

## Our Present Dilemma Regarding the Values of the Natural Constants $e$ , $m$ and $h$

### A New Graphical Method of Presentation

JESSE W. M. DUMOND  
*California Institute of Technology, Pasadena, California*  
 (Received April 7, 1939)

Fourteen experiments each of which determines some function of one or more of the constants  $e$ ,  $m$  and  $h$ , are classified, according to the function determined, into nine groups. The latter number is reduced to seven by the elimination of five of the thirteen experiments as unreliable in the light of our present knowledge. Three cardinal good experiments, the directly measured x-ray values of  $e$ , the many concordant measurements of  $e/m$  and the values of  $h/e$  from the limit of the continuous x-ray spectrum are exhibited on a new type of diagram showing separately results of all independent reliable determinations to portray the consistencies graphically. The indubitable nature of the discrepancy is thus made evident and the remaining four experimental values also plotted on the diagram seem to give better support to the first two named cardinal experiments than to the last. The new type of diagram is an isometric projection of a cartesian coordinate system whose axes represent the relative deviations of  $e$ ,  $m$  and  $h$  from conventionally assumed values. Planes in this space represent the different experimental determinations of functions of  $e$ ,  $m$  and  $h$  and the discussion brings out the absolute geometric property that five of these planes are "cozonal" (parallel to a common axis). Though there are seven equations it thus appears that to solve for  $e$ ,  $m$  and  $h$  either the direct  $e$  equation or alternately the  $R_\infty$  equation is indispensable.

The results are briefly recounted of a careful examination both by the author and others of numerous possible

experimental and theoretical sources of the discrepancy. Modified diagrams are shown for (a) the assumption that the velocity of propagation of radiation in vacuum suffers a slow decline with increasing frequency (the velocity for 20,000-volt x-rays is assumed to be  $\frac{1}{2}$  percent lower than the low frequency optical value); (b) the alternative assumption that the Bohr-Rydberg formula must be revised to include a factor  $(1-\alpha)$ . Both assumptions rectify the diagram as regards the three cardinal experiments but throw the remaining experiments out of line with them. Since the two assumptions named are about equally objectionable both on theoretical grounds and in their results on the diagram they are now rejected as unlikely. The author concludes that in all probability some unsuspected theoretical or experimental flaw in the determination of  $h/e$  by the continuous x-ray spectrum method is the source of the discrepancy and emphasizes the need for renewed study of this experiment along with x-ray ionization and excitation potentials over a wider range of voltages with better spectral resolution. The weakness of low voltage measurements of ionization and excitation potentials is pointed out. It is emphasized that when systematic error is as glaringly evident as at present the temptation to obtain compromise or "best" values of the natural constants by least-squares methods should be strongly resisted for it is not inconceivable that the discrepancy may reveal some important error of principle or theory.

#### I. THE NATURE OF THE DIFFICULTY

**E**XPERIMENTAL determinations of the natural atomic constants,  $e$ ,  $m$  and  $h$ , can be classified into nine types listed in Table I. In each case the experiment yields the determination of a numerical value,  $A$ , for some natural constant or a function of two or more natural constants as indicated.

#### Reliability of the different experiments

From this list we exclude for our present conclusions the oil-drop determinations, the photoelectric determinations, the determinations of the radiation constants  $c_2$  and  $\sigma$  and the fine structure constant determination<sup>13a</sup> because at the present time these results are subject to much greater experimental uncertainties than the

others. The two radiation constants yield results falling so far away from the others as to cast grave suspicion upon these two.<sup>2a</sup> Three independent and painstaking oil-drop determinations combined with five equally independent careful determinations of the viscosity of air listed by Robinson<sup>2</sup> combine to give fifteen possible values of  $e$  ranging from  $4.752 \times 10^{-10}$  to  $4.854 \times 10^{-10}$  e.s.u. Recent careful experimental studies<sup>5</sup> of the distribution in energy of photoelectrons ejected by monochromatic optical light have only served to emphasize the great difficulties and uncertainties attending determinations of  $h/e$  by this method.

Determinations,  $2b$ , of  $h/e$  by ionization and excitation potentials<sup>4</sup> are sufficiently doubtful to be placed on the border line of the excluded class.

TABLE I

|   |                                       |
|---|---------------------------------------|
| 1. Direct determinations of $e$ either by (a) the ruled-grating and crystal x-ray method <sup>1</sup> or by (b) the oil-drop method. <sup>2</sup>   | $A_1 = e$                             |
| 2. (a) Measurements of the continuous x-ray spectrum limit. <sup>3</sup> (b) Measurements of ionization and excitation potentials of atoms. <sup>4</sup> (c) Photoelectric effect. <sup>5</sup> (d) Radiation constant $c_2$ . <sup>6</sup> | $A_2 = h/e$                           |
| 3. Electron diffraction measurements involving the voltage of the electrons and their wavelength. <sup>7</sup>  | $A_3 = h/(em)^{\frac{1}{2}}$          |
| 4. (a) Electron diffraction in which the speed of the electrons is measured. <sup>8</sup> (b) Compton shift measurements. <sup>9</sup>  | $A_4 = h/m$<br>$A_5 = e/m$            |
| 5. Specific charge of the electron (spectroscopic, deflection or otherwise). <sup>10</sup>  | $A_6 = e^2/(mh)$                      |
| 6. X-ray photoelectrons ejected with known quantum energies $h\nu$ and measured by magnetic deflection. <sup>11</sup>   | $A_7 = me^4/h^3 = R_\infty(c/2\pi^2)$ |
| 7. Determinations of the Rydberg wave number equated to Bohr's formula. <sup>12</sup>   | $A_8 = e/h^{\frac{1}{2}}$             |
| 8. Determinations of Stefan-Boltzmann radiation constant $\sigma$ . <sup>13</sup>   | $A_9 = e^2/h = (c/2\pi)\alpha$        |
| 9. Determinations of the fine structure constant <sup>13a</sup> $\alpha$ .  |                                       |

