On Diffraction and Radiation of Electromagnetic Waves

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Inasmuch as it is rarely possible to treat diffraction of electromagnetic waves exactly, the Kirchhoff formulation of Huygens' Principle has been frequently used in approximate calculations. If the Kirchhoff formula is applied directly to the field intensities of the incident wave over the aperture, the diffracted field is found to be inconsistent with Maxwell's equations. If, on the other hand, this formula is applied to some auxiliary vector potential from which the diffracted field is subsequently deduced by differentiation, the result (although consistent with Maxwell's equations) depends on the particular choice of the auxiliary vector and in some instances, at least, is obviously unreasonable (Appendix III). The calculations of diffracted fields and radiation fields, based either on the Equivalence Principle or on the more general Induction Theorem, depend upon *a priori verifiable* approximations to the actual fields in the neighborhoods of the sources of the diffracted and radiated waves. For this reason we feel that these methods are preferable to those based on the Kirchhoff formula.

THE purpose of this paper is to describe several methods for dealing with diffraction and radiation of electromagnetic waves, which are believed to have certain advantages over the methods based on the Kirchhoff formula. These methods arise out of certain "induction" and "equivalence" theorems. At present, it is rarely possible to solve diffraction and radiation problems exactly. Either the usual methods or the methods to be described in this paper require approximations. The point in favor of the latter methods is our ability to use physical intuition and available knowledge, theoretical and experimental, as a guide in making the necessary approximations.

The essential feature of the following theorems is *identification of certain portions of given electromagnetic fields as the fields of appropriate fictitious electric and magnetic current sheets.*

THE INDUCTION THEOREM

It is not easy to state the Induction Theorem in its most general form and it is more expedient to formulate its variants to fit special conditions. In one form, the theorem was enunciated in a previous paper.[1] We shall now state it in the form directly applicable to the problem of radiation from an open end of a perfectly conducting metal pipe.

Let E^0, H^0 be the field which would exist over the surface of the aperture if the pipe were continued to infinity; let E', H' be the "reflected" field, that is, the field which has to be added to the "impressed" field E^0, H^0 in order to obtain the actual field in region (1) just within the pipe when it is terminated; and, finally, let E'', H'' be the "transmitted" or the actual field in region (2) just beyond the end of the pipe. Consider the field $\bar{E}$, $\bar{H}$ composed of E', H' in region (1) and of E'', H'' in region (2). We assert that the field $\bar{E}$, $\bar{H}$ could be produced, in the presence of the pipe, by electric and magnetic current sheets over the aperture and that the densities of the sheets are

$$J^0 = n\times H^0, \quad M^0 = -n\times E^0, \tag{1}$$

J being the density of the electric current sheet, M the density of the magnetic current sheet and n the unit normal to the sheets pointing into region (2). Evidently E^0 and H^0 could be replaced by their components tangential to the surface of the sheets.

While a formal proof of this theorem will be discussed in Appendix I, a few details will be given here in the belief that they will elucidate the meaning of this theorem and the manner in which it can be modified to suit different conditions. The electric and magnetic intensities of the total field around the pipe, under actual conditions, must be continuous across the aperture; thus over the aperture we have

$$E^0 + E' = E'', \quad H^0 + H' = H'';$$

or

$$E'' - E' = E^0, \quad H'' - H' = H^0. \tag{2}$$

By our assumptions the field $\bar{E}$, $\bar{H}$ satisfies

[1] S. A. Schelkunoff, Bell Syst. Tech. J., pp. 92–112, January, 1936.

Maxwell's equations everywhere except over the aperture and the surface of the pipe. This field also satisfies the proper boundary conditions over the surface of the pipe. Over the aperture of the pipe, the field $\bar{E}$, $\bar{H}$ and, in particular, its tangential components are discontinuous. It follows from the integral form of Maxwell's equations (or directly from the laws of Ampère and Faraday) that discontinuities in the components of magnetic and electric intensities, tangential to a surface S, exist if and only if there are corresponding superficial electric and magnetic current sheets over S. The density of the electric current sheet is equal to the discontinuity in the tangential component of the magnetic intensity and the density of the magnetic current sheet is equal to the discontinuity in the tangential component of the electric intensity.[2]

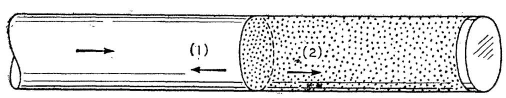

FIG. 1. Pipe enclosing two different dielectric media.

