On the Self-Energy and the Electromagnetic Field of the Electron

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The charge distribution, the electromagnetic field and the self-energy of an electron are investigated. It is found that, as a result of Dirac's positron theory, the charge and the magnetic dipole of the electron are extended over a finite region; the contributions of the spin and of the fluctuations of the radiation field to the self-energy are analyzed, and the reasons that the self-energy is only logarithmically infinite in positron theory are given. It is proved that the latter result holds to every approximation in an expansion of the self-energy in powers of e^2/hc. The self-energy of charged particles obeying Bose statistics is found to be quadratically divergent. Some evidence is given that the "critical length" of positron theory is as small as $h/(mc)\cdot\exp(-hc/e^2)$.

I. Introduction and Discussions of Results

THE self-energy of the electron is its total energy in free space when isolated from other particles or light quanta. It is given by the expression

$$W=T+(1/8\pi)\int(H^2+E^2)d\mathbf{r}. \quad (1)$$

Here T is the kinetic energy of the electron; H and E are the magnetic and electric field strengths. In classical electrodynamics the self-energy of an electron of radius a at rest and without spin is given by $W\sim mc^2+e^2/a$ and consists solely of the energy of the rest mass and of its electrostatic field. This expression diverges linearly for an infinitely small radius. If the electron is in motion, other terms appear representing the energy produced by the magnetic field of the moving electron. These terms, of course, can be obtained by a Lorentz transformation of the former expression.

The quantum theory of the electron has put the problem of the self-energy in a critical state. There are three reasons for this:

(a) Quantum kinematics shows that the radius of the electron must be assumed to be zero. It is easily proved that the product of the charge densities at two different points, $\rho(\mathbf{r}-\xi/2)\times\rho(\mathbf{r}+\xi/2)$, is a delta-function $e^2\delta(\xi)$. In other words: if one electron alone is present, the probability of finding a charge density simultaneously at two different points is zero for every finite distance between the points. Thus the energy of the electrostatic field is infinite as

$$W_{st}=\lim_{(a=0)}e^2/a.$$

(b) The quantum theory of the relativistic electron attributes a magnetic moment to the electron, so that an electron at rest is surrounded by a magnetic field. The energy

$$U_{mag}=(1/8\pi)\int H^2\mathbf{dr}$$

of this field is computed in Section III and the result is

$$U_{mag}=e^2h^2/(6\pi m^2c^2a^3).$$

This corresponds to the field energy of a magnetic dipole of the moment $eh/2mc$ which is spread over a volume of the dimensions a. The spin, however, does not only produce a magnetic field, it also gives rise to an alternating electric field. The closer analysis of the Dirac wave equation has shown[1] that the magnetic moment of the spin is produced by an irregular circular fluctuation movement (Zitterbewegung) of the electron which is superimposed to the translatory motion. The instantaneous value of the velocity is always found to be c. It must be expected that this motion will also create an alternating electric field. The existence of this field is demonstrated in Section III by the computation of the expression

$$U_{el}=(1/8\pi)\int E_s^2\mathbf{dr}.$$

There E_s is the solenoidal part (div. $E_s=0$) of the electric field strength created by the electron. The fact that the above expression does not vanish for an electron at rest proves the existence

[1] E. Schroedinger, Berl. Ber. 1930, 418 (1930).

of a solenoidal field[2] apart from the irrotational electric field of the charge. The energies of the electric and magnetic fields of the spin are found to be equal. The spin movement does not, of course, give rise to a radiation. The time average of the Poynting vector is zero.

The electromagnetic field of the spin does not contribute to the self-energy of the electron. It is shown in Section IV, that the charge dependent part of the self-energy to a first approximation is given by

$$W'=\tfrac{1}{2}\int\left(\rho\phi-\frac{1}{c}\mathbf{i}\cdot\mathbf{A}\right)d\mathbf{r}=\frac{1}{8\pi}\int(E^2-H^2)d\mathbf{r}.$$

Here ρ and $\mathbf{i}$ are the charge and current densities, ϕ and $\mathbf{A}$ are the scalar and vector potential, respectively. If the self-energy is expressed in terms of the field energies, the electric and magnetic parts have opposite signs,[3] so that the contributions of the electric and of the magnetic fields of the spin cancel one another.

(c) The quantum theory of the electromagnetic field postulates the existence of field strength fluctuations in empty space. These give rise to an additional energy, which diverges more strongly than the electrostatic self-energy. The following crude calculation may demonstrate how this particular part of the self-energy arises: Let us consider an electron with radius a. The field fluctuations in a volume a^3 are of the order $E^2\sim hc/a^4$.[4] The mean frequency of the fluctuations is $\nu\sim c/a$. This field induces the electron to perform vibrations with an amplitude $x\sim eE/m\nu^2$ and an energy $W_{\text{fluct}}\sim e^2E^2/m\nu^2\sim e^2h/mca^2$. This energy diverges quadratically for infinitely small radius. The exact value is calculated in Section IV and is $W_{\text{fluct}}=\lim_{(a=0)}e^2h/\pi mca^2$.

A new situation is created by Dirac's theory of the positron: The self-energy diverges only logarithmically with infinitely small radius. This fact has been proved[5] only for the first approximation of the self-energy expanded in powers of e^2/hc. However it will be shown in Section VI that the divergence is logarithmic in every approximation. The main purpose of this paper is to show the physical significance of the logarithmic divergence and to demonstrate the reasons of its occurrence.

Let us consider the case of one electron embedded in the vacuum as described by the positron theory. The vacuum is represented by the state in which all negative energy states are filled with electrons. The charge density of these "vacuum electrons" is not observable in the unperturbed state of a field-free vacuum. However, the differences between the actual density and the unperturbed density are observable.

The presence of an electron in the vacuum causes a considerable change in the distribution of the vacuum electrons because of a peculiar effect of the Pauli exclusion principle. According to this principle it is impossible to find two or more electrons in a single cell of a volume h^3 in the phase space. If two electrons of equal spin are brought together to a small distance d, their momentum difference must be at least h/d. This effect is similar to a repulsive force which causes two particles with equal spin not to be found closer together than approximately one de Broglie wave-length.

As a consequence of this we find at the position of the electron a "hole" in the distribution of the vacuum electrons which completely compensates its charge. But we also find around the electron a cloud of higher charge density coming from the displaced electrons, which must be found one wave-length from the original electron. The total effect is a broadening of the charge of the electron over a region of the order h/mc as it is indicated schematically in Fig. 1. The product $\rho(\mathbf{r}-\xi/2)\times\rho(\mathbf{r}+\xi/2)$ is no longer zero for a finite distance ξ, and is given by the function

$$G(\xi)=e^2\frac{mc}{h}\frac{1}{\xi}\frac{\partial}{\partial\xi}\frac{i}{2\pi}H_0^{(1)}(imc\xi/h)$$

(Section II). Here $H_0^{(1)}(x)$ is the Hankel function of first kind. $G(\xi)$ has still a quadratic singularity

[2] A solenoidal electric field is necessarily an alternating field for its time average vanishes in a stationary state, whereas the time average of a magnetic field does not vanish if stationary currents are present.

[3] This at first sight unfamiliar result is connected with the well-known fact that a system of steady currents increases its magnetic field energy if it performs mechanical work, whereas a system of charges decreases its field energy by performing mechanical work.

[4] The fluctuations are of the order of magnitude of the field-strength of one light quantum with wave-length.

[5] V. Weisskopf, Zeits. f. Physik **89**, 27 (1934); **90**, 817 (1934).

for $\xi=0$. It is shown quantitatively in Section II, that this broadening of the charge distribution is just sufficient to reduce the electrostatic self-energy to a logarithmically divergent expression.

