbf{F}$, causes the distribution of representative points to drift in the direction of the force, since rate of change of the state of an electron is given by

$$d\mathbf{k}/dt = \mathbf{F}/\hbar.$$

When we assume the Boltzmann equation in the form of Eq. (1), we are really assuming that the effect of the collisions of the electrons with the ions is such that the actual distribution, f, differs at each point from the normal distribution, f_0, by an amount that is proportional to the rate at which f would tend to be changed by the drift produced by the force. We are not assuming that a representative point drifts for a certain time or distance, on the average, before a collision changes its state.

Our series solution of Boltzmann's equation is really a solution by successive approximations Rather than consider the effect of the drift produced by the external forces on the final unknown distribution, we first calculate the change in distribution on the assumption that only the representative points of the undisturbed function, f_0, drift. Hence from Eq. (1) we see that in the first approximation we must add to the equilibrium distribution function the term

$$-\tau\mathbf{F}\cdot\mathrm{grad.}_{\mathbf{k}}\, f_0.$$

We get the second approximation by considering the drift of those points given by the first approximation and obtain

$$-\tau\mathbf{F}\cdot\mathrm{grad.}_{\mathbf{k}}[-\tau\mathbf{F}\cdot\mathrm{grad.}_{\mathbf{k}}\, f_0].$$

Each additional term is found by considering the action of the forces on the representative points given by the previous term.

Since the forces on conduction electrons due to magnetic fields in metals are, in general, much greater than those due to electric fields, we include the electric forces in the first term only of this series of approximations. We do not include the magnetic forces in this first term since the drift produced by them is perpendicular to $\mathbf{v}=\mathrm{grad.}_{\mathbf{k}}\, E/\hbar$. It is, therefore, along the surfaces of constant energy and produces no change in the undisturbed distribution. The first approximation gives the ordinary conductivity, the second gives the Hall effect, and the third gives the change of resistance in a magnetic field.

Let us consider our picture of the free electron case. We may assume complete degeneracy since Sommerfeld and Frank[3] have shown that the departure from complete degeneracy gives only about one-ten-thousandth of the observed change of resistance in a magnetic field. Therefore when no external field is applied, the density of representative points is uniform inside a certain sphere and zero outside. When an external field is applied, the change in the distribution function is given by a surface density of representative points on this sphere. We can show this in Fig. 1 by taking a cross section in **k**-space through the plane $k_3=0$. The circle represents the interaction of the sphere with this plane and the width of the surrounding band indicates the surface density of representative points. Black indicates a positive and white a negative surface density. For the case in which the electric field is in the x direction and the magnetic field is in the z direction, Fig. 1 (a) shows the effect of the electric field on the undisturbed distribution. Fig. 1 (b) gives the second approximation and shows the effect produced by the action of the magnetic field on the surface density of representative points shown in (a). Fig. 1 (c) gives the third approximation and shows the effect produced by the magnetic field on the points that make up the second approximation. The actual distribution, shown in (d), is obtained by superposing these three terms.

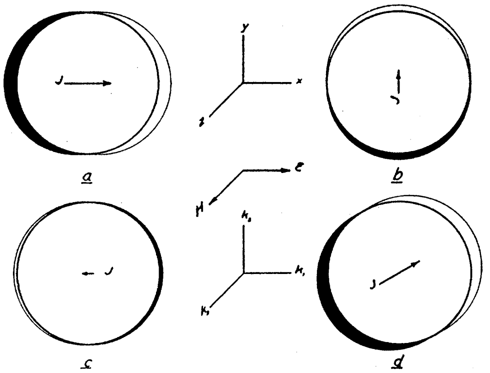

FIG. 1. The free electron case. Since the charge of the electron is taken to be negative, the representative points drift in a direction opposite to that of the electric field.

It is evident that if the electric field is in the x direction, then the current is not. But all the effects are defined for the case in which the current is in the same direction, with and without the magnetic field. Hence the Hall effect arises from the fact that to eliminate the current indicated in (b) we will need to apply an electric field in the negative y direction. Fig. 1 (c) would seem to indicate that the current is less when we have a magnetic field, that is, that the magnetic field increases the resistance. But if we consider the Hall effect of the electric field introduced to suppress the y component of the current, we see that if we have spherical symmetry, this effect completely cancels out and there is no change of resistance.