<sup>1</sup> The absolute scale of wave-length of x-ray lines having been determined relative to the Siegbahn scale ( $\lambda_0/\lambda_s = 1.00203$  according to Bearden's results) by means of ruled grating diffraction, the absolute grating constant of a crystal such as calcite can be calculated from Bragg or Laue diffraction data. This gives the absolute volume of the unit cell of the crystal lattice and from this and the measured crystal density the absolute atomic weight, Avogadro's number and the electronic charge are computed (the latter by means of the Faraday). For work on  $\lambda_0/\lambda_s$  see J. H. Bearden, Phys. Rev. **37**, 1210 (1931); **47**, 883 (1935); **48**, 385 (1935); E. Bäcklin, Zeits. f. Physik **93**, 450 (1935); M. Södermann, Nature **135**, 67 (1935). Certain earlier criticisms of this method of determining  $e$  have been eliminated by work described in the following papers: J. W. M. DuMond and V. L. Bollman, Phys. Rev. **50**, 524 (1936); **54**, 1005 (1938). Regarding the auxiliary constants such as the velocity of light, etc., a thorough examination by the author, here omitted for brevity, shows that such revisions turn out to be far too small to bear materially on the difficulty under discussion. Throughout the present paper I treat the grating wave-length of x-rays as beyond question. In connection with this important experiment even the electrochemically determined value of the Faraday (G. W. Vinal, Cong. Inter. d'Electr. Paris (1932), Vol. 3, Section 2, p. 117; also Nat. Bur. Stand. J. Research **8**, 735 (1932)) has been questioned at times. To accuracy which for our present dilemma is quite sufficient we have, however, a hitherto unnoticed check on the Faraday independent of electrochemistry in the work on  $e/m$ ; for in the six spectroscopic determinations involving H<sup>1</sup> and H<sup>2</sup> lines (or H<sup>1</sup> and He<sup>+</sup> lines) the electrochemical Faraday is used to compute  $e/m$  while by the six other methods it is not used. The agreement as to  $e/m$  between these two classes gives confidence that the Faraday is not seriously in error. I owe the value  $e = 4.8029 \pm 0.0005 \times 10^{-10}$  e.s.u. to R. T. Birge (Nature **137**, 187 (1936)) who used Molecular wt. calcite 100.085, Faraday 9648.9,  $c = 2.99776$ , calcite density 2.71030,  $\varphi(\beta) = 1.09594$  for the volume factor.

<sup>2</sup> H. R. Robinson, Nature **142**, 159 (1938) has reviewed the situation concerning the oil-drop experiment and the related measurements of air viscosity.

<sup>2a</sup> After the manuscript of this paper had gone to the editors the author was informed in a letter from H. T. Wensel of the National Bureau of Standards, Pyrometry Section, that optical pyrometer measurements yield a value of  $c_2$  which when combined with  $c$ , (light velocity)  $F$ , (the Faraday) and  $R$ , (the gas constant) lead to a value of  $h/e$  whose reliability is indeed amply sufficient to warrant including it in our considerations. This result is  $h/e = (1.3772 \pm 0.0006) \times 10^{-17}$  which is in striking agreement with the "low" values of  $h/e$  obtained by the continuous x-ray spectrum limit method. The reader can readily plot this important result for himself on the diagram of Fig. 3 by means of the scale which reads directly

in terms of  $h/e$ . This seems to render an experimental error in the continuous x-ray values of  $h/e$  a somewhat less likely explanation of the discrepancy than it appeared to the author when the body of this paper was written. See H. T. Wensel, Nat. Bur. Stand. J. Research **22**, 387 (1939), especially Eq. (14).

<sup>3</sup> Duane, Palmer and Yeh, J. Opt. Soc. Am. **5**, 213 (1921); H. Feder, Ann. d. Physik **51**, 497 (1929); P. Kirkpatrick and P. A. Ross, Phys. Rev. **45**, 454 (1934); G. Schaitberger, Ann. d. Physik **24**, 84 (1935); J. W. M. DuMond and V. L. Bollman, Phys. Rev. **51**, 400 (1937); E. Brunner, Phys. Rev. **53**, 457 (1938) gives an explanation of the knees in the isochromats.

<sup>4</sup> E. O. Lawrence, Phys. Rev. **28**, 947 (1926); L. C. Van Atta, Phys. Rev. **38**, 876 (1931); **39**, 1012 (1932); J. E. Roberts and R. Whiddington, Phil. Mag. **12**, 962 (1931); R. Whiddington and E. G. Woodroffe, Phil. Mag. **20**, 1109 (1935).

<sup>5</sup> P. Lukirsky and S. Prilezaev, Zeits. f. Physik **49**, 238 (1928); A. R. Olpin, Phys. Rev. **36**, 251 (1930). For the difficulties caused by the energy distribution of the photoelectrons see M. M. Mann and L. A. DuBridge, Phys. Rev. **51**, 120 (1937); W. V. Houston, Phys. Rev. **52**, 1047 (1937); C. F. J. Overhage, Phys. Rev. **52**, 1040 (1937). While it is Houston's opinion that the difficulties can eventually be removed, the work up to the present is, in the author's opinion, made quite unreliable by them.

<sup>6</sup> For an account of this constant and its experimental value see R. T. Birge, Phys. Rev. Supp. (Rev. Mod. Phys.) **1**, 54 (1929).

<sup>7</sup> S. von Friesen, Proc. Roy. Soc. **A160**, 424 (1937), also Inaugural Dissertation, Uppsala, 1936. This author combines his results with the Bohr-Rydberg formula and a value of  $e/m$  so as to compute separate values of  $e$  and  $h$ . On the present Birge-Bond diagrams, however, his results are reduced to a value for  $h/(em)^{\frac{1}{2}}$ , the function really measured in his experiments independent of any assumptions as to  $R$  and  $e/m$ .

<sup>8</sup> J. Gnan, Ann. d. Physik (5) **20**, 361 (1934); R. v. Meibom and E. Rupp, Ann. d. Physik (5) **13**, 725 (1932).

<sup>9</sup> P. A. Ross and P. Kirkpatrick, Phys. Rev. **45**, 223 (1934). These authors showed that when in the Compton shifted scattering process, *bound* electrons are ejected from atoms the momentum imparted by the photon is not given to the electron alone but is slightly shared with the remainder of the atom. A slight decrease in shift results which these authors showed should and does approximately diminish with the square of the wave-length. F. Bloch, Phys. Rev. **46**, 674 (1934) has treated the theory with certain simplifying assumptions as have also G. Burkhardt, Ann. d. Physik **266**, 567 (1936); W. Franz, Zeits. f. Physik **90**, 623 (1934); **95**, 652 (1935). With solid scatterers the exact applicability of the theory is somewhat questionable. A precision determination of the shift at this laboratory with gaseous scatterers is under way. Shift measure-

These exclusions leave us with the first seven of the eight types listed and of these, number seven, the Rydberg wave number,<sup>12</sup> is by far the most precise and consistent determination. Next to it in precision and consistency and in decreasing order as to these qualities come numbers 1a, 5 and 2a. Now since there are only the three unknowns  $e$ ,  $m$  and  $h$  the seven equations yield considerable over-determination and a general statement of the difficulty with which this paper deals is made when we say that on substituting the numerical results of experimental measurements the seven equations fail to be mutually consistent by amounts uncomfortably exceeding the best estimates of their precision.

#### Representation of the discrepancy

More specifically if we take the three cardinally good measurements 1(a) The value of  $e$  directly determined by x-ray methods. 2(a) The value of  $h/e$  determined by the inverse photoelectric effect (limit of the continuous x-ray spectrum) and (5) Specific charge measurements, and substitute the resulting values of  $e$ ,  $h/e$  and  $e/m$  into

ments of  $h/m$  are difficult and uncertain not only because of the above correction but also by reason of the relatively great breadth of the shifted line (which, however, can also be diminished by using shorter primary wave-lengths) and because of the distorting effect on the shape of the shifted line of certain relativity corrections (J. W. M. DuMond and H. Kirkpatrick, Phys. Rev. **52**, 433 (1937)) and of multiple scattering. The author believes he has selected a probable error for (4 b) in keeping with these uncertainties.

