Note on the Scattering of Neutrons by Protons

C. KITTEL AND G. BREIT
University of Wisconsin, Madison, Wisconsin

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Expansions for the computation of s wave scattering for square wells are arranged in a form convenient for numerical substitution. Effects of p wave scattering are estimated using Bethe's neutral form of meson theory. It is found that effects of the order of 50 percent in the angular distribution may be expected for 16-Mev neutrons.

INTRODUCTION

OBSERVATIONS on the scattering of neutrons by protons have been made[1–5] in the energy range 0–16 Mev. The present note compares the experimental results with theoretical calculations. Wigner's approximate formula[6] for a "square well" (potential having a constant value through a distance r_0) is amplified by the addition of three more terms in the range of force. It converges satisfactorily in the above energy region. The virtual level for the singlet state is introduced in such a way that corrections for the range of force can be conveniently made. The above calculations take account of s scattering only. The observations on fast neutrons fall below the theoretical curve for $r_0=e^2/mc^2$, the value for r_0 suggested[7] by experiments on the scattering of protons by protons.[8] These calculations are of a provisional character, since the interactions obtained from field theories should supplant the use of arbitrary potentials. The "cutting off" of potentials which has so far been found necessary in field theories makes the prediction of phase shifts doubtful, however, and the formula for the square well has been thought to be of interest as a temporary way of estimating effects of range and neutron energy.

[1] E. O. Salant, R. B. Roberts and P. Wang, Phys. Rev. **55**, 984 (1939).
[2] H. Aoki, Phys. Rev. **55**, 795 (1939).
[3] R. Ladenburg and M. H. Kanner, Phys. Rev. **52**, 911 (1937).
[4] E. T. Booth and C. Hurst, Proc. Roy. Soc. **A161**, 248 (1937).
[5] W. H. Zinn, S. Seely and V. W. Cohen, Phys. Rev. **56**, 260 (1939).
[6] E. Wigner, Zeits. f. Physik **83**, 253 (1933).
[7] G. Breit, H. M. Thaxton and L. Eisenbud, Phys. Rev. **55**, 1018 (1939). This paper is referred to as B. T. E. in the text.
[8] R. G. Herb, D. W. Kerst, D. B. Parkinson and G. J. Plain, Phys. Rev. **55**, 998 (1939).

The p wave phase shifts have been calculated using Bethe's "neutral" form of meson theory.[9] For a meson mass $\mu=177$ m the p wave scattering cross section is 0.12×10^{-24} cm^2 at 16-Mev neutron energy. This is to be compared with the experimental scattering cross section 0.6×10^{-24} cm^2 which includes the effects of all angular momenta and the s wave cross section 0.56×10^{-24} cm^2 for the square well with range e^2/mc^2. The p wave effects are seen to be appreciable and perhaps ultimately detectable for this μ. Proton-proton scattering experiments suggest,[10] on the other hand, $\mu=330$ m. Estimates for this shorter range of force $(\sim\hbar/\mu c)$ give 0.01×10^{-24} cm^2 for the scattering cross section due to the p wave at a neutron energy of 16 Mev. This amount is practically undetectable. The presence of p scattering can be looked for by studying the angular distribution of scattered neutrons or of recoil protons. Some modifications of the usual phase shift analysis are necessary on account of the spin-orbit-spin coupling. These are discussed at the end of the note. An effect of $\sim$50 percent on the angular distribution is expected at 16 Mev using $\mu=180$ m and roughly $\frac{1}{3}$ of this amount for $\mu=330$ m.

NOTATION

M = mass of proton or neutron (neutron-proton mass difference will be neglected throughout).
μ = mass of meson.
m = mass of electron.
E = kinetic energy of incident neutron.
E' = energy in frame of center of gravity $=E/2$.

