

On the Derivation of the Dispersion Formula for Nuclear Reactions

A. J. F. SIEGERT*

Stanford University, Stanford University, California

(Received August 31, 1939)

Kapur and Peierls gave a derivation of the nuclear dispersion formula which differed from other treatments in defining the compound states by means of wave functions in coordinate space. We have carried through the derivation starting out in a similar way, but avoiding certain disadvantages of their treatment, which mainly consist in too restricting assumptions.

THE dependence of the cross section for nuclear scattering on the energy of the incident particle is described by the nuclear dispersion formula. In this formula the cross section consists of two parts: One of them, the so-called "potential scattering," is a smooth function of the energy. The other part, the "resonance scattering," consists of a set of terms with resonance denominators containing the energy and decay constant of the compound states which cause the resonance phenomenon.

Derivations of this formula have been given by Breit and Wigner,¹ Bethe and Placzek,² Kalckar, Oppenheimer and Serber,³ and Kapur and Peierls.⁴ The authors mentioned first treat the problem by perturbation methods similar to the Weisskopf-Wigner theory of the dispersion of light by atomic systems. Kapur and Peierls have given a different treatment which they characterize as a perturbation calculation with perturbation of the boundary conditions rather than of the Hamilton operator. Their method promises to give a derivation which introduces the compound states in a simpler manner, by differential equations and boundary conditions in coordinate space rather than by inequalities between matrix elements.

The advantage of their treatment is that they can start from a simple model: one particle inside a potential wall. They show how one can generalize this model to cover all cases of nuclear reactions. As far as this generalization is concerned we take over their results. But we would

like to improve on the treatment of the model itself, because there are certain disadvantages connected with their way of treating it.

The main objection is that Kapur and Peierls have made more stringent assumptions than the other authors to prove the dispersion formula. They assumed in fact that the width of all the compound states does not exceed their distance.⁵ This would prohibit the application of the dispersion formula on actual nuclei and is, as we shall prove, an unnecessary assumption.

Another objection against Kapur and Peierls' treatment lies in the following fact: For mathematical convenience only, a radius r_0 is introduced outside of which the potential vanishes. Their distinction between resonance and potential scattering as well as their compound states turn out to be dependent on this parameter r_0 : An increase of r_0 changes both resonance and potential scattering and only the total result is unchanged. We shall change the definition of the compound states slightly, thereby avoiding also another disadvantage of Kapur and Peierls' treatment, which consists in the dependency of energies and wave functions of the compound states on the energy of the incoming particle.

Our way of treatment differs from Kapur and Peierls' perturbation calculation mainly in that it is not an expansion in an orthogonal system. We investigate the singularities of the cross section which occur at certain complex values of the energy. Those singularities which lie near enough to the real axis, cause a sharp resonance maximum on the real axis and we can replace the cross section there by its singular part added to a smooth function of the energy. States with these singular values of the energy we call the

* Now with The Texas Company, Houston, Texas.

¹ G. Breit and E. Wigner, *Phys. Rev.* **49**, 519 (1936).

² H. A. Bethe and G. Placzek, *Phys. Rev.* **51**, 450 (1937).

³ F. Kalckar, J. R. Oppenheimer and R. Serber, *Phys. Rev.* **52**, 273 (1937).

⁴ P. L. Kapur and R. Peierls, *Proc. Roy. Soc.* **A166**, 277 (1938).

⁵ Reference 4, p. 285.

compound states of the model. Their physical significance is easily seen: The cross section being the ratio of the intensities of outgoing and incoming wave becomes singular if there is only an outgoing wave. Our compound states are therefore the "radioactive" states of the system.