Therefore, if we start with the current sheets (1) and wish to determine the field produced by them in the presence of the pipe, we must regard the field $\bar{E}$, $\bar{H}$ as a possible solution. If we accept for the moment that any solution of Maxwell's equations which satisfies given boundary conditions and behaves in the proper manner at infinity is unique, then $\bar{E}$, $\bar{H}$ is the field which we would calculate from the current sheets (1) no matter what particular method of calculation we are likely to adopt.

Let us consider another situation. Let a perfectly conducting metal pipe enclose two different dielectric media which are separated by a plane boundary (Fig. 1) and let a progressive wave E^0, H^0 move from left to right in region (1). By the Induction Theorem, the wave reflected from the interface between the two media and the wave transmitted into region (2) could be produced by electric and magnetic current sheets over the interface, whose densities are given by (1). In this instance, the exact solution can be obtained directly as well as from the Induction Theorem, thus affording a verification of the theorem.

Consider now a system with one degree of freedom. An example of such a system is an ordinary transmission line terminated in some impedance (Fig. 2). Let $V^0(x)$, $I^0(x)$ be the voltage and the current that would exist in the progressive wave moving from left to right if the line were terminated in its characteristic impedance instead of the actual impedance. Choose some particular point x_0 on the actual terminated line. If $V''(x)$, $I''(x)$ is the actual voltage and current to the right of this section and $V'(x)$, $I'(x)$ is the voltage and the current which must be added to $V^0(x)$, $I^0(x)$ in order to obtain the actual voltage and current to the left of the section, then the continuity conditions at $x = x_0$ require

$$V^0(x_0) + V'(x_0) = V''(x_0), \qquad I^0(x_0) + I'(x_0) = I''(x_0);$$

or

$$V''(x_0) - V'(x_0) = V^0(x_0), \qquad I''(x_0) - I'(x_0) = I^0(x_0). \quad (3)$$

In this case, the Induction Theorem consists in asserting that the wave $\bar{V}(x)$, $\bar{I}(x)$, equal to V', I' on the left of $x = x_0$ and to V'', I'' on the right of it, *could be produced by a zero impedance generator in series with the line and an infinite impedance generator (constant current generator) in shunt with the line, the voltage of the series generator and the current through the shunt generator being, respectively,* $V^0(x_0)$ *and* $I^0(x_0)$.

EQUIVALENCE THEOREMS

Broadly speaking, the equivalence theorems differ from the corresponding induction theorems in that the latter specify the current sheets capable of producing both the reflected field E', H' and the transmitted field E'', H'' while the former specify the current sheets capable of producing only the transmitted field. Thus, if in

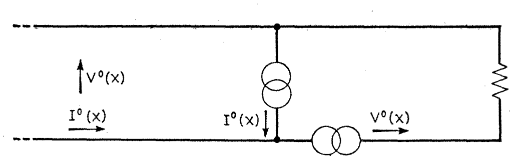

FIG. 2. Transmission line terminated in an impedance.

[2] It also follows from the integral form of Maxwell's equations that the discontinuities in the *normal* components of electric and magnetic *current densities* are determined by the discontinuity in the tangential field *intensities*.

region (1), we postulate a zero field in place of the reflected field E', H' and two current sheets (over the surface of the aperture) defined by

$$J''=n\times H'', \quad M''=-n\times E'', \qquad (4)$$

to take care of the discontinuities in the total field E, H comprised of the zero field in region (1) and the actual transmitted field E'', H'' in region (2). It should be noted that in equivalence formula (4) the densities of the current sheets are not known in advance as is the case in the corresponding Induction Formula (1).

Another equivalence theorem may be obtained by choosing a closed surface S comprised of the surface of the aperture and the outer surface of a horn, postulating a zero field inside the surface and E'', H'' outside, and introducing electric and magnetic current sheets over S of densities given by (4). We assert, then, that these sheets will produce the postulated field. This theorem may be called the "free-space" equivalence theorem in order to emphasize that in carrying out the calculations, the horn must be ignored and the response must be calculated by the "free-space" retarded potentials in contrast with the previous equivalence theorem according to which the response to the current sheets had to be calculated subject to the boundary conditions at the surface of the horn. On the other hand, the current sheets required by the free-space equivalence theorem extend over the outer surface of the horn as well as over the aperture while in the other theorem they extend only over the aperture. In a previous paper[1] the "free-space" equivalence theorem has been named the Equivalence Principle[3] in the belief that it is a precise expression and a generalization of a physical principle enunciated by Huygens.