[9] H. A. Bethe, Phys. Rev. **55**, 1261 (1939). We are indebted to Professor Bethe for communicating his results to us before publication.
[10] S. S. Share, L. E. Hoisington and G. Breit, Phys. Rev. **55**, 1130 (1939).

v = relative velocity of proton and neutron before collision.
r = distance between proton and neutron.
$\Lambda = h/(Mv/2)$; $k = 2\pi/\Lambda$; $\rho = kr$.
r_0 = radius of square well.
$F = r$ times radial wave function, normalized to unit amplitude at ∞.
D_1, D_3 = depths of square wells representing, respectively, proton-neutron interaction in the singlet and triplet states.
K_{01}, K_{03} = phase shifts for singlet and triplet partial waves of zero orbital angular momentum.
E_1 = energy of ${}^1S$ virtual level of deuteron.
$(-E_3) = |$binding energy of deuteron in normal state$|$.
a_1, a_3 = intercepts on axis of r of tangents to F for zero energy neutrons, for singlet and triplet states. A positive a corresponds to tangent cutting axis on left of origin.
$\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_3$ = scattering cross sections for singlet and triplet states.
θ = scattering angle in center of mass system.
$\sigma(\theta)$ = total scattering cross section per unit solid angle.
$\boldsymbol{\sigma} = \int \sigma(\theta) d\Omega$.
$E_1{}^0$ defined by Eq. (8); ϵ defined by Eq. (10).

$\chi = (rdF/Fdr)_{r=r_0}$.
$\gamma_1 = 1 + E'/E_1$; $\gamma_3 = 1 - E'/E_3$.
$x_1 = [ME_1/\hbar^2]^{\frac{1}{2}} r_0$; $x_3 = [M(-E_3)/\hbar^2]^{\frac{1}{2}} r_0$.

S-Scattering

The total neutron-proton scattering cross section is given by

$$\boldsymbol{\sigma} = \tfrac{1}{4}[3\boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_1].$$

If the neutron energy is sufficiently low so that only the s $(L=0)$ partial wave need be considered,

$$\boldsymbol{\sigma}_1 = (\Lambda^2/\pi)\sin^2 K_{01}; \quad \boldsymbol{\sigma}_3 = (\Lambda^2/\pi)\sin^2 K_{03}.$$

For a square well, where the potential energy has the constant amount $-D$ for $r<r_0$ and is zero for $r>r_0$, one obtains

$$\sin^2 K_0 = (\rho\cos\rho - \chi\sin\rho)^2/(\rho^2+\chi^2), \quad (1)$$

where the value of ρ for $r=r_0$ is used. Here

$$\chi = (rdF/Fdr)_{r=r_0} = [(M(D+E')/\hbar^2)^{\frac{1}{2}} r_0] \cot \times [(M(D+E')/\hbar^2)^{\frac{1}{2}} r_0]. \quad (2)$$

Triplet state

One wishes to obtain $\boldsymbol{\sigma}_3$ as a function of the binding energy $(-E_3)$, E and r_0. It is first necessary to eliminate the well depth D_3; this is performed with the help of the series[7]

$$\frac{D_3}{E_3} = \frac{\pi^2}{4x_3^2} + \frac{2}{x_3} + 1 - \frac{4}{\pi^2} + \left(\frac{32}{\pi^4} - \frac{8}{3\pi^2}\right)x_3 + \left(\frac{32}{\pi^4} - \frac{320}{\pi^6}\right)x_3^2 + \cdots. \quad (3)$$

Expanding Eq. (1) in a series in x_3, one obtains after a straightforward calculation:

$$\frac{\sin^2 K_{03}}{\rho^2} = \frac{1}{\gamma_3 x_3^2}[1 + x_3 + G_2 x_3^2 + G_3 x_3^3 + G_4 x_3^4 + \cdots], \quad (4)$$

where

$$G_2 = 1 - 4/\pi^2 - \gamma_3/4 = 0.5947 - 0.25\gamma_3;$$
$$G_3 = \tfrac{1}{3} - 4/\pi^2 + 32/\pi^4 + (1/\pi^2 - \tfrac{1}{3})\gamma_3 = 0.2566 - 0.2320\gamma_3;$$
$$G_4 = \tfrac{1}{3} - 4/\pi^2 + 48/\pi^4 - 320/\pi^6 + (-\tfrac{1}{4} + 2/\pi^2 - 8/\pi^4)\gamma_3 + (-1/48 + 1/2\pi^2)\gamma_3^2 = 0.0880 - 0.1295\gamma_3 + 0.0298\gamma_3^2.$$