<sup>10</sup> W. V. Houston, Phys. Rev. **30**, 608 (1927); L. E. Kinsler and W. V. Houston, Phys. Rev. **45**, 104 (1934); **46**, 533 (1934); C. D. Shane and F. H. Spedding, Phys. Rev. **47**, 33 (1935); R. C. Gibbs and R. C. Williams, Phys. Rev. **48**, 971 (1935); **45**, 475 (1934); **45**, 221 (1934); **44**, 1029 (1933); R. C. Williams, Phys. Rev. **54**, 568 (1938); J. A. Bearden, Phys. Rev. **54**, 698 (1938); **55**, 584 (1939); W. V. Houston, Phys. Rev. **51**, 446 (1937); W. V. Houston and C. F. Robinson (as yet unpublished); D. Y. Chu, Phys. Rev. **55**, 175 (1939); A. E. Shaw, Phys. Rev. **54**, 193 (1938); F. G. Dunnington, Phys. Rev. **52**, 498 (1937); F. Kirchner, Ann. d. Physik **8**, 975 (1931); **12**, 503 (1932); R. T. Birge, Phys. Rev. **54**, 972 (1938).

<sup>11</sup> Robinson, Andrews and Irons, Proc. Roy. Soc. **A143**, 48 (1933). H. R. Robinson, Phys. Soc. Proc. **46**, 693 (1934). H. R. Robinson and Clews, Proc. Roy. Soc. **A149**, 587 (1935). H. R. Robinson, Phil. Mag. **22**, 1129 (1936) (a summary); G. G. Kretschmar, Phys. Rev. **43**, 417 (1933). Kretschmar's results have not been used here because of uncertainties regarding the interpretation of his observations.

<sup>12</sup> R. T. Birge, Phys. Rev., Supp. (Rev. Mod. Phys.) **1**, 60-61 (1929).

<sup>13</sup> R. Ladenburg, *Handbuch der Physik* (2) **23**, 20 (1933); C. Müller, Zeits. f. Physik **82**, 1 (1933).

<sup>13a</sup> A method due to R. T. Birge may have promise of yielding more information on  $\alpha$  though it seems subject at present to some uncertainty on the side of theory.

(7), the Bohr formula <sup>14a</sup> for the Rydberg wave-number  $R_{\infty} = 2\pi^2 e^2 / (h/e)^3 (e/m) c^2$  we obtain a value differing from the measured value of  $R_{\infty}$  by something from one-half to three-quarters of a percent.<sup>14</sup> The intolerable nature of the situation makes it of great interest to try to frame an intelligent guess as to where the trouble lies, be it either of a theoretical or experimental nature. The well-known Birge-Bond diagram<sup>15</sup> has two disadvantages for this purpose: (1) The data are treated in an arbitrary asymmetrical way (certain of the determinations are mixed with Eq. (7), while others are not) so that it becomes difficult to foresee (without replotting) the result of certain changes either from theory or experiment. (2) Relative variations of the same magnitude appear to very different scales on different ordinates.

The special function of the new chart here described<sup>15a</sup> is to indicate the *consistency* of the determinations and of the nine classes listed above we shall plot only the first seven. It has the advantages (a) that each of the seven types of determination appears to about the same scale of relative variation and (b) that a variation in some particular experiment affects only the one line on the chart corresponding thereto.

<sup>14</sup> To the author's knowledge R. T. Birge was the first to point out this discrepancy. R. T. Birge, Phys. Rev. **48**, 918 (1935); Nature **137**, 187 (1936); J. W. M. DuMond and V. L. Bollman, Phys. Rev. **51**, 400 (1937); R. T. Birge, Phys. Rev. **52**, 241 (1937); R. Ladenburg, Ann. d. Physik (5) **28**, 458 (1937); J. W. M. DuMond, Phys. Rev. **52**, 1251 (1937); R. A. Millikan, Ann. d. Physik (5) **32**, 34 (1938).

<sup>14a</sup> Throughout this formula  $e$  is in e.s.u. save in the parenthesis containing  $e/m$ . There it is in e.m.u.

<sup>15</sup> R. T. Birge, Phys. Rev. **40**, 228 (1932). The use of such diagrams was suggested by Bond and extended, improved and widely applied by Birge following a suggestion of R. Brode. The facility with which a complex situation can be appreciated at a glance has led to the facetious name for the diagram "A Birge-Eye-View of the Atomic Constants."

<sup>15a</sup> After the author had developed the three-dimensional representation and its isometric projection and had sent a copy to R. T. Birge, the latter sent him a copy of a chart independently designed by R. A. Beth which bears some resemblance to the one here described. Beth's chart conceived about December, 1937 is essentially a view of the  $R_{\infty}$  plane on which may be plotted lines, the intersections with that plane of the other planes, representing as many other types of determination as are desired, including of course the  $e$ ,  $h$  and  $m$  planes themselves. The advantages of using the isometric projection based on the cozonal property of five types of determination so as to exhibit their consistency independent of the Bohr-Rydberg relation were not recognized by Beth but he was certainly the first to try to avoid the asymmetrical treatment of the data that is an objection in the Birge-Bond diagram.

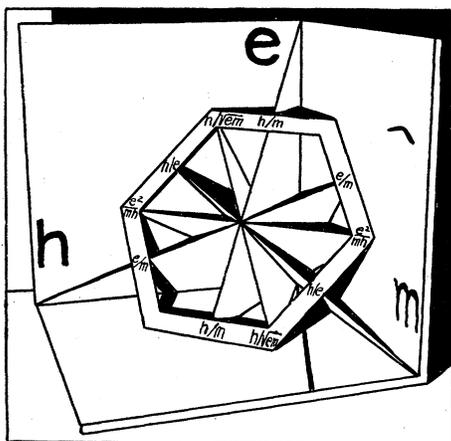


FIG. 1. Perspective view of the three-dimensional space with Cartesian coordinates  $x_e, x_m, x_h$  where  $x_e = (e - e_0)/e_0$ ;  $e_0$  is a conventional value of  $e$  near to the true one, with similar definitions for  $m$  and  $h$ . The coordinates, to appropriate scale, may also be regarded as  $e, m$  and  $h$  directly; the origin is the point  $e_0, m_0, h_0$ . An experimental determination of any function of  $e, m, h$  such as  $h/(em)^2 = A_3$  is represented as a surface in this space which in the region of interest near  $e_0, m_0, h_0$  can with ample accuracy be regarded as a plane whose orientation is fixed by the function determined. A change in the numerical value  $A_i$  determined shifts the plane parallel to itself. Of the seven most reliable types of determination five turn out to be represented by planes which are all parallel to a common axis or normal to a common plane (the plane of the hexagon in the figure). These are here shown, to avoid confusion, as though they passed through a common axis as well as being parallel to it. This cozonal axis is equally inclined to the axes of  $e, m$  and  $h$  and in isometric projection appears as a point while the planes appear as lines in this projection. Obviously this cozonal property of the five types of determination is an absolute one invariant to any change of variable. The meaning of the heavy oblique line in the foreground is made clear in Fig. 2.