Huygens' principle is commonly stated as follows:[4] Each element of any wave front acts as a new source of disturbance, sending out secondary waves, and these secondary waves combine to form the new wave front. I am inclined to believe that if with the aid of a hypothetical perfectly absorbing screen we could isolate the disturbance produced by an individual secondary source, we should find this disturbance identical with that produced by the elementary source (4). It is impossible to prove this contention any more than to prove that the energy flow per unit area in an electromagnetic field is expressed by $E\times H$. Just as in the latter case, it can be proved only that the *total flow* of energy across a *closed* surface can be obtained by integrating the Poynting vector, in the former case it can be proved only that when we integrate a certain expression over a closed surface we obtain the correct field. In either case it is possible to modify the integrand and still obtain the same results *when the integration is performed over the closed surface.* The choice of a particular integrand must necessarily be made on other than mathematical grounds.

Let us now consider a perfectly conducting metal tube and a progressive wave moving from left to right (Fig. 3). The actual field at some point P to the right of some cross section S_1 could be produced by electric and magnetic current sheets given by (4), or by (1) since in this case $E''=E^0$ and $H''=H^0$. The action of these current sheets must be calculated subject to the boundary conditions at the surface of the pipe. This field could also be calculated by the free-space equivalence principle by using the current sheets spread over a *closed* surface S_2 surrounding point P and whose densities are still given by (1).

Radiation from Horns

Preceding theorems can be used for the approximate calculation of the power radiated by horns and the radiation patterns of those horns. Three methods of approach present themselves:

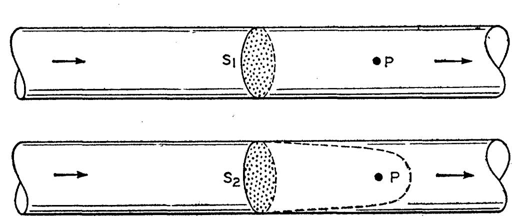

Fig. 3. Perfectly conducting metal tube; progressive wave moving from left to right.

[3] The Equivalence Principle was originally discovered by A. E. H. Love [Phil. Trans. Roy. Soc. **A197**, 1–45 (1901)] and subsequently by H. M. MacDonald [*Electric Waves* (1902), p. 16]. The latter supplied two proofs [MacDonald, *Electric Waves* (1902), p. 16; MacDonald, Proc. London Math. Soc. Series E, **10**, 91–95 (1911)] of the theorem, of which only the second is valid. A few years ago, the theorem was rediscovered by the present author as one of a group of theorems treated in a previous paper (reference 1) and here.

[4] A. E. Caswell, *An Outline of Physics* (1929), p. 544.

1. In accordance with the free-space equivalence principle the sources within the horn *and* the horn itself[5] are replaced by electric and magnetic current sheets, surrounding the horn, with densities given by (4); then, the free-space field of these sources is calculated by means of the following formulae.

$$E = -i\omega\mu A + \frac{1}{i\omega\epsilon}\,\text{grad. (div. } A) - \text{curl } F,$$
$$H = -i\omega\epsilon F + \frac{1}{i\omega\mu}\,\text{grad. (div. } F) + \text{curl } A, \qquad (5)$$

where the magnetic vector potential A and the electric vector potential F are given by

$$A = \int_{(S)} \frac{J'' e^{-i\beta r}}{4\pi r} dS, \quad F = \int_{(S)} \frac{M'' e^{-i\beta r}}{4\pi r} dS. \qquad (6)$$

These formulae become much simpler if we are interested only in the field at great distances from the horn.[6]

In practical applications neither J'' nor M'' are known exactly and suitable approximations must be made, with available theoretical and experimental knowledge as a guide. The situation is similar to that in which we find ourselves when calculating the power radiated by antennas and by transmission lines from the electric current distribution. The latter is not known exactly but available evidence points to the fact that it is nearly sinusoidal.

If the aperture of the horn is so large that the cut-off frequency of the horn is appreciably below the operating frequency, practically all energy reaching the aperture passes into the outer medium. In this case, it is reasonable to suppose that the field over the aperture will be substantially the same, except near the edges, as if the horn were infinitely long. Thus, we can assume that over the aperture J'' and M'' are approximately equal to the *known* quantities J^0 and M^0. Over the surface of the horn M'' is *known* to be very small (it would vanish for perfectly conducting horns) and J'' is *assumed* to be small.