The broadening effect also changes the magnetic field distribution of the spin moment. In positron theory the magnetic field energy is given by

$$U_{\text{mag}}=\lim_{(a=0)}[e^2h/(2\pi mca^2) - e^2mc/(4\pi h)\cdot \lg (h/mca)]. \quad (2)$$

This is equal to the field energy of a momentum distribution spread over a finite region, which is proportional to the spread of charge described above. The divergence, which is less strong than in the one-electron theory,[6] comes from the quadratic singularity of the distribution. The electric field energy of the spin, however, is not equal to the magnetic field energy because of the following effect, which is again based upon the exclusion principle. The vacuum electrons which are found in the neighborhood of the original electron, fluctuate with a phase opposite to the phase of the fluctuations of the original electron. This phase relation, applied to the circular fluctuation of the spin, decreases its total electric field by means of interference, but does not change the magnetic field of the spins since the latter is due to circular currents and is not dependent on the phase of the circular motion. Thus the total solenoidal electric field energy is reduced by interference if an electron is added to the vacuum. The electric field energy U_{el} of an electron in positron theory is therefore *negative* since it is the difference between the field energy of the vacuum plus one electron, and the energy of the vacuum alone. The exact calculations of Section III give $U_{\text{el}}=-U_{\text{mag}}$. Thus the contribution of the spin to the self-energy does not vanish in positron theory and is by Eq. (2)

$$W_{\text{sp}}=-2U_{\text{mag}}=-\lim_{(a=0)}[e^2h/(\pi mca^2) - e^2mc/(2\pi h)\cdot \lg h/(mca)].$$

The broadening effect cannot, however, be applied to the energy W_{fluct}, which is the energy of the action of the electromagnetic field fluctuations upon the electron. The effect of an external field upon an electron in positron theory is to a first approximation the same as one expects for an electron with infinitely small radius, since the effect of the field upon the displaced vacuum electrons can be neglected. For instance, no destructive interference effect would occur in the interaction with a light wave whose wave-length is smaller than h/mc. The exclusion principle does not alter the interaction of an electron with the field as long as one considers that action to a first approximation to be the sum of independent actions at every point; it has only an effect on the probability of finding one particle in the neighborhood of another.

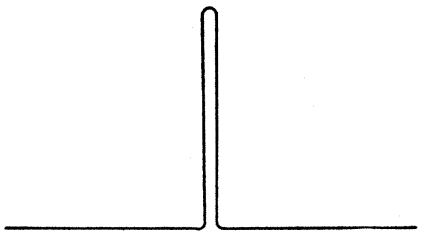

FIG. 1a. Schematic charge distribution of the electron.

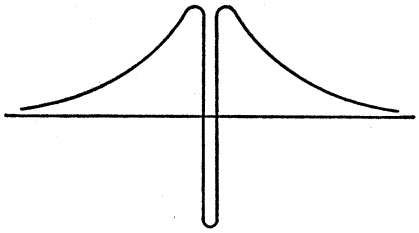

FIG. 1b. Schematic charge distribution of the vacuum electrons in the neighborhood of an electron.

The energy W_{fluct} in positron theory is therefore not different from the same quantity in one electron theory as shown in Section IV. In the former theory, however, it is balanced by the spin energy W_{sp} the most strongly divergent terms of which are just oppositely equal to W_{fluct}. The sum of W_{sp} and W_{fluct} is only logarithmically divergent.

Thus according to positron theory the self-energy of an electron consists of three parts:

(a) The energy W_{st} of the Coulomb field, which diverges logarithmically because of the characteristic spread of charge.

(b) The energy W_{sp} of the oscillatory motion which produces the spin. This energy, although

[6] We use the term "one-electron theory" for the description of the electron by means of the Dirac wave equation without filling up the negative energy states, in order to distinguish it from the "positron theory."

zero in the one-electron theory, is negative and quadratically divergent in the positron theory. This is because of the negative contribution of the magnetic field and the interference effect of the electric field of the vacuum electrons.

(c) The energy W_{fluct} of forced vibrations under the influence of the zero-point fluctuations of the radiation field. The energies (b) and (c) compensate each other to a logarithmic term.

It is interesting to apply similar considerations to the scalar theory of particles obeying the Bose statistics, as has been developed by Pauli and the author.[7] Here the probability of finding two equal particles closer than their wave-lengths is *larger* than at longer distances. The effect on the self-energy is therefore just the opposite. The influence of the particle on the vacuum causes a higher singularity in the charge distribution instead of the hole which balanced the original charge in the previous considerations. It is shown in Section V that this gives rise to a quadratically divergent energy of the Coulomb field of the particle. Thus the situation here is even worse than in the classical theory. The spin term obviously does not appear and the energy W_{fluct} is exactly equal to its value for a Fermi particle.

A few remarks might be added about the possible significance of the logarithmic divergence of the self-energy for the theory of the electron. It is proved in Section VI that every term in the expansion of the self-energy in powers of e^2/hc

$$W=\sum_n W^{(n)} \tag{3}$$

diverges logarithmically with infinitely small electron radius and is approximately given by

$$W^{(n)}\sim z_n mc^2(e^2/hc)^n[\lg\,(h/mca)]^t, \quad t\leqq n.$$

Here the z_n are dimensionless constants which cannot easily be computed. It is therefore not sure, whether the series (3) converges even for finite a, but it is highly probable that it converges if $\delta=e^2/(hc)\cdot\lg\,(h/mca)<1$. One then would get $W=mc^2O(\delta)$ where $O(\delta)=1$ for a value of $\delta<1$. We then can define an electron radius in the same way as the classical radius e^2/mc^2 is defined, by putting the self-energy equal to mc^2. One obtains then roughly a value $a\sim h/(mc)\cdot\exp\,(-hc/e^2)$ which is about 10^{-58} times smaller than the classical electron radius. The "critical length" of the positron theory is thus infinitely smaller than usually assumed.

The situation is, however, entirely different for a particle with Bose statistics. Even the Coulombian part of the self-energy diverges to a first approximation as $W_{\text{st}}\sim e^2h/(mca^2)$ and requires a much larger critical length that is $a=(hc/e^2)^{-\frac{1}{2}}\cdot h/(mc)$, to keep it of the order of magnitude of mc^2. This may indicate that a theory of particles obeying Bose statistics must involve new features at this critical length, or at energies corresponding to this length; whereas a theory of particles obeying the exclusion principle is probably consistent down to much smaller lengths or up to much higher energies.

II. The Charge Distribution of the Electron

The charge distribution in the neighborhood of an electron can be determined from the expression

$$G(\xi)=\int\rho(\mathbf{r}-\xi/2)\rho(\mathbf{r}+\xi/2)d\mathbf{r}\,; \tag{4}$$

here $\rho(\mathbf{r})$ is the charge density at the point $\mathbf{r}$. $G(\xi)$ is the probability of finding charge simultaneously at two points in a distance ξ. If applied to a situation in which one electron alone is present, direct information can be drawn from this expression concerning the charge distribution in the electron itself. The charge density is given by

$$\rho(\mathbf{r})=e\{\psi^*(\mathbf{r})\psi(\mathbf{r})\}-\sigma, \tag{5}$$

where $\psi(\mathbf{r})$, the wave function, is a spinor with four components ψ_μ, $\mu=1, 2, 3, 4$. We write

$$\{\psi^*\psi\}=\sum_{\mu=1}^{4}\psi_\mu{}^*\psi_\mu$$

for the scalar product of two spinors. σ is the charge density of the unperturbed electrons in the negative energy states which is to be subtracted in the positron theory. In the one-electron theory σ is zero. The wave function ψ can be expanded in wave functions φ_q of the

[7] W. Pauli and V. Weisskopf, Helv. Phys. Acta **7**, 709 (1934).

stationary states q of a free electron:

$$\psi(\mathbf{r})=\sum_q a_q\varphi_q(\mathbf{r}). \tag{6}$$

The following relation holds for the φ_q

$$\{\varphi^*{}_q(\mathbf{r})\varphi_q(\mathbf{r})\}=1/V, \tag{7}$$

where V is the total volume of the system. We denote functions with positive energy values by φ_{+q} and with negative energy values by φ_{-q}.