Finally

$$\boldsymbol{\sigma}_3 = (\Lambda^2/\pi)\sin^2 K_{03} = \frac{4\pi\hbar^2}{M(E'-E_3)} \times [1 + x_3 + G_2 x_3^2 + G_3 x_3^3 + G_4 x_3^4 + \cdots]. \quad (5)$$

For $\gamma_3 = 2.3$ the term $G_2 x_3^2$ vanishes and the first two terms are a good approximation.

Singlet state

There is, according to present views, no stable singlet proton-neutron state. The depth D_1 can be determined from a knowledge of $\boldsymbol{\sigma}_1(E=0)$, which results from the experimental value of $\boldsymbol{\sigma}$ for thermal neutrons and the computed $\boldsymbol{\sigma}_3(E=0)$. (For the relevant formula see Eq. (8.6) of B.T.E.) It is, however, possible to avoid using D_1 in the expansion for $\boldsymbol{\sigma}_3$ by the proper introduction of a virtual level. The definition of a virtual level is somewhat arbitrary; and the actual definition chosen for the problem in hand is designed to give the expansion for $\sin^2 K_{01}$ a form closely similar to Eq. (4) for $\sin^2 K_{03}$. Let the energy of the virtual level be defined as a positive energy E_1 such that

$$[(M(D_1-E_1)/\hbar^2)^{\frac{1}{2}} r_0] \cot \times [(M(D_1-E_1)/\hbar^2)^{\frac{1}{2}} r_0] = (ME_1/\hbar^2)^{\frac{1}{2}} r_0. \quad (6)$$

This definition may be compared with the condition for the stable ^3S normal level of the deuteron, namely

$$[(M(D_3+E_3)/\hbar^2)^{\frac{1}{2}}r_0]\cot\times[(M(D_3+E_3)/\hbar^2)^{\frac{1}{2}}r_0] = -(M(-E_3)/\hbar^2)^{\frac{1}{2}}r_0.$$

From Eq. (6) one finds

$$\frac{\sin^2 K_{01}}{\rho^2} = \frac{1}{\gamma_1 x_1^2}[1-x_1+G_2x_1^2 - G_3x_1^3+G_4x_1^4-\cdots],$$

which is similar to Eq. (4), and finally

$$\boldsymbol{\sigma}_1 = (\Lambda^2/\pi)\sin^2 K_{01} = \frac{4\pi\hbar^2}{M(E'+E_1)}\times[1-x_1+G_2x_1^2-G_3x_1^3+G_4x_1^4-\cdots], \quad (7)$$

where the G's are defined as in Eq. (4), but with γ_1 substituted for γ_3.

The virtual level is determined by the range and the quantity E_1^0, defined by

$$4\pi\hbar^2/(ME_1^0) = \boldsymbol{\sigma}_1(E=0). \quad (8)$$

The quantity E_1^0 is the energy of the *virtual level for zero range.* One can evaluate $\boldsymbol{\sigma}_1(E=0)$ from the experimental $\boldsymbol{\sigma}$(thermal) and the computed $\boldsymbol{\sigma}_3(E=0)$. Comparing Eqs. (7) and (8),

$$(1/E_1^0) = (1/E_1)(1-x_1+G_2^0x_1^2-G_3^0x_1^3+\cdots),$$

where the upper suffixes on the G's indicate that they are to be evaluated at $E=0$. One obtains the series