[5] That is, the sources induced in the horn.
[6] S. A. Schelkunoff, *A General Radiation Formula* (to be published).

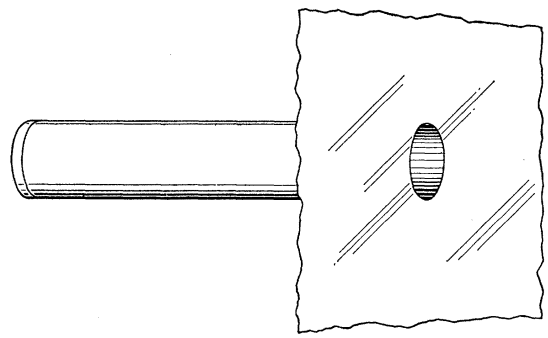

FIG. 4. Pipe ending in a perfectly conducting plane.

It may be pointed out that in theory it is possible to calculate the radiation pattern and the radiated power from the conduction current in the horn. In practice, however, no basis has been found for making a reliable *a priori* assumption with regard to this current distribution.

2. The second method of approach consists in replacing the sources within the horn by electric and magnetic current sheets with the same densities (4) as in the preceding case but extending only over the aperture. These current sheets are supposed to act in the presence of the horn as a reflecting surface. Thus, A and F are given by

$$A = \int_{(S_a)} J'' \cdot \psi_1(x, y, z; x', y', z') dS,$$
$$F = \int_{(S_a)} M'' \cdot \psi_2(x, y, z; x', y', z') dS, \qquad (7)$$

where ψ_1 and ψ_2 are two diadics and the integration is extended over the surface S_a of the aperture. The scalar product of the moment of an electric current element and diadic ψ_1 is the magnetic vector potential of the electric current element and, similarly, the scalar product of the moment of the magnetic current element and diadic ψ_2 is the electric vector potential of the element.

In applying (7) we no longer need approximate estimates of the field over the outer surface of the horn; instead we have to approximate ψ_1 and ψ_2. In the case of the horn considered above, we would say that the effect of the horn, as a reflecting surface, on the radiation field is probably small and that we could replace ψ_1 and ψ_2 by the free-space transmission factors. In this case we obtain the same result whether we follow the first or the second method of approach.

The second method may, however, be more valuable than the first in other situations. For example, consider a pipe and an infinite perfectly conducting plane (Fig. 4). The first method will give us the same result as if the plane were absent since, not knowing *a priori* J'' in the plane, we should be forced to ignore it. The second method will supply us with a better result since we shall deal with electric and magnetic current sheets over a perfectly conducting plane.[7] This problem can be solved by the free-space retarded potentials. The image of the electric current sheet in the plane is negative and will cancel the effect of the sheet itself. The image of the magnetic current sheet is positive and will double the effect of the sheet. In other words, instead of (6) we now have

$$F=\int_{(S_a)}\frac{M''e^{-i\beta r}}{2\pi r}dS,\quad A=0. \tag{8}$$

Since M'' is determined by the tangential components (E_x'', E_y'') of the electric intensity, we can obtain from (5) and (8)

$$\begin{aligned} E_x&=\frac{1}{2\pi}\int_{(S_a)}E_x''\left(i\beta+\frac{1}{r}\right)\frac{e^{-i\beta r}}{r}\cos\theta dS,\\ E_y&=\frac{1}{2\pi}\int_{(S_a)}E_y''\left(i\beta+\frac{1}{r}\right)\frac{e^{-i\beta r}}{r}\cos\theta dS,\\ E_z&=-\frac{1}{2\pi}\int_{(S_a)}(E_x''\cos\varphi+E_y''\sin\varphi)\\ &\qquad\times\left(i\beta+\frac{1}{r}\right)\frac{e^{-i\beta r}}{r}\sin\theta dS. \end{aligned} \tag{9}$$

These results agree with Sommerfeld's modification of the Kirchhoff formula.[8]

3. By the Induction Theorem the field $\bar{E}$, $\bar{H}$ consisting of the reflected field E', H' inside the horn and the transmitted field E'', H'' outside the horn can be produced by electric and magnetic current sheets with densities given by (1). Thus, we have

$$A=\int_{(S_a)}J^0\cdot\psi_1 dS,\quad F=\int_{(S_a)}M^0\cdot\psi_2 dS, \tag{10}$$

where ψ_1 and ψ_2 are the same diadics as in (7).