Define the relative variation  $x_e = (e - e_0)/e_0$ , in which  $e_0$  is an arbitrarily chosen value near to the true one, with similar definitions for  $x_m$  and  $x_h$ . The measured constants  $A_i$  similarly are to be discussed in terms of numerics  $a_i = (A_i - A_{i0})/A_{i0}$  in which  $A_{i0}$  is consistent with  $e_0, m_0, h_0$ . In the three-dimensional cartesian system of coordinates,  $x_e, x_m, x_h$ , each of the equations (1) to (7) is a surface which with ample accuracy can be replaced near  $x_e = x_m = x_h = 0$  by its tangent plane found by appropriate Taylor's expansion. The orientation of each such plane depends on the particular function of  $e, m$  and  $h$  involved in each type of experiment while a change in the experimental value  $A_i$  shifts the plane parallel to itself. Three or more planes agree to a set of values  $x_e, x_m, x_h$  when they intersect in a common point having these coordinates.

In the new variables the equations are (1)  $x_e = a_1$ ; (2)  $x_h - x_e = a_2$ ; (3)  $x_h - \frac{1}{2}x_e - \frac{1}{2}x_m = a_3$ ; (4)  $x_h - x_m = a_4$ ; (5)  $x_e - x_m = a_5$ ; (6)  $2x_e - x_m - x_h = a_6$ ; (7)  $x_m + 4x_e - 3x_h = a_7$  and it is easy to see by making an isometric projection that Eqs. (2) to (6) inclusive all stand for planes which are parallel to a common axis. The term "cozonal" borrowed from crystal structure conveniently describes this absolute property. The axis in question is equally inclined to the three positive axes  $x_e, x_m, x_h$  and in the isometric projection would be seen on end as a point while the five planes are seen on edge projected as lines.

Figure 1 is a perspective view of a three-dimensional model showing the orientations of the five cozonal planes which here for simplicity are not only parallel to but also intersect in a common cozonal axis. The white hexagon lies in the plane of the isometric projection normal

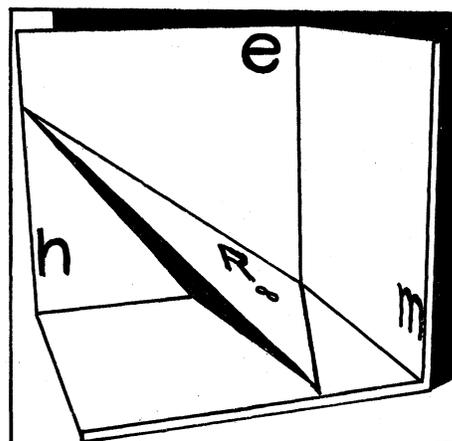


FIG. 2. The same space of  $x_e, x_m$  and  $x_h$  (or  $e, m, h$ ) as in Fig. 1 is here shown to illustrate the orientation of the two exceptional planes representing the direct determination of  $e$  (base plane) and the determination of the Rydberg fundamental frequency equated to Bohr's theoretical formula (inclined plane marked  $R_\infty$ ). These planes intersect in an oblique line (which is also shown in Fig. 1) and it is evident that parallel displacements of either the  $R$  plane or the  $e$  plane will impose parallel displacements on this line of intersection. When this line or its isometric projection (on the plane of the hexagon of Fig. 1) coincides with the cozonal axis common to two or more planes of the cozonal set, the necessary and sufficient condition for consistency of the  $R$  and  $e$  determinations with the two or more cozonal determinations is fulfilled. Thus a consistency diagram can be constructed in two dimensions in the plane of the hexagon of Fig. 1 consisting of the projections of the five cozonal planes (seen on edge) and the projection of the line of intersection of the  $e$  plane with the  $R_\infty$  plane. When all these lines intersect in a common point we have complete consistency of the seven types of determination.