$$E_1 = E_1^0[1-(E_1^0/\epsilon)^{\frac{1}{2}}+0.8447(E_1^0/\epsilon) - 0.6666(E_1^0/\epsilon)^{\frac{3}{2}}+\cdots], \quad (9)$$

where

$$\epsilon = \hbar^2/(Mr_0^2). \quad (10)$$

The above series gives the virtual level for an assumed range in terms of the virtual level for zero range. For zero range, $E_1=E_1^0$, and the virtual level has a fairly direct physical meaning: for vanishing range of interaction the position of the virtual level determines $\boldsymbol{\sigma}_1(E=0)$ in the same way as the binding energy $(-E_3)$ determines $\boldsymbol{\sigma}_3(E=0)$.

Numerical formulae

Using the values of the fundamental physical constants as given by B. T. E., p. 1022, one may rewrite Eqs. (5) and (7) in a form convenient for numerical computation, if the range is kept fixed and the energy is varied.

$$\boldsymbol{\sigma}_3 = (\Lambda^2/\pi)\sin^2 K_{03} = \frac{5.21\times10^{-24}}{(E'-E_3)\text{ Mev}}\times[1+x_3+0.5947x_3^2+0.2566x_3^3+0.0880x_3^4+\cdots+\gamma_3(-0.25x_3^2-0.2320x_3^3-0.1295x_3^4)+0.0298\gamma_3^2x_3^4+\cdots]\text{ cm}^2, \quad (11)$$

$$\boldsymbol{\sigma}_1 = (\Lambda^2/\pi)\sin^2 K_{01} = \frac{5.21\times10^{-24}}{(E'+E_1)\text{ Mev}}\times[1-x_1+0.5947x_1^2-0.2566x_1^3+0.0880x_1^4+\cdots+\gamma_1(-0.25x_1^2+0.2320x_1^3-0.1295x_1^4)+0.0298\gamma_1^2x_1^4\cdots]\text{ cm}^2. \quad (12)$$

The fundamental physical constants are involved only in the number 5.21×10^{-24} cm^2. The pure numbers occurring inside the square brackets are independent of these constants. Graphs of the square brackets against energy are sufficiently linear to allow graphical interpolation. In each case the square bracket represents a correction for the range of force.

Using $E_3=-2.17$ Mev, and $\boldsymbol{\sigma}$(thermal) $=14.8\times10^{-24}$ cm^2 reported by Simons,[11] one finds, with $r_0=e^2/mc^2$, the value $E_1=0.0978$ Mev for the position of the virtual level, while for zero range $E_1^0=0.112$ Mev. For these values of r_0, E_1 and E_3,

$$\boldsymbol{\sigma}_3 = \frac{5.21\times10^{-24}}{(E/2+2.17)\text{ Mev}}[1.974-0.1880\gamma_3+0.00513\gamma_3^2-\cdots]\text{ cm}^2; \quad (13)$$

$$\boldsymbol{\sigma}_1 = \frac{5.21\times10^{-24}}{(E/2+0.0978)\text{ Mev}}[0.8738-0.00413\gamma_1+0.0000104\gamma_1^2-\cdots]\text{ cm}^2; \quad (14)$$

$$\gamma_3 = 1+E/4.34; \quad \gamma_1 = 1+E/0.1956.$$

These series have been checked against exact solutions. For $E<10$ Mev, Eq. (13) gives results less than one percent too small, and Eq. (14) less than one percent too large. For $E=12$ Mev, $\boldsymbol{\sigma}_{(\text{exact})}=0.789\times10^{-24}$ cm^2; $\boldsymbol{\sigma}_{(\text{series})}=0.780\times10^{-24}$ cm^2. In Fig. 1, $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}_1$, and $\boldsymbol{\sigma}_3$ are plotted, using the above series, in the energy range 0–16 Mev.

[11] L. Simons, Phys. Rev. 55, 792 (1939).