We use the same model and notation as Kapur and Peierls. For higher angular momenta, we only consider one particle moving with zero angular momentum in a potential $V(r)$. It is assumed that $V(r)=0$ for $r>r_0$. $\phi_E(r)$ is the wave function multiplied by r ; it is a solution of the Schrödinger equation

$$\hbar^2\phi_E''/2m+(E-V)\phi_E=0 \quad (1)$$

with the boundary condition $\phi_E=0$ for $r=0$. For $r>r_0$ we have

$$\phi_E=(I/k)\sin kr+Se^{ikr}$$

with $k=(2mE/\hbar^2)^{1/2}$. The scattering cross section is given by

$$\sigma=4\pi|S/I|^2.$$

We consider now S/I as function of E , including complex values of E . Expressing S and I by $\phi_E(r_0)$ and $\phi_E'(r_0)$ we have

$$\frac{S}{I}=\frac{\phi_E(r_0)\cos kr_0-\phi_E'(r_0)(\sin kr_0/k)}{\phi_E'(r_0)-ik\phi_E(r_0)}e^{-ikr_0}.$$

We look for singularities arising from vanishing denominator. The energies W_n for which the denominator vanishes are defined by the eigenvalue problem:

$$\hbar^2\phi_n''/2m+(W_n-V)\phi_n=0 \quad (2)$$

with the boundary conditions

$$\phi_n=0 \quad \text{at} \quad r=0$$

and⁶ $\phi_n'-ik_n\phi_n=0$ at $r=r_0$
with $k_n=(2mW_n/\hbar^2)^{1/2}$.

To obtain S/I in the neighborhood of a singularity W_n , we multiply (1) by ϕ_n and (2) by ϕ_E and subtract. We integrate the resulting equation

$$(\hbar^2/2m)(\phi_n''\phi_E-\phi_E''\phi_n)+(W_n-E)\phi_n\phi_E=0$$

⁶ Taking here k_n instead of k makes the compound states independent of the energy of the incoming particle, and as we shall show later, it makes the distinction between resonance and potential scattering invariant against changes of r_0 .

between $r=0$ and $r=r_0$ and obtain, using the boundary conditions,

$$\begin{aligned} & \frac{\hbar^2}{2m}\phi_n(r_0)[ik_n\phi_E(r_0)-\phi_E'(r_0)] \\ & + (W_n-E)\int_0^{r_0}\phi_n\phi_E dr=0. \end{aligned}$$

This gives for the denominator in S/I :

$$\begin{aligned} & \phi_E'(r_0)-ik\phi_E(r_0) \\ & = \frac{W_n-E}{(\hbar^2/2m)\phi_n(r_0)}\int_0^{r_0}\phi_n\phi_E dr+i(k_n-k)\phi_E(r_0) \\ & = \frac{W_n-E}{(\hbar^2/2m)\phi_n(r_0)}\left\{\int_0^{r_0}\phi_n\phi_E dr+i\frac{\phi_n(r_0)\phi_E(r_0)}{k_n+k}\right\}, \end{aligned}$$

using

$$k_n-k=\frac{2m}{\hbar^2}\frac{W_n-E}{k_n+k}.$$

Assuming that the eigenvalue W_n is not degenerate we have in the limit $E\rightarrow W_n$: $\phi_E\rightarrow\phi_n$ and

$$\phi_E'-ik\phi_E\rightarrow(W_n-E)\frac{2m}{\hbar^2\phi_n}\left\{\int_0^{r_0}\phi_n^2 dr+i\frac{\phi_n^2}{2k_n}\right\}.$$

For the numerator of S/I we obtain

$$\phi_E(r_0)\cos kr_0-\phi_E'(r_0)\sin kr_0/k\rightarrow\phi_n(r_0)e^{-iknr_0}$$

and finally

$$S/I=\frac{1}{W_n-E}\frac{(\hbar^2/2m)\phi_n^2(r_0)e^{-2iknr_0}}{\int_0^{r_0}\phi_n^2 dr+i\frac{\phi_n^2(r_0)}{2k_n}}+f(E), \quad (3)$$

where $f(E)$ is a regular function in the surrounding of W_n .

The factor multiplying $1/(W_n-E)$, and W_n itself, depend only on properties of the model not on E and they are invariant against changes of r_0 , as long as $V(r)=0$ for $r>r_0$. To show this let us define a wave function φ_n by changing r_0 into $r_1>r_0$ in the definition of ϕ_n ; this definition will be satisfied by $\varphi_n=A\phi_n$. The boundary condition is fulfilled since

$$\varphi_n(r)=A\phi_n(r)=A\phi_n(r_0)e^{ik_n(r-r_0)} \quad \text{for} \quad r>r_0.$$