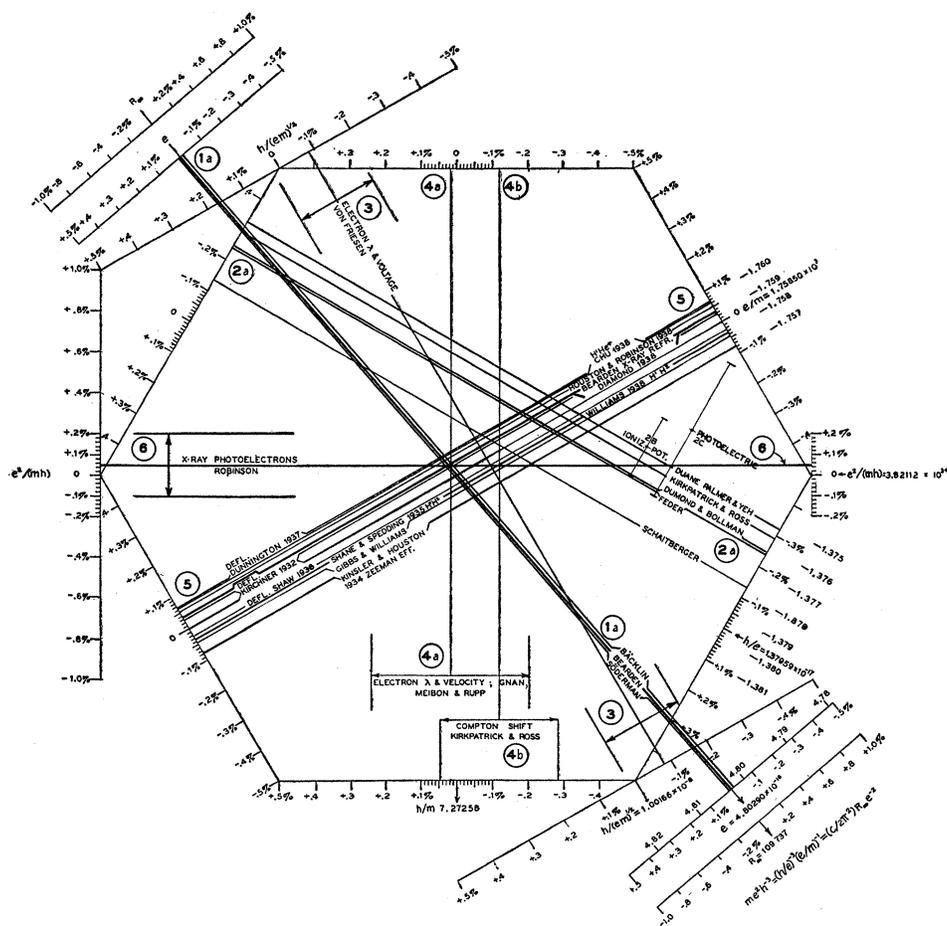


FIG. 3. A consistency graph of the determinations of the atomic constants constructed as explained in the captions to Figs. 1 and 2. The scales for displacement of the lines are expressed in percent deviation from the standard conventional values adopted for construction of the chart, an intersection in the exact center of the chart indicating that the conventional values  $e_0$ ,  $m_0$ ,  $h_0$  are the true ones. In the most interesting cases (for  $e/m$ ,  $h/e$  and  $e$ ) scales reading directly in terms of these quantities are also attached and it is quite obvious that the entire diagram could be constructed to read directly in this way if desired. The conventional or central values are as follows:  $e_0 = 4.80290 \times 10^{-10}$  e.s.u.;  $m_0 = 9.11096 \times 10^{-28}$  gram;  $h_0 = 6.62602 \times 10^{-27}$  erg sec. and in consequence  $e_0/m_0 = 1.75850 \times 10^7$  e.m.u./gram;  $h_0/e_0 = 1.37959 \times 10^{-17}$  e.s.u.;  $h_0/m_0 = 7.27258$ ;  $e_0^2/(m_0 h_0) = 3.82112 \times 10^{21}$ ;  $h_0/(e_0 m_0) = 1.00166 \times 10^{-3}$ ;  $R_\infty = 2\pi^2 e^2 c^{-2} (h/e)^{-3} (e/m) = 109737$   $\text{cm}^{-1}$ ;  $c = 2.99776 \times 10^{10}$  cm/sec. As can be seen a region near the central values of this chart is pretty well indicated by all the experiments within the precision of each except for experiment 2a (the five measurements of the short wave-length limit of the continuous x-ray spectrum). A little care is required in interpreting the  $e$  and the  $R_\infty$  scales. Each of these scales shows the displacement that the projected line of intersection of the  $e$  and  $R$  planes would suffer if the variable of the one scale alone were changed; the other variable is left at its central or conventional value. A change in both variables then calls for the algebraic sum of the displacements indicated on each scale. *Note added to proof:*—F. Kirchner informs the author that the work of R. v. Meibom and E. Rupp is probably completely unreliable. This changes the position of 4a to  $-0.31 \pm 0.41$  percent corresponding to J. Gnan's value  $h/m = 7.25 \pm 0.03$ .

to the cozonal axis. The great convenience of this projection for our purpose is now evident for on a plane we may plot the five types of lines with their appropriate orientations each with the correct displacement from a central standard point to represent the results of each measurement. (See Fig. 3.) Three or more lines passing through a common point indicate consistency of

the corresponding equations or determinations. To represent the exceptional Eqs. (1) and (7) ( $e = A_1$ ,  $R_\infty = (2\pi^2/c)A_7$ ) whose planes are not cozonal with the rest we note that these intersect in a line in three-dimensional space (see Fig. 2) which line is projected on the plane of Fig. 3 (e.g. the line marked Bearden). Scales at the ends of this line show the displacements it would

suffer for changes from the standard values in either  $e$  or  $R_\infty$ . It is evident that the necessary and sufficient condition for consistency of Eqs. (1) and (7) with any set of two or more of the cozonal equations is that this line as seen in the isometric projection of Fig. 3 shall coincide with the mutual intersection of the two or more cozonal planes (seen on edge). Were there more than these two exceptional planes not belonging to the cozonal set the consistency of each with the rest could be indicated similarly by plotting the projection on our isometric plane of the line of intersection of each of them with the  $R_\infty$  plane. Scales for variations of both the function in question and  $R_\infty$  would be attached. In particular this device makes it perfectly easy to attach scales for  $m$  and  $h$  similar to the scale for  $e$  permitting values of these variables corresponding to any intersection point on the chart to be read off directly. Since they are not essential to the purposes of this paper these and many other possible scales have been omitted to avoid overburdening the diagram.<sup>15b</sup>

The standard values adopted in constructing Fig. 1 were  $e_0 = 4.80290 \times 10^{-10}$ ,  $m_0 = 9.11096 \times 10^{-28}$ ,  $h_0 = 6.62602 \times 10^{-27}$ ,  $R_\infty = 109737$ . The order of uncertainty has been indicated in Fig. 3 in some cases but when a large number of determinations is plotted the mutual consistency of the determinations is relied upon as a guide to judgment. The thickness of the lines is adjusted to indicate roughly the weight they should receive.

The failure of intersection at a common point

<sup>15b</sup> The practical mode of construction of scales for any function of, say, the ratios  $e/m$  and  $h/e$  is as follows. Take for example the scales  $e$  and  $R_\infty$ . We have  $(h/e)^{-3}(e/m)^{-1} = (c/2\pi^2)R_\infty e^{-2}$ . Evidently the function  $(h/e)^{-3}(e/m)^{-1}$  calls for a scale laid off in such a direction that the displacement of 0.3 percent along it corresponds to a displacement of -0.1 percent in  $h/e$  ( $e/m$  constant) or to a displacement of -0.3 percent in  $e/m$  ( $h/e$  constant). This fixes both direction and magnitude of the percent scale of  $(h/e)^{-3}(e/m)^{-1}$  which is identical to the percent scale of  $R_\infty$ . Obviously the scale of  $e$  will be parallel to this last but an interval on it of 0.1 percent will be twice as long and in the opposite sense. This procedure can be followed for the construction of a scale for any function whatever (say  $F$ ) of  $e$ ,  $m$  and  $h$ . We can always write  $(h/e)^i(e/m)^j = (c/2\pi^2)R_\infty F^k$  and upon substitution of the Bohr expression for  $R_\infty$  the equation becomes an identity permitting the determination of the powers  $i$ ,  $j$  and  $k$ . In particular this method leads to the following equations for setting up direct reading scales of  $m$  and  $h$ .  $(h/e)^{-3}(e/m) = (c/2\pi^2)R_\infty m^{-2}$ ;  $(h/e)^{-3}(e/m)^{-1} = (c/2\pi^2)R_\infty h^{-2}$ . This is the analytical counterpart of the projective geometrical construction described in the main text of this paper.

of the three cardinal groups of lines shows strikingly the nature of the discrepancy and the three remaining groups of lines seem to give more support to groups 1 and 5 than to group 2. The dispersion of the measurements in group 2 also casts more suspicion upon the reliability of this group for our purpose.

## II. ANALYSIS OF THE DIFFICULTY

### Least-squares methods premature

First let it be emphasized that with systematic error so patently present as this diagram indicates the temptation to obtain compromise or "best" values of the natural constants should be resisted, for it is not inconceivable that the discrepancy may reveal some important error of principle or theory.

Rigorously a least squares determination of  $e$ ,  $h$  and  $m$  for all possible fruitful groupings of the equations of Table I by threes, fours, fives, sixes, and sevens might be made.<sup>16</sup> A comparison of the results when different experiments having some feature in common were omitted from the determination might then throw light on the source of the discrepancy. So much labor, however, hardly seems justified when we recall that it is at present quite beside the point to hope for a numerical solution for the "best" values of  $e$ ,  $h$  and  $m$  by least squares. Our present efforts must be directed rather toward revealing the *source* of the discrepancy, be it theoretical or experimental, and a great deal of detective work can be done without recourse to such exaggerated difficulties as elaborate least-squares solutions.