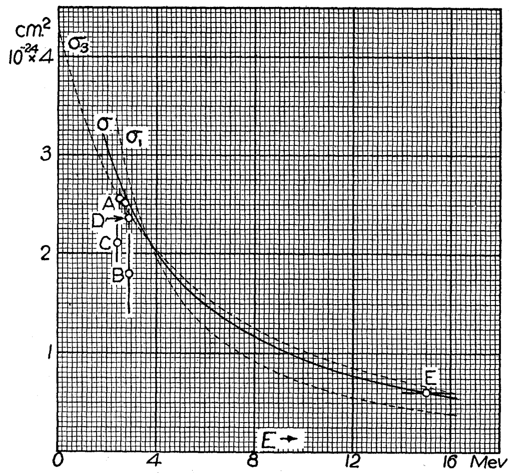

FIG. 1. *Abscissae:* energy of incident neutrons in Mev. *Ordinate:* neutron-proton scattering cross section for square well according to Eqs. (13), (14), with range $r_0=e^2/mc^2$. Only the effect of s waves is included. Curves $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_3$ are for singlet and triplet states, and $\boldsymbol{\sigma}=\boldsymbol{\sigma}_1/4+3\boldsymbol{\sigma}_3/4$. *Experimental points:* *A*—Aoki; *B*—Booth and Hurst; *C*—Ladenburg and Kanner; *D*—Zinn, Seely and Cohen; *E*—Salant, Roberts and Wang.

With the same values of r_0, E_1 and E_3, one finds $D_1=11.52$ Mev, $D_3=21.0(2)$ Mev for the well depths; and $a_1=1.918\times10^{-12}$ cm, $a_3=-0.585\times10^{-12}$ cm for the intercepts.

The s wave scattering cross section for a square well with $r_0=e^2/mc^2$ is plotted in Fig. 1 as a function of the energy. The experimental value for the point E includes an unknown amount of d and p scattering. Estimates regarding the latter will now be discussed.

P-SCATTERING

The quadrupole moment of the deuteron indicates that nuclear potentials contain terms of the form $f(r)[3(\boldsymbol{\sigma}_1\mathbf{r})(\boldsymbol{\sigma}_2\mathbf{r})-r^2(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)]$ so that different phase shifts are expected for $^3P_0$, $^3P_1$, $^3P_2$. The phase shift analysis of the experimental data must be made, therefore, taking into account the differences of these three phase shifts. It must also be modified[12] to take account of the fact that a d wave arises out of the s wave, and that for this d wave the angular distribution contains no terms in $(3\cos^2\theta-1)^2$, but only terms in $(3\cos^2\theta-1)$. The usual formulas for the angular distribution are not applicable, and the general formulas are complicated. Nevertheless it is possible to look for the effects of the p wave by determining the term in $\cos\theta$ in an analysis of the scattering cross section per unit solid angle, in the center of gravity system, in powers of $\cos\theta$.

Taking into account s and p waves only, one obtains for the cross section per unit solid angle in the center of gravity system

$$
\begin{aligned}
\sigma(\theta)=&\frac{3}{4}\left(\frac{\Lambda}{2\pi}\right)^2\Bigg\{\sin^2 K_{03}\\
&+6\sin K_{03}\left[\frac{5}{9}\cos(\delta_2-K_{03})\sin\delta_2\right.\\
&+\frac{3}{9}\cos(\delta_1-K_{03})\sin\delta_1\\
&\left.+\frac{1}{9}\cos(\delta_0-K_{03})\sin\delta_0\right]\cos\theta\\
&+(5\sin^2\delta_2+3\sin^2\delta_1+\sin^2\delta_0)\cos^2\theta\\
&+\left[-\frac{3}{4}(\sin\delta_1-\sin\delta_2)^2-\frac{1}{3}(\sin\delta_0-\sin\delta_2)^2\right.\\
&-3\sin\delta_1\sin\delta_2\sin^2\frac{\delta_1-\delta_2}{2}\\
&\left.-\frac{4}{3}\sin\delta_0\sin\delta_2\sin^2\frac{\delta_0-\delta_2}{2}\right](3\cos^2\theta-1)\Bigg\}\\
&+\frac{1}{4}\left(\frac{\Lambda}{2\pi}\right)^2\Big\{\sin^2 K_{01}+6\sin K_{01}\cos(K_1-K_{01})\\
&\qquad\times\sin K_1\cos\theta+9\sin^2 K_1\cos^2\theta\Big\}. \qquad (15)
\end{aligned}
$$