In theory, this theorem is more powerful than either of the two preceding theorems. It does not require any approximations to the field over the surface of integration and it gives both the reflected and the transmitted field. In practice, however, the major value of this theorem is to furnish a basis for approximating the field over the aperture. In transmission systems admitting of only one transmission mode, the Induction Theorem leads to a definite solution (see Appendix II). We then assume that the known results in the simple case may be taken as a first approximation in more complex cases.

The Kirchhoff Formula

It is only fitting that we should not pass in silence the well-known Kirchhoff formula which has been universally used for solving diffraction problems and recently has been applied to problems of radiation.[9, 10, 11] This formula is

$$V=\int_{(S)}\left[\left(i\beta+\frac{1}{r}\right)V\cos(n,r)-\frac{\partial V}{\partial n}\right]\frac{e^{-i\beta r}}{4\pi r}dS, \tag{11}$$

where V is a wave function, S any closed surface surrounding a source-free region, and n is the normal to S looking into this region.

When applying (11) to electromagnetic problems, we are confronted with greater difficulties than we were when applying previous methods. In either case, we have to make approximations to the integrand over the surface of integration but while before we were aided in making the necessary approximations by physical intuition and experience, we have nothing to guide us, at least at present, in the case of the Kirchhoff formula. We could interpret, for example, V as a cartesian component of the magnetic vector potential, but we have no knowledge of its

[7] While the actual plane has a hole in it, in applying the second method we are permitted to plug the hole with a perfectly conducting disk since the field due to the postulated current sheets is known to be identically zero on the side of the pipe.

[8] A. Sommerfeld, Göttinger Nachr. 1894, Nr. 4.

[9] R. Darbord, L'Onde Elec. **11**, 53–82 (1932).

[10] H. Diamond and F. W. Dunmore, Nat. Bur. Stand. J. Research **19**, 1–19 (1937). (Paper RP1006.) Proc. I. R. E. **25**, 1542–1560 (1937).

[11] W. L. Barrow and F. M. Greene, Proc. I. R. E., **26**, 1498–1519 (1938).

behavior over the outer surface of the horn. We can expect no help from the experiment since we cannot measure the magnetic vector potential either directly or indirectly. Likewise, we could apply (11) to the electric vector potential but with no better success. If we simply ignore the contributions coming from the outer surface of the horn, we get different radiation patterns, depending upon the choice of V. If we apply (11) directly to cartesian components of E and H and ignore the contributions of the outer surface of the horn, we obtain a result which is inconsistent with Maxwell's equations. In other words, while (11) is correct when applied to a *closed* surface, there seems to be no way to adapt it to practical needs when we are forced to make approximations. An interesting example of the difficulties which may confront us is given in Appendix III.

At this point a few words must be said about the comparison made by Barrow and Greene[11] between experimental radiation patterns and those calculated with the aid of the Kirchhoff formula from the magnetic vector potential. As the authors point out the agreement is satisfactory only in the horizontal plane (in the plane perpendicular to the electric vector). The agreement would have been satisfactory in the vertical plane (but not in the horizontal) if (11) were applied to the electric vector potential. In the case considered by Barrow and Greene formula (6) agrees satisfactorily with experimental results in both planes. Fig. 5 represents a comparison between an experimental vertical radiation pattern and patterns calculated by different methods.

Conclusion

It is hardly necessary to emphasize that the proposed methods of dealing with radiation and diffraction problems are based, at least in the important practical cases, on certain approximations. Fortunately these approximations are susceptible of direct experimental verification. It is hoped that this paper will stimulate such experiments.

These experiments should be made preferably on fairly small apertures. As the aperture becomes larger, the difference (but not the ratio) of the radiation intensities obtained from the Equivalence Principle on one side, and from the Kirchhoff formula on the other, becomes small; this is because the "space factor" of the array of secondary sources over the aperture becomes the dominant factor and the precise nature of the secondary sources loses its importance. In fact, one could make almost arbitrary assumptions with regard to the secondary sources and obtain the radiation patterns, for large apertures, which would differ but little.