The method of representing the various determinations of functions of  $e$ ,  $m$  and  $h$  in three-dimensional space here described brings forth very clearly a situation which to the author's knowledge has never before been brought to public notice. Although we have seven equations for

<sup>16</sup> This would of course be far more laborious than necessary but I am informed that two independent workers, Beth and Dunnington, have indeed performed the least-squares solutions with the *two most interesting* exclusions, (a) the Bohr-Rydberg formula and (b) the latter and also  $h/e$  from the continuous x-ray spectrum limit. Such commendable devotion was however not more richly rewarded with conclusions than the author's more homely procedure. The danger in quoting numerical results of least-squares computations made with data known to contain large systematic errors is that the unwary will take them for gospel when in the present case they were computed merely to obtain clues to the location of the discrepancy.

determining only three unknowns, five of these (the cozonal set) are insufficient by themselves to determine  $e$ ,  $m$  or  $h$  but are only sufficient to determine ratios between these quantities. Only with the help of Eqs. (1) or (7) in conjunction with two or more of the five cozonal equations can sets of values for  $e$ ,  $m$  and  $h$  be determined. The exceptional importance of the direct x-ray determinations of  $e$  and of the Bohr expression for the experimentally determined Rydberg wave number is thus clear. *The planes corresponding to these two equations are the only ones that can intersect the cozonal axis in a point.*

Thus if either one of the two last mentioned determinations is dropped from the set of seven under discussion the remaining six *are only just sufficient* to determine  $e$ ,  $m$  and  $h$  and the desirable over-determination implied in the words "least squares" is absent as far as these three variables are concerned.

The author believes that a method of visualizing the rather complicated situation, as to inconsistency of results such as Fig. 3 *in which to the greatest possible extent all reliable original data are separately visible* is much to be preferred to any method in which original data are concealed behind averages or least-squares solutions. *No blind mathematical process of averaging should in his opinion precede an opportunity for the exercise of intelligent judgment.*

#### Examination of the experiments themselves

The first obvious thing to try is a careful examination of each of the experiments to see if some unsuspected source of systematic error can be found. The author is but one of many physicists who have undertaken this over a considerable period of time and the result so far is completely negative. For brevity we must omit most of this careful analysis save for a few brief comments. Line 1a on the diagram is even more certain and satisfactory than it ever was since the values of Bäcklin<sup>1</sup> and Söderman<sup>1</sup> have been revised by the discovery by Haglund<sup>17</sup> of a small

<sup>17</sup> C. F. Robinson, Phys. Soc. Progress Reports, 1937, p. 212. In explaining the necessity of a revision of Bäcklin and Söderman's data Robinson makes a slight but obvious slip quoting an error which is really five parts in one hundred thousand, as one part in five thousand, and an error in  $e$  of 1/7000 as 1/1700. P. Haglund (Zeits. f. Physik 94, 369 (1935)) found an erroneous correction for refractive index previously made by Larsson, in his determination of

error in certain fundamental measurements of Larsson which Bäcklin and Söderman used. The results of the two latter, recomputed, now agree astoundingly with the result of the monumental work of Bearden.<sup>1</sup> Line 5 has been determined in many independent ways, notably lately by Bearden's beautiful x-ray method of refractive index of diamond,<sup>18</sup> and the results are so distributed that there seems little reason any longer to distinguish two values of  $e/m$  (spectroscopic and deflection).<sup>18a</sup>

Line 2a is one on which the author has firsthand knowledge. Possible sources of error were carefully considered in a recent paper<sup>19</sup> and in Fig. 14 of that reference the points  $W_1$  and  $W_2$  indicating where the quantum limit would have to appear in order to reconcile the discrepancy vividly show how impossible with the present data such a reconciliation by this method is. The reader's attention is called to the fact that the sign of the discrepancy is such that if the present experiment is held responsible we would conclude that quanta  $h\nu$  appear in the x-ray tube at electron energies  $e\nu$  too low to excite them—a result much harder to account for by experimental errors than the reverse would be. Since this latter paper was written a very likely theoretical explanation<sup>20</sup> has been given for the knees  $K_1K_2$  (Fig. 14) in the isochromats. It seems possible that the somewhat less reliable (condenser) method of measuring the voltage used by Schaitberger<sup>3</sup> might well account for his higher value of  $h/e$ , departing as it does rather radically from all the others. In the work of DuMond and Bollman the potentiometer method used to measure the voltage *was also the method whereby the voltage was held constant* during the x-ray measurements so that there could be no

AIK $\alpha$  on the conventional Siegbahn scale (A. Larsson, Dissertation, Uppsala, 1929). The accord between Bearden, Bäcklin and Söderman after this revision is the more striking in that the three workers were ignorant of the necessity for the correction when they made their original observations. The greater weight given to Bearden's more numerous observations prevents any change in the final mean of  $\lambda_g/\lambda_s$ . The unfortunate adoption of two conventional wave-length scales other than Siegbahn's, one by Bearden, another by Bäcklin and Söderman, requires vigilance on the part of the student.

<sup>18</sup> J. A. Bearden, Phys. Rev. 48, 698 (1938).

<sup>18a</sup> R. T. Birge, Phys. Rev. 54, 972 (1938), but see J. A. Bearden, Phys. Rev. 55, 584 (1939).

<sup>19</sup> J. W. M. DuMond and V. L. Bollman, Phys. Rev. 51, 416 (1937).

<sup>20</sup> E. Brunner, Phys. Rev. 53, 457 (1938).

doubt that the voltage measured obtained while the x-ray observations were being taken.

Recently the suggestion has arisen in several quarters that the "low" value of  $h/e$  obtained from the continuous x-ray spectrum limit is the result of unsuspected oscillations in the x-ray tube such that the maximum voltage imparted to the electrons exceeds the average voltage measured by the potentiometer method. In the paper by DuMond and Bollman,<sup>19</sup> however, this possibility was carefully considered and rejected for three reasons. (1) An exploration with the cathode-ray oscillograph at lower frequencies and with a very sensitive aperiodic short wave detector for the short and ultra-short wave region failed to reveal any oscillations capable of producing the discrepancy. (2) The experiment as DuMond and Bollman performed it consisted in determining *the voltage at which the short wave-length limit of the continuous x-ray spectrum just coincides with the wave-length of the peak or center of the transmission band defined by the two-crystal x-ray monochromator*. Now if oscillations in the voltage applied to the tube were superposed upon the steady voltage measured by the potentiometer these would not change the *average* wave-length of the short wave limit. If the amplitude of such oscillations were sufficiently small they would merely increase the symmetrical blurring or "smearing" effect of the finite resolving power of the spectrometer. Oscillations of amplitude sufficient to explain the voltage discrepancy observed in our 20,000-volt case (about 50 volts) should distort the shape of our isochromats so that they would exhibit *two* regions where the curve is concave upward, corresponding to the maximum and minimum values of the voltage, one of these being 50 volts above and one 50 volts below the point marked  $W_1$  in Fig. 14 of the reference in question. Such a distortion above  $W_1$  was not observed. (3) The fact that determinations of  $h/e$  at *two different voltages* (10,000 and 20,000) by this method agree in giving the same "low" value of  $h/e$  makes the explanation by oscillations unlikely as the amplitude of such oscillations would have to be proportional to the applied voltage.

Preparations are now well advanced at this laboratory for a repetition of this important experiment over a wider range of voltages with

much better x-ray spectral resolving power (attainable thanks to our possession of a 30-kw x-ray outfit). Any departure from linearity of the relation between  $h\nu$  and  $eV$  will be very carefully looked for in the new work for reasons which will soon be apparent.