Here δ_0, δ_1, δ_2 are the phase shifts for $^3P_0$, $^3P_1$, $^3P_2$; and K_1 is the phase shift for $^1P_1$. The part of the expression in curly braces with the coefficient $\frac{3}{4}$ is due to triplet scattering. The s phase shifts for triplets and singlets are K_{03}, K_{01}, respectively. The last set of terms due to triplet scattering contains the factor $(3\cos^2\theta-1)$. These terms do not vanish for $\theta=\pi/2$ as is the case for $\delta_0=\delta_1=\delta_2$. The necessity of terms of this type may be seen by considering the special case $\delta_0\neq0$, $\delta_1=\delta_2=0$. The terms in $\sin^2\delta_0\cos^2\theta$ are then seen to cancel as they should since the P_0 part of the incident wave has no total angular momentum. The interference term in $\sin K_{03}\sin\delta_0\cos\theta$ is present, however, in agree-

[12] J. Schwinger, Phys. Rev. 55, 235 (1939).

ment with the fact that the $^3S_1$ part of the incident wave has an angular momentum. The terms in $3\cos^2\theta-1=2P_2(\cos\theta)$ do not affect the integral of the cross section over all solid angles, and thus have no influence on absorption measurements. Since the d wave which is coupled to the s wave also gives rise to a term in $P_2(\cos\theta)$ for $\sigma(\theta)$ it is presumably not practical to try to disentangle these two contributions. The terms in $\cos\theta$ are, on the other hand, not affected by the d wave, and can be used as a direct test of a combination of effects of δ_0, δ_1, δ_2, K_1. Using the neutral form of theory given by Bethe, with $\mu=177$ m, numerical integrations for $E=16$ Mev give $\delta_0=-18.7°$, $\delta_1=23.7°$, $\delta_2=-5.2°$, $K_1=10.7°$. For these values of the phase shifts one has as the combined effect of s and p waves:

$$4\pi\sigma(\theta)=[1+0.56\cos\theta+0.62\cos^2\theta -0.17(3\cos^2\theta-1)]\times 0.58\times 10^{-24}\text{ cm}^2.$$

Here the first term in brackets represents the combined effect of ^3S and ^1S. The term $0.56\cos\theta$ is due to the interference between s and p waves, and the last two terms in brackets are due to the p waves. The contribution to $4\pi\sigma(\theta)$ in $\cos\theta$ is,

$$0.3(2)\times 10^{-24}\cos\theta\text{ cm}^2,$$

which can be compared with the experimental value

$$\boldsymbol{\sigma}=0.6\times 10^{-24}\text{ cm}^2$$

for the average over solid angles of $4\pi\sigma(\theta)$. The analysis of $\sigma(\theta)$ into powers of $\cos\theta$ may be expected, therefore, to show an appreciable term in $\cos\theta$.

The above effect is sensitive to the range. Using a meson mass $\mu\sim 330$ m numerical integrations give $\delta_0=-6.1°$, $\delta_1=3.7°$, $\delta_2=-1.8°$, $K_1=4.0°$. For these values the terms in $\cos\theta$ contribute $\sim 0.1\times 10^{-24}\cos\theta$ cm² to $4\pi\sigma(\theta)$, which is about $\frac{1}{3}$ of the amount for the cosmic-ray mass of the meson and at the maximum $\sim\frac{1}{6}$ of the solid angle average. The effect is obviously sensitive to the range of force; the important contributions to the p phase shifts come roughly from the regions $r=\hbar/\mu c$ to $r=3\hbar/\mu c$. The effect is not sensitive to the choice of the method of "cut-off" ("straight" or "zero").