Appendix I

The essential parts of our proofs of induction and equivalence theorems are the appropriate uniqueness theorems, that is, theorems asserting the uniqueness of the field defined by a given set of conditions. If the given set of conditions is known to lead to a unique solution, we are assured of obtaining the same solution no matter what method we happen to use even if we were simply to guess the solution and then to verify it. For instance, in dealing with free oscillations inside a perfectly conducting spherical sheet we may obtain a certain solution subject to the condition that the tangential component of the electric intensity vanishes at the surface of the

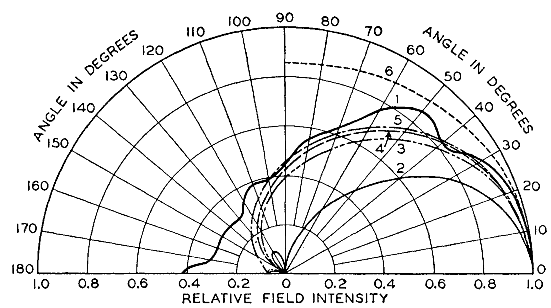

FIG. 5. (1) The experimental vertical radiation pattern obtained by Barrow and Greene for the open end of a rectangular tube; this curve is taken from Fig. 10B of reference 11; (2) the preferred theoretical curve calculated by Barrow and Greene by applying the Kirchhoff formula to a Hertzian vector; (3) the theoretical curve calculated by Barrow and Greene by applying the Kirchhoff formula directly to the electric vector and disapproved by them because the direction of the electric vector in the radiation field is wrong; (4) the theoretical curve obtained from the Equivalence Principle on the assumption that the ratio of E to H over the aperture is the same as for plane waves in free space; (5) the theoretical curve obtained from the Equivalence Principle on the assumption that the field over the aperture is the same as would have been over the same surface in case the tube were continued indefinitely; (6) the theoretical curve for the case in which a perfectly conducting infinite flange is added to the open end of the tube.

spherical sheet. The field outside the sphere may be taken equal to zero. Having obtained the solution, we find that the tangential component of the magnetic intensity is discontinuous across the sphere. In accordance with Maxwell's equations, this discontinuity implies electric current in the sphere, the density of this current being equal to the above-mentioned discontinuity. If we were to calculate the field of this current sheet by the retarded potential method, we expect to obtain the field from which we started.

An exhaustive analysis of uniqueness theorems of electromagnetics would require considerable space and could properly be regarded as a subject in itself. We shall restrict ourselves to the simplest case of monochromatic waves in dissipative media. This case is really sufficient for our purposes since we can assume the conductivity of the medium to be so slight that its effect is negligibly small and yet be sure that the needed theorems are applicable.

We shall write Maxwell's equations in the following form

$$\begin{aligned} \text{curl } E &= -i\omega\mu H - M, \\ \text{curl } H &= (g+i\omega\epsilon)E + J, \end{aligned} \tag{12}$$

where J and M are the densities of the applied electric and magnetic currents. Let us multiply scalarly the first equation by the conjugate of H, the conjugate of the second equation by E, and subtract; thus we obtain

$$\begin{aligned} H^*\cdot\text{curl } E - E\cdot\text{curl } H^* = &-M\cdot H^* - E\cdot J^* \\ &-gE\cdot E^* + i\omega\epsilon E\cdot E^* - i\omega\mu H\cdot H^*, \end{aligned} \tag{13}$$

where the asterisks designate the conjugate complex numbers. Integrating over a volume enclosed by a surface S and taking into account that

$$\begin{gathered} H^*\cdot\text{curl } E - E\cdot\text{curl } H^* = \text{div. } E\times H^*, \\ \int_{(v)} \text{div. } E\times H^* dv = \int_{(S)} (E\times H^*)_n dS, \end{gathered} \tag{14}$$

we obtain

$$\begin{aligned} -\int_{(v)} (E\cdot J^* + M\cdot H^*)dv = g\int_{(v)} E\cdot E^* dv \\ + i\omega\int_{(v)} (\mu H\cdot H^* - \epsilon E\cdot E^*)dv + \int_{(S)} (E\times H^*)_n dS, \end{aligned} \tag{15}$$

where n is the normal to S pointing away from the volume under consideration. In (15) the region v may be a multiply connected region enclosed by surface S_0 externally and by surfaces S_1, S_2, etc. internally; in this case, S consists of S_0, S_1, S_2, etc.