W_n is therefore unchanged. For the factor of $1/(W_n - E)$ we get

$$\frac{\varphi_n^2(r_1)e^{-2ik_nr_1}}{\int_0^{r_1} \varphi_n^2 dr + i \frac{\varphi_n^2(r_1)}{2k_n}} = \frac{\phi_n^2(r_0)e^{2ik_n(r_1-r_0)} \cdot e^{-2ik_nr_1}}{\int_0^{r_0} \phi_n^2 dr + \phi_n^2(r_0) \frac{e^{2ik_n(r_1-r_0)} - 1}{2ik_n} + i\phi_n^2(r_0) \frac{e^{2ik_n(r_1-r_0)}}{2k_n}},$$

which is obviously the same factor as in the original formula. In defining in the limit of long-life compound states contributions to the cross section due to the singular part of S/I as "resonance scattering," and those due to the regular part as "potential scattering," we have made a definition which is independent of r_0 . We therefore can choose r_0 as large as we want and do not need to carry out additional considerations to include a screened Coulomb field or centrifugal forces, as Kapur and Peierls had to do.⁷

We can write the resulting Eq. (3) in a different form: From Eq. (2) and its conjugate complex we derive the equation

$$\frac{\hbar^2}{2m} |\phi_n(r_0)|^2 = \frac{\gamma_n}{k_n + k_n^*} \int_0^{r_0} |\phi_n|^2 dr, \quad (4)$$

where γ_n is defined by $E_n - (i\gamma_n/2) = W_n$ with E_n and γ_n real. For sufficiently small values of γ_n we can multiply ϕ_n by a suitable constant A of absolute magnitude 1 so as to make $A\phi_n$ real in the region where it contributes (except for terms of the order γ_n) to the integral $\int_0^{r_0} |\phi_n|^2 dr$.⁸ We can thus write in this limit

$$\int_0^{r_0} A^2 \phi_n^2 dr = \int_0^{r_0} |\phi_n|^2 dr.$$

With this fixed value for A we have then

$$A\phi_n(r_0) = |\phi_n(r_0)| e^{i(k_nr_0 + \frac{1}{2}\delta_n)},$$

where δ_n is a phase determined entirely by the properties of the compound state n and independent of r_0 . We thus obtain from (3)

$$S/I = \frac{1}{W_n - E} \times \frac{(\hbar^2/2m) |\phi_n(r_0)|^2 e^{i\delta_n}}{\int_0^{r_0} |\phi_n|^2 dr + \frac{i}{2k_n} |\phi_n(r_0)|^2 e^{2i(k_nr_0 + \frac{1}{2}\delta_n)}}$$

or, using (4)

$$S/I = \frac{\gamma_n e^{i\delta_n}}{W_n - E} \frac{1}{k_n + k_n^* + \frac{im\gamma_n}{k_n \hbar^2} e^{2i(k_nr_0 + \frac{1}{2}\delta_n)}}. \quad (5)$$

As γ_n tends towards zero k_n becomes real and the last term in the denominator of (5) becomes negligible, so that we have in this limit

$$S/I = \frac{\gamma_n}{E_n - \frac{1}{2}i\gamma_n - E} \frac{e^{i\delta_n}}{2k_n} + f(E). \quad (6)$$

We now subtract from S/I the sum of all singularities with small γ_n . The difference

$$F(E) = S/I - \sum'_n \frac{\gamma_n}{(E_n - E) - \frac{1}{2}i\gamma_n} \frac{e^{i\delta_n}}{2k_n}$$

is a smooth function of the energy for real E . For the cross section σ we obtain the dispersion formula:

$$\sigma = 4\pi |S/I|^2 = 4\pi \left| \sum'_n \frac{\gamma_n}{(E_n - E) - \frac{1}{2}i\gamma_n} \frac{e^{i\delta_n}}{2k_n} + F(E) \right|^2$$

$F(E)$ is called the potential scattering term although it actually consists—at least partly—of contributions from broad levels.

We have thus proven the dispersion formula in the limit of long lifetime of the levels contributing to the resonance part. The existence of broad levels overlapping each other and any number of resonance levels does not affect our derivation.

The author is indebted to Professor F. Bloch and his nuclear seminar for the stimulus to this paper and for valuable discussions.

⁷ Reference 4, p. 286.

⁸ This is possible if the phase varies slowly in regions which contribute the main part of the integral. That this is the case can be easily proved.