The presence of p scattering affects $\boldsymbol{\sigma}$. The contribution to $\boldsymbol{\sigma}$ at $E=16$ Mev due to the p wave is 0.12×10^{-24} cm² for $\mu=177$ m and 0.007×10^{-24} cm² for $\mu=330$ m, in a total of $\sim 0.6\times 10^{-24}$ cm². For the cosmic-ray mass of the meson one should, therefore, decrease the experimental value by ~ 20 percent at 16 Mev in order to obtain the net effect of the s and d waves. For the proton-proton range of force the correction is of the order of one percent and is insignificant. The interaction of $^3P_2$ and $^3G_2$ has been neglected above.

Appendix I

Effect of p wave

The usual scattering theory must be modified if the phase shifts are different for $^3P_0$, $^3P_1$, $^3P_2$, $^1P_1$. Since the meson interaction energy is diagonal in the spin of the two particles the incident wave may be considered as a statistical mixture in the proportions $\frac{1}{4}$, $\frac{3}{4}$ for singlet and triplet states. The cross section is then obtained as

$$\sigma(\theta)=\tfrac{1}{4}\sigma_1(\theta)+\tfrac{3}{4}\sigma_3(\theta).$$

For $\sigma_1(\theta)$ the usual considerations apply. For $\sigma_3(\theta)$ the incident wave may be considered as a statistical mixture of the three states

$$e^{ikz}S_1,\quad e^{ikz}S_0,\quad e^{ikz}S_{-1}$$

with weights of $\frac{1}{3}$ for each. Here S_1, S_0, S_{-1} are spin functions for the two particles corresponding to magnetic quantum numbers 1, 0, -1. The s and p parts of the above waves can be expressed in terms of linear combinations of products of angular and spin functions corresponding to the states $^3P_0$, $^3P_1$, $^3P_2$. Thus, for example,

$$\rho e^{ikz}S_1=F_0S_1+i(12\pi)^{\frac{1}{2}}F_1[(^3P_2)_1+(^3P_1)_1]/2^{\frac{1}{2}}.$$

Here $(^3P_j)_m$ is an angular-spin function for a ^3P state with total angular momentum j and magnetic quantum number m. The normalization is such that the integral over all solid angles of the sum over spin coordinates is unity. The customary notation for F_0, F_1 is used.[7] The interaction between the particles changes the differential equations satisfied by the radial factors. For $^3P_0$, $^3P_1$ there are then separate differential equations. For $^3P_2$ there is besides a coupling to $^3G_2$ which is neglected here. The scale with which the radial solutions must be introduced in place of F_0, F_1 must be chosen so that the resultant wave is e^{ikz}+part in $e^{i\rho}$. For large ρ this condition determines completely the form of the radial functions in terms of the phase shifts.[13] Thus for instance $F_0=\sin\rho$ is replaced by

$$\sin\rho+e^{i(\rho+K_{03})}\sin K_{03}.$$

Proceeding in this way one obtains

$$(\rho e^{ikz}S_1)_{sc}=e^{i(\rho+K_{03})}\sin K_{03}S_1+(12\pi)^{\frac{1}{2}}e^{i\rho}\{2^{-\frac{1}{2}}(^3P_2)_1e^{i\delta_2}\sin\delta_2+2^{-\frac{1}{2}}(^3P_1)_1e^{i\delta_1}\sin\delta_1\};$$

$$(\rho e^{ikz}S_0)_{sc}=e^{i(\rho+K_{03})}\sin K_{03}S_0+(12\pi)^{\frac{1}{2}}e^{i\rho}\{(2/3)^{-\frac{1}{2}}(^3P_2)_0e^{i\delta_2}\sin\delta_2-3^{-\frac{1}{2}}(^3P_0)_0e^{i\delta_0}\sin\delta_0\};$$

$$(\rho e^{ikz}S_{-1})_{sc}=e^{i(\rho+K_{03})}\sin K_{03}S_{-1}+(12\pi)^{\frac{1}{2}}e^{i\rho}\{2^{-\frac{1}{2}}(^3P_2)_{-1}e^{i\delta_2}\sin\delta_2-2^{-\frac{1}{2}}(^3P_1)_{-1}e^{i\delta_1}\sin\delta_1\}.$$