In dissipative media the field at great distances from the sources vanishes as an exponential function of the distance; hence, the last term in (15) vanishes in the limit as S_0 recedes to infinity. If there are no impressed currents or if in the regions occupied by impressed currents, the components of the field in the directions of these currents vanish, then the left member of (15) is equal identically to zero. The first term on the right is real and the second is a pure imaginary; therefore, in absence of impressed currents these terms must vanish separately. Since $E\cdot E^*$ is essentially positive, the volume integral can vanish only if E vanishes everywhere; then H also vanishes everywhere and the electromagnetic field is identically zero.

Let us now prove with the aid of the above result the free-space equivalence principle (4). The assertion is that the current sheets (4) over a surface S enclosing given sources produce the field $\bar{E}$, $\bar{H}$ which is equal to the field E'', H'' on the source-free side of S and to the field 0, 0 on the other side of S. This synthetically obtained field is certainly one solution of Maxwell's equations subject to the proper boundary conditions across S and behaving in the appropriate manner at infinity. The question is whether it is the only one. In other words, shall we obtain the same field $\bar{E}$, $\bar{H}$ if we solve (12) by the retarded potential method or shall we obtain a different field $\bar{E}_1$, $\bar{H}_1$? Let us suppose that the latter is the case. Then we have

$$\begin{aligned} \text{curl } \bar{E} &= -i\omega\mu\bar{H} - M'', \\ \text{curl } \bar{E}_1 &= -i\omega\mu\bar{H}_1 - M'', \\ \text{curl } \bar{H} &= (g+i\omega\epsilon)\bar{E} + J'', \\ \text{curl } \bar{H}_1 &= (g+i\omega\epsilon)\bar{E}_1 + J''. \end{aligned} \tag{16}$$

On subtracting these equations we find that the difference field $\bar{E}-\bar{E}_1$, $\bar{H}-\bar{H}_1$ satisfies the homogeneous form of (12), that is, the form in which $M=J=0$. Therefore, this field vanishes identically throughout the entire space and $\bar{E}_1=\bar{E}$, $\bar{H}_1=\bar{H}$.

Similarly we can prove an equivalence

theorem for the case in which the horn is retained and the impressed current sheets (4) extend only over the mouth of the horn. The Induction Theorem (1) is also proved in the same manner.

In connection with the above theorems, the reader has undoubtedly noted an interesting paradox. The electric and magnetic current sheets on S imply only *discontinuities* in the magnetic and electric intensities across the surface. How is it that in accordance with the above theorems we know the actual intensities on either side of the sheet and not only their differences? The explanation lies in the fact that in these theorems the densities of the superficial current sheets *are not independent*, these densities are obtained from functions satisfying the homogeneous Maxwell's equations and theoretically either the electric field alone or the magnetic field alone suffice for defining *both* current sheets.

Appendix II

In the case of a simple transmission line,[12] a direct verification of the Induction Theorem is possible and the results are useful as an aid to the study of more complex cases. As before we shall designate by a single prime the reflected voltage and current and by a double prime the transmitted voltage and current. The subscript $_1$ will be used to designate those parts of the voltage and the current which are produced by the impressed series voltage V^0 and the subscript $_2$ will refer to the voltage and current produced by the shunt current I^0.

We shall consider the case of a semi-infinite transmission line, with characteristic impedance Z_0, terminated in the impedance Z at some point $x=x_0$. Carrying out the calculations in accordance with the Induction Theorem, we obtain

$$
\begin{aligned}
I_1'(x_0) &= V^0/(Z_0+Z),\\
I_2'(x_0) &= -ZI^0/(Z_0+Z),\\
I_1''(x_0) &= V^0/(Z_0+Z),\\
I_2''(x_0) &= Z_0I^0/(Z_0+Z),\\
V_1'(x_0) &= -Z_0V^0/(Z_0+Z),\\
V_2'(x_0) &= Z_0ZI^0/(Z_0+Z),\\
V_1''(x_0) &= ZV^0/(Z_0+Z),\\
V_2''(x_0) &= Z_0ZI^0/(Z_0+Z).
\end{aligned}
\tag{17}
$$

Thus, the total reflected voltage and current and the total transmitted voltage and current are

$$
\begin{aligned}
I'(x_0) &= \frac{Z_0-Z}{Z_0+Z}I^0, \quad V'(x_0) = \frac{Z-Z_0}{Z+Z_0}V^0,\\
I''(x_0) &= \frac{2Z_0}{Z_0+Z}I^0, \quad V''(x_0) = \frac{2Z}{Z_0+Z}V^0.
\end{aligned}
\tag{18}
$$

These expressions are precisely the same as would be obtained directly from the boundary conditions at point $x=x_0$.