If the magnetic field is parallel to the current, the application of an electric field yields, if we have spherical symmetry, a first approximation that has cylindrical symmetry about an axis

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[3] A. Sommerfeld and N. H. Frank, Rev. Mod. Phys. **3**, 1 (1931).

parallel to the magnetic field. Therefore the magnetic field produces no change in the distribution function and hence no change of resistance.

If we do not have spherical symmetry, we must remember that the velocity is in a direction perpendicular to the surfaces of constant energy. The interaction of effects is very complicated, but, by a sufficient extension of this kind of argument, it can be seen in a qualitative way that the effect of all departures from spherical symmetry is a tendency to increase the resistance in a magnetic field. If the departures are small, the Hall effect is not greatly changed. If the departures from spherical symmetry are large, then by proper choice of the shapes of the surfaces of constant energy it is easy to explain the fact that the Hall coefficient is positive in some metals.

§4. Calculations

In the case of the monovalent metals we expect that the surface of the Fermi distribution will lie entirely within the first Brillouin zone and will be nearly spherical. By expressing the dependent variables in terms of spherical harmonics, we can evaluate the integrals of §2 and get simple expressions for the various coefficients.

Our integrals are all of the form

$$I=\int(\partial f_0/\partial E)F(\mathbf{k})dV,$$

where $dV=k^2\sin\theta dkd\theta d\varphi$ is an element of volume in $\mathbf{k}$-space. If we use as our independent variables E, θ and φ, this becomes

$$I=\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{\partial f_0}{\partial E}F(\mathbf{k})k^2\sin\theta\left(\frac{\partial k}{\partial E}\right)_{\theta,\,\varphi}dEd\theta d\varphi.$$

We can do this since E is a monotonic function of k in the cases in which we are interested. Near the surface of the Fermi distribution all the factors in the integrand vary slowly compared to $\partial f_0/\partial E$, while elsewhere $\partial f_0/\partial E$ is practically zero. Hence

$$I=-\int_0^{2\pi}\int_0^{\pi}\left[\frac{\partial k}{\partial E}F(\mathbf{k})k^2\right]_{E=E_0}\sin\theta d\theta d\varphi, \qquad (10)$$

where E_0 is the energy at the surface of the Fermi distribution.

The partial derivatives that we find in our integrands can be changed to the new variables by means of the following formulas from the calculus:

$$\begin{aligned}E_{k1}&=(k\sin^2\theta\cos\varphi-k_\theta\sin\theta\cos\theta\cos\varphi\\&\qquad+k_\varphi\sin\varphi)(kk_E\sin\theta)^{-1},\\E_{k2}&=(k\sin^2\theta\sin\varphi-k_\theta\sin\theta\cos\theta\sin\varphi\\&\qquad-k_\varphi\cos\varphi)(kk_E\sin\theta)^{-1},\\E_{k3}&=(k\cos\theta+k_\theta\sin\theta)(kk_E)^{-1},\end{aligned}$$

where E_{k1} means the partial of E with respect to k_1, keeping k_2 and k_3 constant; while k_E means the partial of k with respect to E, keeping θ and φ constant.

Now the lattices of all the monovalent metals have cubic symmetry. Consequently when we expand $k(E,\theta,\varphi)$ and $\tau(E,\theta,\varphi)$ in series of surface harmonics, we need include only those surface harmonics that have cubic symmetry. The first two such surface harmonics are

$$Y_0^c=1$$

and $Y_4^c=P_4(\cos\theta)+\cos 4\varphi P_4^4(\cos\theta)/168$;

the next is of the sixth degree. P_n^n is Ferrers' associated Legendre function. Neglecting terms of the sixth degree and higher in our expansion, we write

$$\begin{aligned}k&=\alpha_0(E)+\alpha_1(E)Y_4^c,\\\tau&=\tau_0(E)+\tau_1(E)Y_4^c.\end{aligned}\qquad(11)$$