We apply the method of quantized waves and consider the ψ's as operators acting on eigenfunctions $c(\cdots N_q\cdots)$ whose variables are the numbers N_q of electrons in different states q. If the ψ's are written in the form (6), the a's are operators which fulfill the well-known relations:

$$a_q{}^*a_q=N_q, \quad a_qa_q{}^*=1-N_q. \tag{8}$$

We now insert (6) into (5) and (5) into (4) and keep only terms which contain the products of two a's of the form (8) or the following combinations of four a's:

$$\begin{aligned} &a_q{}^*a_qa_{q'}{}^*a_{q'}=N_qN_{q'},\\ &a_q{}^*a_{q'}a_{q'}{}^*a_q=N_q(1-N_{q'}). \end{aligned} \tag{9}$$

All other combinations do not contribute to the expectation value $\bar{G}(\xi)$ of $G(\xi)$ because they have no diagonal elements. We obtain then

$$\bar{G}(\xi)=e^2\sum_q\sum_{q'}N_qN_{q'}+e^2\sum_q\sum_{q'}N_q(1-N_{q'})$$

$$\times\int\{\varphi_q{}^*(\mathbf{r}_1)\varphi_{q'}(\mathbf{r}_1)\}\{\varphi_{q'}{}^*(\mathbf{r}_2)\varphi_q(\mathbf{r}_2)\}d\mathbf{r}$$

$$-2\sigma e\sum_q N_q+\sigma^2V. \tag{10}$$

Here and in the following formulas we put $\mathbf{r}_1=\mathbf{r}-\xi/2$, $\mathbf{r}_2=\mathbf{r}+\xi/2$.

We first apply this expression to a single electron. We then put $\sigma=0$, and $N_{q_0}=1$ for $q=q_0$, $N_q=0$ for $q\neq q_0$:

$$\bar{G}(\xi)=e\sum_q\int\{\varphi_{q_0}{}^*(r_1)\varphi_q(r_1)\}\{\varphi_q{}^*(r_2)\varphi_{q_0}(r_2)\}d\mathbf{r}.$$

This expression can be evaluated by inserting the wave functions of a free electron. If q_0 is the state of an electron at rest, one obtains after replacing the sum over q by an integral over the momenta $\mathbf{p}$ of the states, the following result:

$$\bar{G}(\xi)=e^2\int d\mathbf{p}\frac{\exp i(\xi\cdot\mathbf{p})/h}{8\pi^3h^3}=e^2\delta(\xi). \tag{11}$$

Thus in the one-electron theory, $\bar{G}(\xi)$ is equal to the δ-function.

We now apply (10) to the vacuum of the positron theory, that is, we set

$$N_{+q}=0, \quad N_{-q}=1, \quad \sigma=\sum_{-q}N_q.$$

It is easily seen, that the first, the third and the fourth term cancel each other. The terms remaining give

$$\bar{G}_{\text{vac}}(\xi)=e^2\sum_{+q}\sum_{-q'}\int\{\varphi_{-q'}{}^*(r_1)\varphi_{+q}(r_1)\}$$

$$\times\{\varphi_{+q}{}^*(r_2)\varphi_{-q'}(r_2)\}d\mathbf{r}.$$

The fact that this expression is different from zero and even infinite in the vacuum is closely connected with the charge fluctuations of the empty space which have been investigated by Heisenberg and Oppenheimer.[8] Heisenberg has shown that the charge fluctuations are infinite if the region in which they are measured is sharply limited. This result is due to the electron pairs produced when the charge is measured in a sharply defined region.

We are at present interested in the expression $\tilde{G}(\xi)$ corresponding to the charge distribution of one electron. This can be obtained by calculating $G_{\text{vac}+1}(\xi)$ for the state in which one electron in the state $+q_0$ is present ($N_{q_0}=1$ all other $N_{+q}=0$, $N_{-q}=1$), and by subtracting the effect of the vacuum $G_{\text{vac}}(\xi)$:

$$\tilde{G}(\xi)=G_{\text{vac}+1}(\xi)-G_{\text{vac}}(\xi)=$$

$$e^2(\sum_{+q}-\sum_{-q})\int\{\varphi_{q_0}{}^*(r_1)\varphi_q(r_1)\}\{\varphi_q{}^*(r_2)\varphi_{q_0}(r_2)\}d\mathbf{r}.$$

If one inserts the actual solutions φ_q of Dirac's wave equation of the free electron, this expression can be readily evaluated. One obtains after replacing the sum by an integral as before:

$$\tilde{G}(\xi)=e^2mc^2\int d\mathbf{p}\frac{\exp i(\xi\cdot\mathbf{p})/h}{8\pi^3h^3E(p)}; \tag{12}$$

[8] W. Heisenberg, Verh. d. Sächs. Akad. **86**, 317 (1934); J. R. Oppenheimer, Phys. Rev. **47**, 144 (1934).

here $E(p)=c(p^2+m^2c^2)^{\frac{1}{2}}$. This integral can be evaluated and gives:

$$\tilde{G}(\xi)=e^2\frac{mc}{h}\frac{1}{\xi}\frac{\partial}{\partial\xi}\frac{i}{2\pi}H_0^{(1)}\left(\frac{imc}{h}\cdot\xi\right).$$

$H_0^{(1)}(x)$ is the Hankel function of first kind; this function has a logarithmic singularity for $x=0$ and falls off exponentially for $x\gg1$. We obtain thus

$$\tilde{G}(\xi)=\begin{cases}\dfrac{e^2}{4\pi^2}\dfrac{mc}{h}\dfrac{1}{\xi^2} & \text{for } \xi\ll h/mc\\[2ex] e^2\cdot\left(\dfrac{mc}{h}\right)^2(h/2\pi^3mc\xi^3)^{\frac{1}{2}}\cdot e^{-mc\xi/h} & \\[2ex] & \text{for } \xi\gg h/mc.\end{cases}$$

This expression replaces the delta-function of the one-electron theory and indicates a spread of charge over a finite region of the order of h/mc. It is of interest to construct a charge density $\tilde{\rho}(r)$ for which

$$\int\tilde{\rho}(r_1)\tilde{\rho}(r_2)d\mathbf{r}=G(\xi).$$

This density is given by

$$\tilde{\rho}=e\int d\mathbf{p}\left(\frac{mc^2}{E(p)}\right)^{\frac{1}{2}}\frac{\exp i(\xi\cdot\mathbf{p})/h}{8\pi^3h^3},$$

and for

$$r\ll h/mc:\ \tilde{\rho}\sim e\frac{mc}{h}2^{-5/2}\pi^{-3/2}r^{-5/2};$$

for $r\gg h/mc$, $\tilde{\rho}$ falls off exponentially.

In order to show that this "spread of charge" does not reduce the effect of a periodical field with a short wave-length, let us consider the operator

$$\int\rho(\mathbf{r})\exp i\mathbf{k}\cdot\mathbf{r}\,d\mathbf{r},$$

which represents an interaction energy between the charge and a field of wave number k. By inserting (5), this operator can be written in the form

$$e\sum_q a_q^*a_{q'},\quad \mathbf{p}_{q'}=\mathbf{p}_q+\mathbf{k}$$

and gives rise to transitions from any occupied state q' to any unoccupied state q. These transitions take place quite independently of the ratio of h/k to the linear dimensions h/mc of the spread of charge.