Here the index sc means that the contribution to the scattered wave due to e^{ikz} is taken. The choice of signs of the linear combinations in $(^3P_j)_m$ is made so that

$$(^3P_2)_2=Y_1{}^1S_1;\quad (^3P_1)_1=2^{-\frac{1}{2}}(Y_1{}^0S_1+Y_1{}^1S_0);$$
$$(^3P_0)_0=3^{-\frac{1}{2}}(Y_1{}^{-1}S_1-Y_1{}^0S_0+Y_1{}^{-1}S_{-1}).$$

The functions $Y_L{}^m$ are angular functions normalized to unity for integration over solid angles. The orbital angular momentum and magnetic quantum numbers are, respectively, L, m. The function $Y_1{}^0=(3/4\pi)^{\frac{1}{2}}\cos\theta$. The others are determined by the requirement that the angular momentum matrices have the standard form.[14] A computation of $\frac{1}{3}\Sigma_\mu((\rho e^{ikz}S_\mu)_{sc},(\rho e^{ikz}S_\mu)_{sc})$, with the scalar product applying to the spin coordinates only, gives the contribution to the triplet scattered wave which corresponds to the cross section used in the text (Eq. (15)).

Appendix II.

Calculation of small phase shifts due to concentrated potentials

If the potential causing the phase shift is sufficiently small, Taylor's formula is convenient. The meson potential responsible for the p waves is large at small r, but the phase shifts are small. In such cases a transformation of the radial equation is useful. This is first put in the form

$$d^2\mathfrak{F}/dx^2+g_0{}^2(x)\mathfrak{F}+g_1{}^2(x)\mathfrak{F}=0,$$

where x is r expressed in convenient units (such as $\hbar/\mu c$ in the present case). It is supposed that the solution of

$$d^2F/dx^2+g_0{}^2(x)F=0$$

is known, and the problem is to calculate the phase shift K of $\mathfrak{F}$ relative to that of F. Introducing

$$\eta=F^2\left(\frac{d\mathfrak{F}}{\mathfrak{F}dx}-\frac{dF}{Fdx}\right) \tag{II.1}$$

one finds

$$\frac{d\eta}{dx}+\frac{\eta^2}{F^2}+F^2g_1{}^2=0; \tag{II.2}$$

$$\tan K=\frac{-x/\rho}{(1/\eta)+(xG/\rho F)}. \tag{II.3}$$

One also has

$$\frac{d}{dx}\left(\frac{1}{\eta}+\frac{x}{\rho}\frac{G}{F}\right)=\frac{F^2g_1{}^2}{\eta^2}. \tag{II.4}$$

Eq. (II.2) is used to obtain η by numerical integration. The occurrence of $F^2g_1{}^2$ is convenient, since in Taylor's approximation $\int F^2g_1{}^2dx$ is proportional to K. The method has an apparent disadvantage if the potential in addition to being large in absolute value locally has also moderate values over an appreciable distance. One must then play safe and integrate to sufficiently large x. If there is any phase shift the term in η^2/F^2 in Eq. (II.2) makes η variable. It is then convenient to use Eq. (II.4) because (a) the desirability of carrying the integration farther can be judged by estimating $\int F^2g_1{}^2\eta^{-2}dx$ which determines the relative change of the denominator in Eq. (II.3); (b) a numerical check can be obtained by carrying the integration of Eq. (II.2) to two large values of x and checking by means of Eq. (II.4) the two values of $1/\eta+xG/\rho F$.

[13] N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Oxford Press, 1933), Chap. II.
[14] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge Press, 1935).