Let us now transform the integrals of Eqs. (5) by the use of Eqs. (10) and (11). If we expand $[\alpha_0(E)+\alpha_1(E)Y_4^c]^n$, $[\alpha_0'(E)+\alpha_1'(E)Y_4^c]^n$, and $[\tau_0(E)+\tau_1(E)Y_4^c]^n$ by the binomial expansion, all the integrations can actually be carried out by means of formulas given by Gaunt.[4] If we write

$$a=\alpha_0(E_0),\quad a'=\left.\frac{d\alpha_0(E)}{dE}\right|_{E=E_0}\quad t=\tau_0(E_0),$$

$$A=\alpha_1(E_0)/\alpha_0(E_0),\quad B=\left.\frac{1}{a'}\frac{d\alpha_1(E)}{dE}\right|_{E=E_0},$$

$$T=\tau_1(E_0)/\tau_0(E_0),$$

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[4] J. A. Gaunt, Trans. Roy. Soc. **228**, 151 (1929), Appendix.

and neglect the cubes of A, B and T compared to unity, we get

$$I_1=I_2=I_3=(4\pi/3)(t/a')a^2\{1+(4/21)\\ \times[21A^2+2A(T-B)-B(T-B)]\},$$

$$I_4=(4\pi/3)^2(4/77)(t/a')^4a^2\\ \times\{597A^2+46A(T-B)+37(T-B)^2\},$$

$$I_5=(4\pi/3)(80/231)(t/a')^3\{3A+T-B\}^2, \qquad (12)$$

$$I_6=(4\pi/3)(t/a')^3\\ \times\{1+(4/231)[2220A^2+600A(T-B)\\ -33B(T-B)+532(T-B)^2]\},$$

$$I_7=(4\pi/3)(t/a')^2a\\ \times\{1+(4/21)[30A^2+22A(T-B)\\ -2B(T-B)+(T-B)^2]\}.$$

Substitution of these values in Eqs. (6)–(9) gives

$$\sigma=(4\epsilon^2/3h^2)(t/a')a^2\{1+(4/21)\\ \times[21A^2+2A(T-B)-B(T-B)^2]\}, \qquad (13)$$

$$R=-(3\pi^2/c\epsilon a^3)\{1+(4/21)\\ \times[-12A^2+18A(T-B)+(T-B)^2]\}, \qquad (14)$$

$$B_t=(4/77)(3\pi^2\sigma/c\epsilon a^3)^2\\ \times\{597A^2+46A(T-B)+37(T-B)^2\}, \qquad (15)$$

$$B_l=(80/231)(3\pi^2\sigma/c\epsilon a^3)^2\{3A+T-B\}^2. \qquad (16)$$

We have eliminated t from the expressions for B_t and B_l by introducing σ from Eq. (13). We are to regard σ as being determined by experiment.

§5. Discussion of Results

In the evaluation of these integrals we have made three approximations. In the integration over E we have, as usual, assumed complete degeneracy. We have stopped at the second term in our expansions in surface harmonics and at the square terms in our binomial expansions of k, k' and τ. If it seemed desirable, more terms could be included in each expansion, but it does not seem likely that the general results would be changed much.

Reasonable values of the parameters, based on Jones and Zener's[1] work on lithium, give the correct orders of magnitude when inserted in the expressions (13)–(16). The consideration of the departures from spherical symmetry have a negligible influence on σ and R, but provide the essential part of B_t and B_l.

Since t and T always enter in the combinations t/a' and $(T-B)$, we get nothing by putting in a relaxation time that varies with direction that we could not obtain by suitable choice of the shape of the surfaces of constant energy. However, the extra parameters would give a means of getting better agreement between theory and experiment if the shape of the energy surfaces were given by other considerations.

If we consider the value of B_t/B_l given by Eqs. (15) and (16), we find that this ratio cannot be less than the value, 4.08, obtained when $A/(T-B)=1/6$, and that if $A/(T-B)$ does not lie between -0.07 and 5.5, B_t/B_l will be larger than 9. This disagrees with Jones and Zener's theoretical results and with the experimental results; in each the ratios are approximately one. There seem to be two ways of explaining this. One can say that the experimental results are not to be trusted, or one can say that some point has been overlooked in the development of the theory of magnetic effects in metals.

The author wishes to express his gratitude to Professor W. V. Houston for his continued interest in the research on which this paper is based, and for his very helpful advice at all stages of the work.