The energy W_{st} of the electrostatic field can be calculated directly from $\tilde{G}(\xi)$:

$$W_{st}=\tfrac{1}{2}\int\frac{\tilde{G}(\xi)}{|\xi|}d\xi.$$

The quadratic singularity of $\tilde{G}(\xi)$ at $\xi=0$ gives rise to a logarithmic divergence of W_{st}. By substituting (12) and by performing the integration over ξ first, we obtain the result

$$W_{st}=\frac{e^2}{4\pi^2}\int d\mathbf{p}\frac{mc^2}{h^3E_{(p)}p^2}$$
$$=\lim_{(P=\infty)}\frac{e^2}{\pi hc}mc^2\lg\frac{P+(P^2+m^2c^2)^{\frac{1}{2}}}{mc}\qquad(13)$$

or by putting $P=h/a$, where a is a length giving the "dimensions" of the electron, we get

$$W_{st}\sim\lim_{(a=0)}\frac{e^2}{\pi hc}m^2c^2\lg\frac{h}{mca}.$$

III. The Electromagnetic Field of the Electron

We calculate in this section the solenoidal part, E_s and H, of the electromagnetic field produced by the electron. It is given by

$$\mathbf{E}_s=-\frac{1}{c}\frac{\partial\mathbf{A}_s'}{\partial t},\quad \mathbf{H}=\operatorname{curl}\mathbf{A}'.$$

Here $\mathbf{A}_s'$ is the solenoidal part of the vector potential $\mathbf{A}$ which is given by

$$\mathbf{A}'(\mathbf{r},t)=\frac{1}{c}\int\frac{\mathbf{i}(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'.$$

A is primed to indicate that this field is produced by the electron. The current density $\mathbf{i}$ is defined by

$$i_x=ec\{\psi^*\alpha_x\psi\},\quad \text{etc.}$$

Here α_i are the well-known Dirac matrices. We consider in our approximation the wave func-

tions ψ to be the solutions of the wave equations of the field-free electron. If we expand the wave functions according to (6) we obtain

$$A_x'(r,t)=e\int \mathbf{dr}'\frac{\sum_q\sum_{q'}a_q^*a_{q'}\{\varphi_q^*(r't')\alpha_x\varphi_{q'}(r't')\}}{|\mathbf{r}-\mathbf{r}'|},$$

$$t'=t-\frac{|r-r'|}{c}.$$

The wave functions φ_q of the free electron are

$$\varphi_q=\frac{1}{V^{\frac{1}{2}}}u_q\exp\frac{1}{h}(i\mathbf{p}_q\cdot\mathbf{r}-iE_qt). \qquad (14)$$

p_q and E_q are momentum and energy of the state q, V is the volume of the space considered, u_q is a normalized spinor. We obtain then for A_x'

$$A_x(r,t)=\frac{2\pi eh^2c^2}{V}\sum_q\sum_{q'}\frac{\{u_q^*\alpha_xu_{q'}\}a_q^*a_{q'}}{E_qE_{q'}-m^2c^4-c^2\mathbf{p}_q\cdot\mathbf{p}_{q'}}\times\exp i[(\mathbf{p}_{q'}-\mathbf{p}_q)\cdot\mathbf{r}-(E_{q'}-E_q)t]/h. \qquad (15)$$

The field strengths can immediately be computed from this expression. The time average of the magnetic field energy, U_{mag} is found to be

$$U_{\text{mag}}=\frac{1}{8\pi}\int(\text{curl }\mathbf{A}')^2d\mathbf{r}$$

$$=\frac{\pi}{2}e^2h^2c^4\sum_q\sum_{q'}(\mathbf{p}_q-\mathbf{p}_{q'})^2\cdot N_q(1-N_{q'})\times\frac{\{u_q^*\alpha_su_{q'}\}\{u_{q'}^*\alpha_su_q\}}{[E_qE_{q'}-m^2c^4-c^2\mathbf{p}_q\cdot\mathbf{p}_{q'}]^2} \qquad (16)$$

if one uses the relations (9). α_s is the component of $\boldsymbol{\alpha}$ which is perpendicular to $\mathbf{p}_q-\mathbf{p}_{q'}$. We apply this expression first to a single electron at rest ($N_q=1$, $q=q_0$; $N_q=0$, $q\neq q_0$):

$$U_{\text{mag}}=\frac{\pi}{2}e^2h^2\sum_q p_q^2\frac{\{u_{q_0}^*\alpha_su_q\}\{u_q^*\alpha_su_{q_0}\}}{m^2(E_q-m^2c^4)^2}.$$

One obtains after averaging over the spin directions and replacing the sum by an integral:

$$U_{\text{mag}}=\pi\frac{e^2h^2}{m^2c^2}\int\frac{\mathbf{dp}}{8\pi^3h^3}=\frac{1}{2\pi}\lim_{(a=0)}\frac{e^2h^2}{3m^2c^2a^3}. \qquad (17)$$

This divergent expression corresponds to the field energy of a magnetic dipole density concentrated in a sphere of infinitely small radius a. If one now calculated expression (16) for the vacuum of the positron theory ($N_{+q}=0$, $N_{-q}=1$) one obtains a highly divergent expression which represents the magnetic field energy U_{mag}(Vac.) produced by the current fluctuations of the vacuum. We are interested in the field energy of one electron at rest which is obtained by calculating U_{mag}(Vac.+1) for the state ($N_{q_0}=1$, $p_{q_0}=0$, $E_{q_0}=mc^2$ all other $N_{+q}=0$, $N_{-q}=1$) and by subtracting the effect of the vacuum:

$$\tilde{U}_{\text{mag}}=U_{\text{mag}}(\text{Vac.}+1)-U_{\text{mag}}(\text{Vac.})$$

$$=2\pi e^2h^2(\sum_{+q}-\sum_{-q})p_q^2\frac{\{u_{q_0}^*\alpha_su_q\}\{u_q^*\alpha_su_{q_0}\}}{m^2(E_q-m^2c^4)^2}.$$

Averaging over the spin directions gives:

$$U_{\text{mag}}=\pi\frac{e^2h^2}{m}\int\frac{\mathbf{dp}}{8\pi^3h^3E(p)}$$

$$=\frac{1}{2\pi}\frac{e^2}{mch}\lim_{(P=\infty)}\left[PP_0-\frac{m^2c^2}{2}\lg\frac{P+P_0}{mc}\right]. \qquad (18)$$

Here P_0 is defined by $P_0=(P^2+m^2c^2)^{\frac{1}{2}}$. This quadratically divergent expression is just what one would expect for the field energy of the magnetic dipole density of the electron if this density is equal to the charge distribution calculated in the previous section.

In order to show directly, that this magnetic field is equal to the "static" field of the magnetic moment of the spin, we consider the magnetic polarization (dipole density) $\mathbf{M}$,

$$M_x=\frac{eh}{2mc}\{\psi^*\beta\alpha_y\alpha_z\psi\},$$

and calculate the function

$$J(\xi)=\int\mathbf{M}(r-\xi/2)\cdot\mathbf{M}(r+\xi/2)d\mathbf{r},$$

which corresponds to $G(\xi)$ (see (4)) and which provides information about the "spin distribution" in the electron. If one evaluates this integral by the method which is used in Section II,

one obtains

$$J(\xi)=\frac{3}{4}\frac{h^2}{m^2c^2}G(\xi),$$

which shows[9] that the spin is distributed in exactly the same way as the charge. The magnetic field energy of this distribution is given by

$$U_{\text{mag}}=\tfrac{1}{2}\int\frac{\text{div. }\mathbf{M}(r_1)\text{ div. }\mathbf{M}(r_2)}{|\xi|}d\mathbf{r}$$

and can be evaluated by the methods used for the other calculations in this section and leads to the expressions (17) for the one-electron theory and to (18) for the positron theory. Hence we are allowed to consider them as the field energy of the magnetic moment.

The energy of the solenoidal electric field strength is given by

$$U_{\text{el}}=\frac{1}{8\pi c^2}\int\left(\frac{\partial A_s}{\partial t}\right)^2 d\mathbf{r}$$

$$=\frac{\pi}{2}\frac{e^2h^2}{c^2}\sum_q\sum_{q'}(E_{q'}-E_q)^2\frac{\{u_q{}^*\alpha_s u_{q'}\}\{u_{p'}{}^*\alpha_s u_q\}}{[E_qE_{q'}-m^2c^4-c^2\mathbf{p}_q\cdot\mathbf{p}_{q'}]^2}N_q(1-N_{q'}).$$

Applied to a single electron at rest, this expression gives

$$U_{\text{el}}=\pi\frac{e^2h^2}{m^2c^2}\int\frac{\mathbf{dp}}{8\pi^3h^3}=U_{\text{mag}}. \tag{19}$$

Applied to an electron at rest in positron theory, one obtains, however,

$$\tilde{U}_{\text{el}}=U_{\text{el}}(\text{Vac.}+1)-U_{\text{el}}(\text{Vac.})$$

$$=\frac{2\pi e^2h^2}{c^2}(\sum_{+q}-\sum_{-q})(E_q-mc^2)^2\times\frac{\{u_{q_0}{}^*\alpha_s u_q\}\{u_q{}^*\alpha_s u_{q_0}\}}{m^2(E_q-m^2c^4)^2}$$

and then

$$\tilde{U}_{\text{el}}=-\pi\frac{e^2h^2}{m}\int\frac{\mathbf{dp}}{8\pi^3h^3E(p)}=-\tilde{U}_{\text{mag}}. \tag{20}$$

The interpretation of this result is given in Section I.