In a recent letter to the editor of the *Physical Review* this author<sup>21</sup> has compared graphically the results of computing a value for  $h/e$  from the data of some of the different experiments listed in Table I. A reasonably direct and easy way was adopted to avoid the labor of a least-squares solution. Just as in the construction of the Birge-Bond diagram Eq. (7) (the Bohr formula for the Rydberg constant) is treated as though it were exact and used to eliminate  $m$  so here the two next most accurate determinations 1a and 5 are also regarded as exact and used in the remaining Eqs. 2b, 2c, 2d, 3, 4b, 6 and 7 to compute  $h/e$  for comparison with  $h/e$  as measured in 2a (continuous x-ray limit).<sup>22</sup> The results showed that whereas the continuous x-ray limit experiment gives a value of  $h/e$  falling satisfactorily near the error ranges of the relatively less accurate other determinations of  $h/e$  by 2b, 2c, 2d, 3, 4b and 6 which grouped themselves well both above and below it, the even more accurate value of  $h/e$  computed in the same way from (7) (the Bohr-Rydberg relation) fell rather far from most of them and strikingly far from their mean. It is indeed difficult, however, to find any flow in the *experimental* determination of the fundamental Rydberg wave number and we are thus led naturally to our next topic. In this letter to the editor Birge's original suggestion<sup>23</sup> that the Bohr-Rydberg formula might require revision by the introduction of a factor  $(1-\alpha)=1-1/137$  was resuggested. ( $\alpha$ =fine structure constant)<sup>24</sup>.

<sup>21</sup> J. W. M. DuMond, Phys. Rev. **52**, 1251 (1937).

<sup>22</sup> This common sense procedure which would be open to criticism if the object were a determination of the "best" values of the atomic constants is perfectly justified when a clue to the location of the discrepancy is all that we desire.

<sup>23</sup> R. T. Birge, Nature **137**, 187 (1936).

<sup>24</sup> In the letter (21) written on a transatlantic crossing without reference books the author confusedly used the notation  $\alpha=137$  instead of  $\alpha=1/137$  as it is used here; but without ambiguity as to meaning. Since writing this letter over a year ago the author's judgment has been modified and he now regards the conclusions of the letter less favorably. Fig. 3 of this paper together with theoretical considerations had much to do with this change. A careful study leads him to attach far less weight to the points

### Examination of theoretical assumptions behind the experiments

The next thing to consider is the possibility of an error in the theoretical assumptions at the foundation of the experiments listed in Table I. The difficulty in such a program is to be exhaustive. One never can be certain that he has thought of all the buried assumptions which our lazy minds have learned to label as "obvious." However a list of a few such is here given.

#### List of assumptions

(1) The familiar ruled grating formula  $n\lambda = d(\cos \theta - \cos \phi)$  may not apply in the x-ray region.<sup>25</sup> (2) The Bragg law  $n\lambda = 2d \sin \theta$  may be incorrect for some reason.<sup>26</sup> (3) The photoelectric equation  $e\nu = h\nu = hc/\lambda$  may not apply rigorously at all energies and frequencies.<sup>27</sup> (4) The velocity of propagation of radiation in vacuum may vary slowly with the frequency, a diminution with increasing frequency of about  $\frac{1}{4}$  percent in going from the optical region to x-rays of 20,000 volts being sufficient for our purpose. (5) The Bohr formula for the Rydberg constant may require revision.

#### Examination of the proposed modifications

Regarding assumption 1, the measurements of x-ray wave-lengths in many different orders with many varieties of gratings and with many grat-

ing labeled Radiation constant  $c_2$ , Photoelectric effect, and Ionization potentials than he did in the letter.

<sup>25</sup> This has been often proposed ever since the x-ray value of  $e$  was found to differ from the old oil-drop value. Eckart (Phys. Rev. **44**, 12 (1933)) has given a very elegant proof, based only on the superposition principle for electromagnetic waves, of the validity of the grating formula without corrections for refractive index, etc. The agreement of x-ray wave-length measurements by Bearden by refraction in a diamond prism with the ruled grating values gives further confirmation.

<sup>26</sup> This however has been very carefully tested *as to the functional relationship* between  $\nu$ ,  $\lambda$  and  $\theta$  by Siegbahn and by Allison and Armstrong including of course the very slight correction for refractive index. A. Eddington, *Relativity Theory of Protons and Electrons* (Macmillan, Cambridge, 1936), pp. 306, 307, makes the weird suggestion that the *number* of atoms per  $\text{cm}^3$  in a crystal may not have objective reality! One might obtain a different result, if they were counted *consecutively* from the result of *simultaneous* counting as in the x-ray measurements! His wording is so cautious however that one feels that even this exponent of the superiority of a *priori* reason would not disdain more experimental information from humble "dirt" physicists before committing himself on this point to a full fledged theory.

<sup>27</sup> J. W. M. DuMond, Phys. Rev. **52**, 1251 (1937). Last sentence in letter; also A. Eddington, *Relativity Theory of Protons and Electrons*, p. 304.

ing constants have given no cause to doubt the grating formula. Also the fact that the ruled grating x-ray wave-lengths are strictly proportional to the crystal wave-lengths measured on the conventional Siegbahn scale by means of the Bragg law with a conventional value of  $d$  over the huge range of wave-lengths of nearly ten to one gives some reassurance regarding both assumptions 1 and 2. Furthermore experiments 1a, 2a, 3, 4a and b, and 6 all of which involve assumption 1 are mutually inconsistent (see Fig. 1) so that a revision of this sort does not seem promising. As to the second assumption (the Bragg law) such experiments as 2a, 5 and 6 which do not essentially involve this law are outstanding examples of the discrepancy as reference to Fig. 3 shows. The remaining theoretical assumptions 3, 4 and 5, however, seem on such *a priori* grounds more promising candidates for examination.

From the present theoretical viewpoint, however, changes in assumptions 3, 4 or 5 are about equally unpalatable. If we change 3 we strike directly at the tap root of modern quantum theory which has been elsewhere at almost countless points so uniformly and quantitatively successful.

A change (4) of  $\frac{1}{4}$  percent downward in the velocity of x-rays at 20,000 volts relative to the low frequency Michelson value applicable in optics has the effect of modifying all the points in the diagram in which an observed x-ray wave-length must be converted to frequency by the relation  $\nu = c/\lambda$ . This would involve experiments 2a, 4b and 6 all of which have been done at or near the 20,000-volt region. (Here the desirability of extending the voltage range of these experiments becomes evident.) Experimental results such as 3 or 4a might suffer very slight modification of second-order smallness because the value of  $c$  to be used in their relativity correction formulae might be modified but this seems unlikely to be important. It is easy to show that the  $\frac{1}{4}$  percent change in  $c$  would shift lines 2a on the diagram of Fig. 3 by just the amount to give agreement with 1 and 5 but would also shift line 4b to the right by the same amount while line 6 would be shifted down by  $\frac{1}{4}$  percent. Fig. 4 shows how the diagram would then look.