If Z is nearly equal to Z_0, V' and I' are small while V'' and I'' are nearly equal to V^0 and I^0. This is the case when practically all power carried by the wave V^0, I^0 is transmitted beyond $x=x_0$. Similarly we expect that in the case of a horn with a large mouth, when practically all power delivered to the mouth passes on into the outer medium, E'' and H'' are substantially equal to E^0 and H^0 as we have previously assumed in our applications of the equivalence theorems.

On the other hand, if Z is either very small compared with Z_0 or very large, then the reflection is nearly complete and one of the quantities V'', I'' nearly vanishes while the other is nearly doubled. In this case the effect across the impedance Z is produced largely either by the series generator alone or the shunt generator alone. In the corresponding three-dimensional case (which occurs when the frequency is near the cut-off frequency) we expect, therefore, that as a first approximation we can calculate the radiation patterns by assuming only one current sheet over the mouth of the aperture. The corresponding radiation patterns will be symmetrical about the plane of the aperture of the horn. The departure from symmetry will be caused by the incomplete reflection.

Appendix III

Consider a perfectly conducting semi-infinite coaxial pair with an open end. Let the radii a and $b(b>a)$ be very small compared with the wavelength. The field inside the coaxial pair is approximately

$$
E_\rho = \frac{60I}{\rho}\cos\beta z, \quad H_\varphi = -\frac{iI}{2\pi\rho}\sin\beta z, \quad \beta = \frac{2\pi}{\lambda}, \tag{1}
$$

where: I is the maximum amplitude of the

[12] A transmission line admitting of only one transmission mode.

current, λ the wave-length, ρ the distance from the axis and z the distance along the axis. The open end is in the plane $z=0$.

The field can be obtained from a vector potential whose components are

$$A_\rho = A_\varphi = 0, \quad A_z = \frac{iI}{2\pi}\log\frac{\rho}{\rho_0}\sin\beta z. \tag{2}$$

This can be verified by substitution in

$$H_\varphi = -\frac{\partial A_z}{\partial\rho}, \quad E_\rho = \frac{1}{i\omega\epsilon}\frac{\partial^2 A_z}{\partial\rho\partial z}. \tag{3}$$

The values of the above wave function and its normal derivative over the aperture are

$$A_z = 0, \quad \frac{\partial A_z}{\partial n} = \frac{i\beta}{2\pi}I\log\frac{\rho}{\rho_0}. \tag{4}$$

Using the Kirchhoff formula in the customary manner (i.e., by applying it to the aperture) we find that at great distances from the aperture the approximate value of A_z is

$$A_z = -\frac{iI}{4\lambda}\left[\left(b^2\log\frac{b}{\rho_0} - a^2\log\frac{a}{\rho_0}\right) - \tfrac{1}{2}(b^2-a^2)\right]\frac{e^{-i\beta r}}{r}. \tag{5}$$

From this we calculate the field and then the radiated power; thus we obtain

$$W = \frac{40\pi^4}{\lambda^4}\left[\left(b^2\log\frac{b}{\rho_0} - a^2\log\frac{a}{\rho_0}\right) - \tfrac{1}{2}(b^2-a^2)\right]^2 I^2. \tag{6}$$

By choosing ρ_0 properly we can make the radiated power W equal to anything from zero to infinity. The approximate value for W found with the aid of the Equivalence Principle corresponds to ρ_0 satisfying

$$b^2\log\frac{b}{\rho_0} - a^2\log\frac{a}{\rho_0} = 0. \tag{7}$$

Note: On Diffraction and Radiation of Electromagnetic Waves

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Dr. Schelkunoff has kindly shown us the manuscript of the preceding paper which is closely allied to ours.[13] Since our formulation of the problem differs somewhat from that of Dr. Schelkunoff in "Some Equivalence Theorems of Electromagnetics and Their Application to Radiation Problems,"[14] and in the preceding paper, it appears worth while to point out that the results obtained by the Equivalence Principle are identical with ours. If the field of the equivalent surface currents is calculated from the vector potential or the Hertz vector, the contour charges need not be introduced explicitly. The contour charges simply ensure the self-consistency of the assumed field on the surface.

[13] J. A. Stratton and L. J. Chu, "Diffraction Theory of Electromagnetic Waves," Phys. Rev. **56**, 99–107 (1939).
[14] S. A. Schelkunoff, Bell Sys. Tech. J. **15**, 92–112 (1936).