[9] The factor $\frac{3}{4}$ is $s(s+1)$ for $s=\frac{1}{2}$.

IV. The Self-Energy of the Electron

The self-energy is calculated in this section by means of a method which is different from the usual perturbation method, in order to outline the physical significance of the different terms. It is similar to the method applied to this problem previously by the author.[5]

The Hamiltonian of a system of charged particles and their electromagnetic field can be written in the form

$$\mathfrak{H}=\tfrac{1}{2}\int\left[\frac{1}{c^2}\left(\frac{\partial\mathbf{A}}{\partial t}\right)^2+\sum_{i,\,k=1}^{3}\left(\frac{\partial A_i}{\partial x_k}\right)^2-\frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2-(\text{grad. }\phi)^2\right]d\mathbf{r}$$

$$+\int\left(\rho\phi-\frac{1}{c}\mathbf{i}\cdot\mathbf{A}\right)d\mathbf{r}$$

$$+c\sum_i\int\{\psi_i{}^*(\boldsymbol{\alpha}\cdot\mathbf{p}_i+\beta mc)\psi_i\}d\mathbf{r}.$$

The summation in the last term is performed over all particles. The solutions of this Hamiltonian are restricted by the condition

$$\partial\phi/\partial t+c\text{ div. }\mathbf{A}=0.$$

By introducing the field strengths instead of the potentials we obtain the expression

$$\mathfrak{H}=\frac{1}{8\pi}\int(E^2+H^2)d\mathbf{r}-\frac{1}{4\pi}\int\phi\cdot\text{div. }E d\mathbf{r}$$

$$+\int\left(\rho\phi-\frac{1}{c}\mathbf{i}\cdot\mathbf{A}\right)d\mathbf{r}$$

$$+c\sum_i\int\{\psi_i{}^*(\boldsymbol{\alpha}\cdot\mathbf{p}_i+\beta mc)\psi_i\}d\mathbf{r}.$$

This is equal to (1) if one uses the relations div. $E=4\pi\rho$ and

$$T=c\sum_i\int\{\psi_i{}^*[\boldsymbol{\alpha}\cdot(\mathbf{p}-(e/c)\mathbf{A})+\beta mc]\psi\}d\mathbf{r}.$$

The interaction energy

$$\int(\rho\phi-(1/c)\mathbf{i}\cdot\mathbf{A})=eH'$$

between matter and field contains the electronic charge e explicitly as a linear factor, so that in the above notation H' is explicitly independent of e; $\partial H'/\partial e=0$. Let us consider the energy W_s of a stationary state s of this Hamiltonian. If the electronic charge e is increased by de the Hamiltonian gets the additional term $de\,H'$. According to perturbation theory the increase of W_s is $de\langle H'\rangle_{\text{Av}}$, where $\langle H'\rangle_{\text{Av}}$ is the time average of H' in the state s, assuming that the electronic charge has its original value e. We therefore get

$$W_s=W_s^{(0)}+\int_0^e\langle H'(e)\rangle_{\text{Av}}de, \tag{21}$$

where $W_s^{(0)}$ is the value of the energy for $e=0$. We now expand $H'(e)$ in a power series of e:

$$H'(e)=H'^{(0)}+eH'^{(1)}+\cdots$$

and get from (21)

$$W_s=W_s^{(0)}+e\langle H'^{(0)}\rangle_{\text{Av}}+\frac{e^2}{2}\langle H'^{(1)}\rangle_{\text{Av}}+\cdots.$$

The second term is zero since W_s cannot depend upon the sign of e, and we obtain by neglecting all terms containing e in a higher power than the second

$$W'=W_s-W_s^{(0)}$$

$$=\frac{e^2}{2}\langle H'^{(1)}\rangle_{\text{Av}}=\frac{e}{2}\langle H'\rangle_{\text{Av}}=\tfrac{1}{2}\int(\langle\rho\phi\rangle_{\text{Av}}-\frac{1}{c}\langle\mathbf{i}\cdot\mathbf{A}\rangle_{\text{Av}})d\mathbf{r}.$$

The potentials can be split into two parts

$$\mathbf{A}=\mathbf{A}_0+\mathbf{A}',\qquad \phi=\phi_0+\phi'.$$

$\mathbf{A}_0$ and ϕ_0 are the potentials of the field when no electron is present. In empty space, A_0 is the potential of the zero-point oscillations and $\phi_0=0$. A' and ϕ' is the field produced by the electron. We then get

$$W'=\tfrac{1}{2}\int\left(\langle\rho\phi'\rangle_{\text{Av}}-\frac{1}{c}\langle\mathbf{i}\cdot\mathbf{A}'\rangle_{\text{Av}}\right)d\mathbf{r}$$

$$-\frac{1}{2c}\int\langle\mathbf{i}\cdot\mathbf{A}_0\rangle_{\text{Av}}d\mathbf{r}.$$

The first term can be transformed by means of the relations

$$\Delta\phi'-\frac{1}{c^2}\ddot{\phi}'=-4\pi\rho,\quad \Delta\mathbf{A}'-\frac{1}{c^2}\ddot{\mathbf{A}}'=-4\pi\mathbf{i},$$

and we obtain

$$W'=\frac{1}{8\pi}\int(\langle E'^2\rangle_{\text{Av}}-\langle H'^2\rangle_{\text{Av}})d\mathbf{r}-\frac{1}{2c}\int\langle\mathbf{i}\cdot\mathbf{A}_0\rangle_{\text{Av}}d\mathbf{r}.$$

We consider now the state s to be the state of an electron at rest. The charge dependent part W' of the self-energy can then be written in the form

$$W'=W_{\text{st}}+W_{\text{sp}}+W_{\text{fluct}}. \tag{22}$$

Here

$$W_{\text{st}}=\frac{1}{8\pi}\int E_{\text{st}}^2d\mathbf{r}$$

is the static field energy of the irrotational field E_{st}; W_{sp} is defined by

$$W_{\text{sp}}=\frac{1}{8\pi}\int(\langle E_s'^2\rangle_{\text{Av}}-\langle H'^2\rangle_{\text{Av}})d\mathbf{r}=U_{\text{el}}-U_{\text{mag}}. \tag{23}$$

It contains the contribution of the field produced by the spin and is calculated in the previous section; W_{fluct} is the energy produced by the fluctuations of the radiation field,

$$W_{\text{fluct}}=-\frac{1}{2c}\int\mathbf{i}\cdot\mathbf{A}_0d\mathbf{r}.$$

In order to calculate W_{fluct}, we divide $\mathbf{i}$ into two parts $\mathbf{i}=\mathbf{i}_0+\mathbf{i}'$, where $\mathbf{i}_0$ is the current density for the field-free case. The term $\int\mathbf{i}_0\cdot\mathbf{A}_0d\mathbf{r}$ vanishes when averaged over the time because of the absence of phase relations between $\mathbf{i}_0$ and $\mathbf{A}_0$. The remaining term

$$W_{\text{fluct}}=-\frac{1}{2c}\int\mathbf{i}'\cdot\mathbf{A}_0d\mathbf{r}$$

can be evaluated as follows:

$\mathbf{i}'$ is to a first approximation given by

$$\mathbf{i}'=\{\psi_0^*\boldsymbol{\alpha}\psi_1\}+\{\psi_1^*\boldsymbol{\alpha}\psi_0\}. \tag{24}$$

Here $\psi_0=\Sigma a_q\varphi_q$ is the wave function unperturbed by the field and ψ_1 is the first approxima-

tion of the perturbed wave function:

$$\psi_1=\sum a_q\varphi_q'.$$

φ_q' can be calculated by means of the ordinary perturbation method. The interaction energy with an arbitrary field $\mathbf{A}$ is given by

$$-e\alpha\mathbf{A}=-e\alpha\sum_k[A_k^{(+)}\exp i(\mathbf{k}\cdot\mathbf{r}+c|k|t)+A_k^{(-)}\exp -i(\mathbf{k}\cdot\mathbf{r}+c|k|t)].$$

The sum is taken over all wave numbers $\mathbf{k}$ of an orthogonal system of plane waves in the volume V. If we write the wave functions φ_q of the free electron in the form (14), we get for φ_q'

$$\varphi_q'=e\sum_{q'}\varphi_{q'}\left[\frac{\{u_{q'}^*\alpha u_q\}A_k^{(+)}}{E_q-E_{q'}+c|k|}e^{+ic|k|t}+\frac{\{u_{q'}^*\alpha u_q\}A_k^{(-)}}{E_q-E_{q'}-c|k|}e^{-ic|k|t}\right],\quad h\mathbf{k}=\mathbf{p}_{q'}-\mathbf{p}_q.$$

This expression is introduced into (24) and gives for the current

$$i_x'=\sum_q\sum_{q'}N_q\{u_q^*\alpha_x u_{q'}\}\{u_{q'}^*\alpha u_q\}\times\left[\frac{\mathbf{A}^{(+)}\exp i(\mathbf{k}\cdot\mathbf{r}+c|k|t)}{E_q-E_{q'}+c|k|}+\frac{\mathbf{A}^{(-)}\exp -i(\mathbf{k}\cdot\mathbf{r}+c|k|t)}{E_q-E_{q'}-c(k)}\right]+\text{conj.}$$

We have retained only terms containing products (8) or (9). It is seen from this expression, that i' is the same in the one-electron theory and in the positron theory. In the latter case we have to consider $\bar{i}=i'(\text{Vac.}+1)-i'(\text{Vac.})$. The actual value can easily be evaluated for an electron at rest. One obtains

$$\mathbf{i}'=-\frac{1}{V}\frac{e^2}{mc}\mathbf{A},$$

which is immediately understood as the forced vibrations of a point charge under the action of an oscillating field $\mathbf{A}$. W_{fluct} is directly obtained if one replaces $\mathbf{A}$ by the field fluctuations $\mathbf{A}_0$ of the vacuum:

$$W_{\text{fluct}}=-\frac{1}{2c}\int\mathbf{i}'\cdot\mathbf{A}_0\mathbf{dr}=\frac{e^2}{2mc^2}\langle A_0^2\rangle_{\text{Av}}.$$

We have

$$\langle A_0^2\rangle_{\text{Av}}=\frac{1}{\pi^2}\int\frac{ch}{|k|}\mathbf{dk}\tag{25}$$

and we get finally, on replacing $|k|$ by p/h,

$$W_{\text{fluct}}=\frac{e^2}{2\pi^2mc^2}\int\frac{c\mathbf{dp}}{h|p|}=\frac{e^2}{\pi hc}\frac{1}{m}\lim_{(P=\infty)}P^2.$$

We now collect the results obtained for the other parts of the self-energy of an electron at rest. The one-electron theory gives [Eqs. (11), (23), (19)]

$$W_{\text{st}}=\lim_{(a=0)}\frac{e^2}{a},\quad W_{\text{sp}}=U_{\text{el}}-U_{\text{mag}}=0.$$

The positron theory gives (Eqs. (13), (23), (20))

$$W_{\text{st}}=\lim_{(P=\infty)}\frac{e^2}{\pi hc}mc^2\lg\frac{P+P_0}{mc},$$

$$W_{\text{sp}}=U_{\text{el}}-U_{\text{mag}}=-\frac{e^2}{\pi hc}\frac{1}{m}\lim_{(P=\infty)}\left[PP_0-\frac{m^2c^2}{2}\lg\frac{P+P_0}{mc}\right].$$

W_{sp} is partly balanced by W_{fluct}. The total self-energy in positron theory is then given by

$$W'=\frac{3}{2\pi}\frac{e^2}{hc}mc^2\lim_{(P=\infty)}\lg\frac{P+P_0}{mc}+\text{finite terms.}\tag{26}$$

The self-energy of a free electron in motion can be obtained by a Lorentz transformation from (26). The direct calculation from the above methods is ambiguous because it leads to a difference of terms, each of which diverges quadratically. The factor of the logarithmically divergent difference of these terms depends essentially on the way in which the infinite terms are subtracted. The calculation of the self-energy of an electron at rest is not so much exposed to these ambiguities because of the spherical symmetry of the problem, which suggests only one natural way of subtracting two divergent integrals over the momentum space, namely, the subtraction of the contributions of concentric spherical shells around the center. It must be expected, that the value of the self-energy of a moving electron can only be covariant to the value (26) if one performs the subtraction appropriately. This is why the expressions for the self-energy obtained in reference 5 are apparently not relativistically covariant.

V. The Self-Energy of a Particle Obeying the Bose-Statistics

It has been shown that the quantization of the scalar wave equation of Klein and Gordon leads to a theory of elementary particles with Bose statistics and charges of both signs. The theory includes a description of pair creation and of all related phenomena. The quantitative results are not very different from the results of Dirac's positron theory. The formalism has been recently applied to particles with intrinsic angular momentum. It will be shown here that the calculation of the self-energy, however, gives results quite different from the positron theory. The energy of the electrostatic field of the electron is found to be more strongly divergent than in the classical theory; the energy of the radiation field diverges quadratically and is equal to the corresponding energy of a single electron in the one-electron theory. The qualitative arguments for this behavior are given in I. The following calculation is based on the formulas derived elsewhere.[7]

The operator of the charge density is given by

$$\rho = ie(\psi^*\pi^* - \psi\pi),$$

where ψ is the wave function and π its conjugate operator:

$$\pi = h\frac{\partial\psi^*}{\partial t},$$

$$\psi(r)\pi(r') - \pi(r')\psi(r) = \delta(r-r').$$

We introduce new variables by means of

$$\psi = \frac{1}{V^{\frac{1}{2}}}\sum_k q(\mathbf{k}) \exp i\mathbf{k}\cdot\mathbf{r},$$
$$\pi = \frac{1}{V^{\frac{1}{2}}}\sum_k p(\mathbf{k}) \exp -i\mathbf{k}\cdot\mathbf{r}. \tag{27}$$

Here $1/V^{\frac{1}{2}}\cdot\exp i\mathbf{kr}$ form a set of orthogonal functions in the volume V.

We further introduce

$$p(k) = \left(\frac{E(k)}{2}\right)^{\frac{1}{2}}(a^*(\mathbf{k}) + b(\mathbf{k})),$$
$$q(k) = -i\left(\frac{1}{2E(k)}\right)^{\frac{1}{2}}(a^*(\mathbf{k}) - b(\mathbf{k})), \tag{28}$$
$$E(k) = c(h^2k^2 + m^2c^2)^{\frac{1}{2}},$$

and obtain for

$$\rho(\mathbf{s}) = \frac{1}{V}\int \rho(\mathbf{r}) \exp(-i\mathbf{s}\cdot\mathbf{r})\mathbf{dr}$$

the following expression:

$$\rho(\mathbf{s}) = \frac{e}{2V}\sum_k \frac{E(k)+E(l)}{[E(l)E(k)]^{\frac{1}{2}}}[a^*(\mathbf{k})a(\mathbf{l}) - b^*(\mathbf{k})b(\mathbf{l})]$$
$$+\frac{E(k)-E(l)}{[E(k)E(l)]^{\frac{1}{2}}}[a(\mathbf{k})b(\mathbf{l}) - a^*(\mathbf{k})b^*(\mathbf{l})], \tag{29}$$

where $\mathbf{l}=\mathbf{k}+\mathbf{s}$.

The $a(\mathbf{k})$ and $b(\mathbf{k})$ fulfill the relations

$$a^*(\mathbf{k})a(\mathbf{k}) = N(\mathbf{k}), \qquad b^*(\mathbf{k})b(\mathbf{k}) = M(\mathbf{k}),$$
$$a(\mathbf{k})a^*(\mathbf{k}) = 1+N(\mathbf{k}), \qquad b(\mathbf{k})b^*(\mathbf{k}) = 1+M(\mathbf{k}).$$

The $N(\mathbf{k})$'s are the numbers of positrons, the $M(\mathbf{k})$'s the number of negatrons in the states with the wave vector $\mathbf{k}$. The electrostatic self-energy is given by:

$$W_{\mathrm{st}} = \tfrac{1}{2}\int \frac{\rho(\mathbf{r}-\xi/2)\rho(r+\xi/2)}{|\xi|}\mathbf{drd}\xi$$
$$= 2\pi V\sum_s \frac{\rho(-\mathbf{s})\rho(\mathbf{s})}{s^2}.$$

We obtain by introducing (29) and retaining only the diagonal terms:[10]

$$W_{\mathrm{st}} = \frac{\pi e^2}{2V}\sum_k\sum_s \frac{1}{s^2}\Big[\frac{(E(k)+E(l))^2}{E(k)E(l)}$$
$$\times(N(\mathbf{k})[N(\mathbf{l})+1]+M(\mathbf{k})[M(\mathbf{l})+1])$$
$$+\frac{(E(\mathbf{k})-E(\mathbf{l}))^2}{E(k)E(l)}$$
$$\times(N(\mathbf{k})M(\mathbf{l})+[N(\mathbf{k})+1][M(\mathbf{l})+1])\Big].$$

This expression does not vanish for the vacuum ($N(\mathbf{k})=M(\mathbf{k})=0$ for every $\mathbf{k}$). We calculate the difference

$$\tilde{W}_{\mathrm{st}} = W_{\mathrm{st}}(\mathrm{Vac.}+1) - W_{\mathrm{st}}(\mathrm{Vac.})$$

[10] The term $s=0$ is omitted. It can easily be shown that this term does not contribute for $V\to\infty$.

and assume that the particle is at rest:

$$\tilde{W}_{\text{st}}=\frac{e^2h}{16mc}\int ds\frac{1+2m^2c^2/h^2s^2}{(s^2+m^2c^2/h^2)^{\frac{1}{2}}}$$

$$=e^2\frac{h}{mc}\frac{\pi}{4}\lim_{(P=\infty)}\left[\frac{P^2}{h^2}+\frac{m^2c^2}{h^2}\lg\frac{P+P_0}{mc}\right].$$

This is an expression which diverges quadratically. By putting $P=h/a$ one obtains

$$\tilde{W}_{\text{st}}\cong\frac{\pi}{4}mc^2\left(\frac{h}{mca}\right)^2.$$

We now show that the particle does not produce a solenoidal electric or magnetic field in the approximation considered here. The current density is given by Eq. (43) of reference 7:

$$\mathbf{i}=\mathbf{i}_1+\mathbf{i}_2,$$

$$\mathbf{i}_1=ihce(\psi\ \text{grad.}\ \psi^*-\psi^*\ \text{grad.}\ \psi),$$

$$\mathbf{i}_2=-2e^2\mathbf{A}\psi^*\psi.$$

We introduce the new variables (27) into i_1 and get

$$\mathbf{i}_1=\sum_s i(\mathbf{s})\exp i\mathbf{r}\cdot\mathbf{s},$$

$$i(\mathbf{s})=\tfrac{1}{2}hce\sum_k\frac{\mathbf{k}+1}{[E(k)E(l)]^{\frac{1}{2}}}(a^*(\mathbf{k})a(1)+b(\mathbf{k})b^*(\mathbf{l})$$

$$-a(\mathbf{k})b(1)-b(\mathbf{k})a(1)),\quad 1=\mathbf{k}+\mathbf{s}.$$

$i_1(\mathbf{s})$ is proportional to $\mathbf{k}+1$ and $\mathbf{i}_1$ is therefore irrotational for all transitions which start or end with a particle at rest. ($\mathbf{k}+1$ is parallel to $\mathbf{s}$ if $\mathbf{k}=0$ or $1=0$). Thus $\mathbf{i}_1$ does not produce a solenoidal field. $\mathbf{i}_2$ is proportional to e^2 so that its field does not come into consideration. The particle does not give rise to a solenoidal field as long as it is at rest since it has no magnetic spin moment.

The remaining term of the self-energy is

$$W_{\text{fluct}}=-\frac{1}{2c}\int\mathbf{i}'\cdot\mathbf{A}_0\mathbf{dr}. \tag{22}$$

$\mathbf{i}'$ is defined as the current density produced by the field $\mathbf{A}_0$. Here the first part $\mathbf{i}_1'$ can be shown to be again irrotational. The integral over the product of $\mathbf{i}_1'$ and the solenoidal vector $\mathbf{A}_0$ vanishes. The second part $\mathbf{i}_2'$ is in the required approximation directly given by

$$i_2'=-2e^2\mathbf{A}_0\psi^*\psi.$$

We obtain then

$$W_{\text{fluct}}=e^2\int\mathbf{A}_0{}^2\psi^*\psi\mathbf{dr}.$$

By introducing the new variables and retaining only diagonal elements we find

$$W_{\text{fluct}}=\frac{e^2}{2}\langle A_0{}^2\rangle_{\text{Av}}\sum_k\frac{N(\mathbf{k})+M(\mathbf{k})+1}{E(k)}$$

and finally

$$\tilde{W}_{\text{fluct}}=W_{\text{fluct}}(\text{Vac.}+1)-W_{\text{fluct}}(\text{Vac.})$$

$$=\frac{e^2}{2mc^2}\langle A_0{}^2\rangle_{\text{Av}}.$$

This is identical with the corresponding expression for the Dirac electron.

VI. The Higher Approximations of the Self-Energy in the Theory of the Positron

It will be proved in this section that the successive approximations of the self-energy of the electron vanish in the limit $m\to 0$. Furthermore it is shown that the divergence of the self-energy is logarithmic in every approximation.[11]

We consider the total system containing the electrons and the radiation field and calculate the energy $W(s)$ of the state s of this system. $W(s)$ can be expanded in a series of approximations $W(s)=\Sigma_n W^{(n)}(s)$ corresponding to an expansion in powers of the parameter e^2/hc. Since this procedure does not give zero for the vacuum in Dirac's positron theory, the self-energy of the electron must be defined as the difference between the energy $W(\text{Vac.}+1)$ of the state in which one electron is present and the energy $W(\text{Vac.})$ of the vacuum alone. We confine ourselves to the calculation of the electrodynamic

[11] Recently A. Mercier (Helv. Phys. Acta **12**, 55 (1938)) has treated the same problem and has obtained a higher divergence. As he does not compute the numerical factors of the divergent expressions, he cannot exclude a factor zero for the highest divergent terms. The following considerations, however, show that the highest nonvanishing terms diverge only logarithmically.

self-energy. The calculation of the electrostatic energy and the mixed terms in higher approximation can be made along the same lines.

We now consider the detailed form of the nth approximation $W_s^{(n)}$ of the energy of the state s. $W_s^{(n)}$ is a sum of terms containing a product of $2n$ matrix elements of the interaction energy which correspond to consecutive transitions of the total system from one state to another starting from the state s and returning to it. The terms have denominators that are products of energy differences between the original state s and intermediate states.

The self-energy of the electron $\tilde{W}^{(n)}$ is given by the difference

$$\tilde{W}^{(n)}=W^{(n)}(\text{Vac.}+1)-W^{(n)}(\text{Vac.}). \quad (30)$$

We now prove that $\tilde{W}^{(n)}=0$ for $m=0$ by comparing it with the self-energy of a positron $\tilde{W}'^{(n)}$ in the same state:

$$\tilde{W}'^{(n)}=W^{(n)}(\text{Vac.}-1)-W^{(n)}(\text{Vac.})=\tilde{W}^{(n)}, \quad (31)$$

which is equal to the self-energy of the electron. There is no loss of generality if we confine our considerations to the self-energy of an electron at rest. The state (Vac.+1) is then specified by: every negative energy state and the lowest positive state s_{+0} occupied; (Vac.−1) means: every negative energy state except the highest one s_{-0} is occupied. We now show that

$$W^{(n)}(\text{Vac.}+1)+W^{(n)}(\text{Vac.}-1) =2W^{(n)}(\text{Vac.}) \quad \text{for} \quad m=0. \quad (32)$$

Comparing $W^{(n)}(\text{Vac.}+1)$ and $W^{(n)}(\text{Vac.}-1)$ with $W^{(n)}(\text{Vac.})$ we notice:

(a) $W^{(n)}(\text{Vac.}-1)$ lacks all terms containing transitions of the electron in the state s_{-0}.

(b) $W^{(n)}(\text{Vac.}+1)$ contains additional terms from transitions of the additional electron in s_{+0}.

(c) $W^{(n)}(\text{Vac.}-1)$ contains additional terms from transitions of one of the vacuum electrons into the empty state s_{-0}.

(d) $W^{(n)}(\text{Vac.}+1)$ lacks terms containing transitions of the vacuum electrons into the state s_{+0} of the additional electron.

We now prove that the missing terms of (a) and (d) are in the limit $m\to 0$ identical with the additional terms of (b) and (c), respectively. The only difference between the pair (a), (b) and the pair (c), (d) consists in the fact that the specified transitions start from a positive or from a negative state, respectively. This fact does not affect the energy differences in the denominators in the limit $m\to 0$, as in this limit the energies of s_{+0} and s_{-0} are equal. It remains to show that the numerators are also unchanged. This can be seen in the following way: the transition matrix elements appearing in the denominator belong to a chain of consecutive transitions starting from and returning to the initial state of the system. Thus the transition elements from or to s_{+0} or s_{-0} belong to a chain of transitions of an electron starting from the state s_{+0} (or from s_{-0}, respectively) and returning to this state. The transition elements form the product:

$$P_\pm=\{\psi^*(\pm 0)H_1\psi(p_1)\}\{\psi^*(p_1)H_2\psi(p_2)\}\cdots \times\{\psi^*(p_{n-1})H_n\psi(\pm 0)\}. \quad (33)$$

Here H_i is the interaction energy with the light quantum which is emitted or absorbed with the ith transition and $\psi(p_i)$ is the wave function of the electron with the momentum p_i which performs the transition. $\psi(\pm 0)$ is the wave function of the state s_{+0} or s_{-0}, respectively. After averaging over the two spin states for every momentum p_i, this product can be written as trace of the following matrix:

$$P_\pm=\text{Trace}\left[H_1\cdot\tfrac{1}{2}\left(1+\frac{\boldsymbol{\alpha}\cdot\mathbf{p}_1+\beta mc}{E_1}\right)\cdot H_2 \cdot\tfrac{1}{2}\left(1+\frac{\boldsymbol{\alpha}\cdot\mathbf{p}_2+\beta mc}{E_2}\right)\cdots H_n\tfrac{1}{2}(1\pm\beta)\right].$$

Here $E_i=\pm(p_i^2+m^2c^2)^{\frac{1}{2}}$ is the energy of the wave function $\psi(p_i)$. If we now go to the limit $m\to 0$, all terms with β disappear except those containing the β in the last parenthesis. (H_i does not contain the operator β.)[12] Since the trace of every expression containing α's but only one β is zero, we are allowed to omit the last also, and it becomes evident that $P_+=P_-$. From (30), (31)

[12] This conclusion does not hold if one or more of the p_i's are $\sim mc$. As $\tilde{W}^{(n)}$ is an integral over all possible intermediate momenta p_i in the volume V, the terms with one or more $p_i\sim mc$ contain one or more factors $(mc)^3$ so that these contributions should be neglected.

and (32) it follows directly that

$$\tilde{W}^{(n)}=0 \quad \text{for} \quad m=0.$$

We now prove that this result infers a logarithmical divergence of $\tilde{W}^{(n)}$ for finite m. The terms of $\tilde{W}^{(n)}$ contain transitions in which n light quanta are emitted and absorbed. We write $W^{(n)}$ as an integral over the wave vectors $\mathbf{k}_1\cdots\mathbf{k}_n$ of these light quanta:

$$\tilde{W}^{(n)}=\int_0^{P/h}\mathbf{dk}_1\cdots\int_0^{P/h}\mathbf{dk}_n F(k_1\cdots k_n) \quad (34)$$

and integrate to a finite limit $P/h\gg mc/h$ in order to make $\tilde{W}^{(n)}$ finite. The properties of $F(k_1\cdots k_n)$ are very simple for all $p_i\gg mc$ where p_i are as above the momenta of the electrons which change their states in the transitions to intermediate states (except of course the momentum $p_0=0$ of the electron under consideration in its initial state). It can be shown that replacing every k_l ($l=1\cdots n$) by $k_l'=xk_l$ gives

$$F(k_1'\cdots k_n')=x^N F(k_1\cdots k_n) \quad \text{if} \quad p_i\gg mc,$$

where N is an integer. This result may be understood in the following way. By means of the substitution $k_l'=xk_l$ the momenta p_i of the excited electrons in the intermediate states are also multiplied by a since the p_i are sums or differences of the momenta of the absorbed or emitted light quanta. If now $k_l\gg mc$, all momenta p_i involved are large compared to mc, and the corresponding energies can be replaced by $c|p_i|$. This neglect of mc compared to p_i has as consequence that all energy differences E_s-E_i between the initial state and the intermediate states are multiplied by x if one replaces k_l by k_l'. Since all terms of $\tilde{W}^{(n)}$ have $2n-1$ energy differences in the denominator, the latter is proportional to x^{2n-1}. The numerators consist of n matrix elements which form expressions like (33). From the fact that every H_i is proportional to $k_i^{-\frac{1}{2}}$ and from the structure of (33) it follows that the numerators are sums of terms proportional to x^{-n}, $x^{-n-1}\cdots x^{-2n}$. We get, therefore,

$$F(k_1'\cdots k_n')=\sum_{z=3n-1}^{z=5n-1}c_z x^{-z} \quad \text{for} \quad p_i\gg mc.$$

Not all of these c_z need to be different from zero. If c_ξ is the first coefficient different from zero, we can write

$$F(k_1'\cdots k_n')=c_\xi x^{-\xi}\leqq x^{-\xi}F(k_1\cdots k_n) \quad (35)$$

because we certainly can neglect the terms with $z>\xi$, which are smaller by the ratio $\sim(mc/p_i)^{z-\xi}$. We are now interested in the function $\tilde{W}^{(n)}(P)$. The relation (35) is not valid in the entire region of integration in (34). The regions in which the conditions $p_i\gg mc$ are not fulfilled are restricted to certain small areas in the $3n$-dimensional space of the wave vectors $\mathbf{k}_1\cdots\mathbf{k}_n$. The contributions of these areas can be neglected for $P\gg mc$. We therefore get

$$\tilde{W}^{(n)}(xP)\leqq\int_0^{P/h}\mathbf{dk}_1'\cdots\int_0^{P/h}\mathbf{dk}_n' F(k_1'\cdots k_n')$$
$$\leqq x^{3n-\xi}\cdot W'^{(n)}(P)+\text{smaller terms.}$$

The additional smaller terms come from the regions of integration in which these considerations are not applicable and from the neglected terms. From this relation follows, that

$$\tilde{W}^{(n)}\leqq c\left(\frac{e^2}{hc}\right)^n\cdot\frac{P^N}{(mc)^N}\cdot\left(\lg\frac{P}{mc}\right)^t\cdot mc^2+\text{smaller terms.}$$

Here $N=3n-\xi$, c is a numerical factor and $0\leqq t\leqq n$ because any of the n integrations might give rise to a logarithm. The factors mc are applied in order to make the dimensions fit. It was proved that

$$\lim_{(m=0)}\tilde{W}^{(n)}=0.$$

This is only possible if $N\leqq 0$. This is equivalent to the fact that $\tilde{W}^{(n)}$ does not diverge stronger than

$$\tilde{W}^{(n)}\sim\left(\frac{e^2}{hc}\right)^n mc^2\left(\lg\frac{P}{mc}\right)^n.$$