Such a change in assumption 4, however,



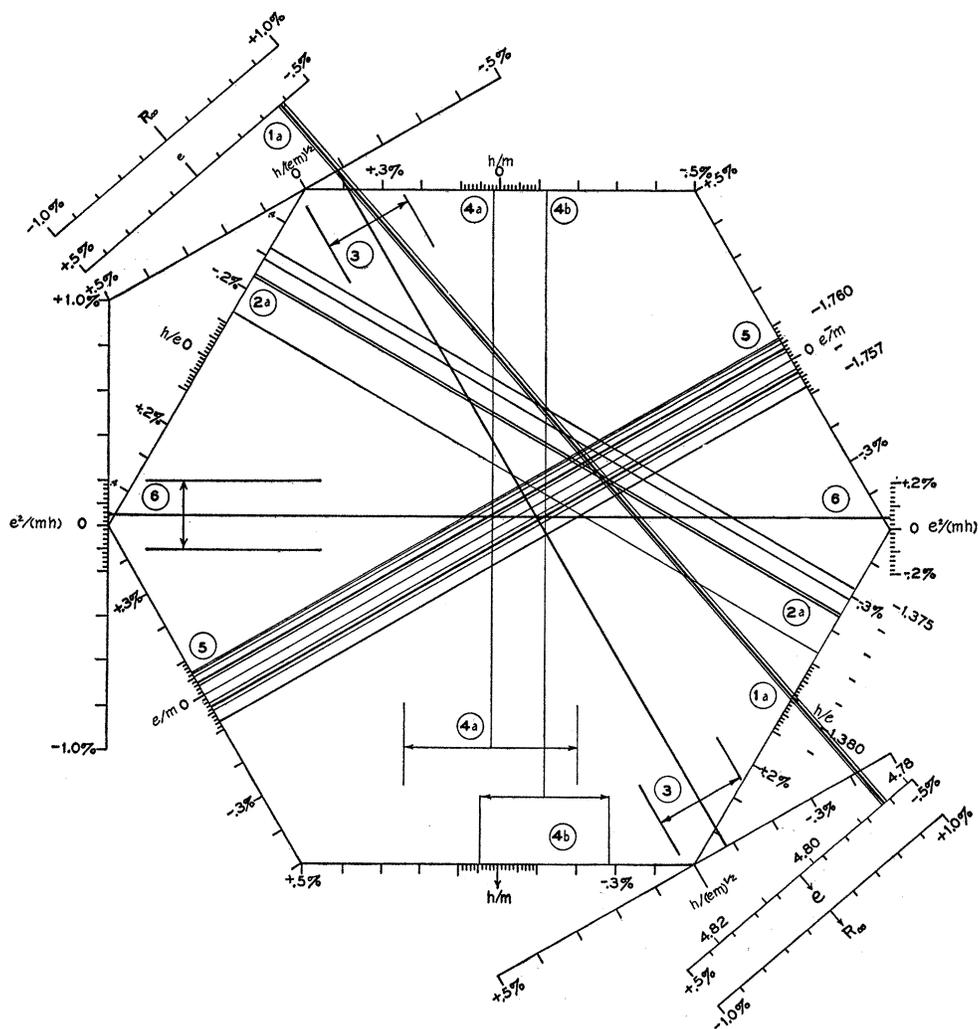


FIG. 5. A consistency graph like Fig. 3 but modified by the assumption that the Bohr formula for the Rydberg wave number should contain an extra factor  $(1-\alpha)$  where  $\alpha = 1/137$  is the fine structure constant of Sommerfeld. This is the same as saying that the familiar Rydberg formula holds but the numerical value of the Rydberg must be increased  $\frac{1}{4}$  percent.

states pervading all otherwise empty space. Unfortunately such an attempt in the hands of the present author to construct a theory of anomalous dispersion with critical frequency at one million volts fails to yield the necessary  $\frac{1}{4}$  percent diminution in  $c$  at 20,000 volts. In fact the best that can be managed seems to be only about 1/12 percent at 20,000 volts instead of  $\frac{1}{4}$  percent.  $v_x/c = [1 - (hv/2mc^2)^2]^{1/2} = 1 - 1/1250$ . The author does not regard the proposed modification of velocity with enthusiasm.

**Unreliability of low voltage determinations of ionization and excitation potentials**

The author attaches little or no importance to the fact that experiment 2b (not plotted in the present diagrams) derived from measurements of critical ionization and excitation potentials (of a few volts only) seems to support the high voltage continuous spectrum x-ray measurements by also giving a "low" value of  $h/e$ . At face value this apparent support would immediately rule out the variation in  $c$  as an ex-

planation of the discrepancy. But the author believes that measurements of ionization and excitation potentials *in the low energy range* are too unreliable to warrant such an argument. It has been shown<sup>29</sup> that in vacuum especially under bombardment, there form on metal surfaces what are probably thin layers of insulating hydrocarbons in such a way that there is never any assurance that a metal surface is an equipotential surface *to within several volts*. Only in the x-ray region do ionization and excitation potential measurements hold any promise.

Figure 4, showing the general effect on our diagram of the proposed modification in the velocity of x-rays, is to be compared with Fig. 5 which exhibits, as an alternative, the effect of introducing the factor  $(1-\alpha)$  (for purely empirical reasons) into the Bohr-Rydberg formula. In Fig. 5 only one set of lines is displaced from the positions of Fig. 3. Probable errors are roughly indicated as before. In this connection it is well to recall the following property of the "error function."

The chance that an error shall occur greater than  $n$  times the "probable error" is given by the fraction in the right-hand column below

|                 |     |       |      |       |        |
|-----------------|-----|-------|------|-------|--------|
| $n$             | 1   | 2     | 3    | 4     | 5      |
| Chance of error | 1/2 | 1/5.6 | 1/23 | 1/143 | 1/1340 |

It is seen from Figs. 4 and 5 that judged by results there is not much to choose between a modification in the velocity of x-rays and a modification in the Bohr-Rydberg formula. It should be noted from Fig. 5 that the rectification of the three cardinal groups of lines,  $1a$ ,  $2a$  and  $5$ , by the empirical introduction of  $(1-\alpha)$  into the Bohr formula is accomplished at the rather heavy expense of throwing  $3$ ,  $4a$  and  $6$  out of coincidence with the newly created intersection of the cardinal groups.

Regarding the theoretical implications of a change in the Bohr-Rydberg formula by the addition of a factor  $(1-\alpha)$  the situation is very discouraging. The author has consulted with many prominent theoretical physicists both in Europe and in the United States on this point and at the present time it seems impossible to modify theory in such a way as to explain such

a factor in the formula (though changes of order  $\alpha^2$  are entirely possible). Kramers<sup>30</sup> has suggested an effect of pair formation in the hydrogen atom which is unfortunately inadequate, we believe, both as to magnitude and sign of the correction. The present author has tried without success the idea that the center of charge of the electron in the hydrogen atom may not coincide with the center of mass. Certain very forced assumptions will indeed permit a correction factor of the required form and sign but these assumptions violate the correspondence principle for large "orbits." A major obstacle to the formulation of any modification in the Bohr formula for  $R$  is the fact that the series formula  $(1/n^2 - 1/m^2)$  is so rigorously obeyed by spectral lines.<sup>21</sup> This obedience makes an escape to the requisite order of magnitude from the Coulomb law of force between nucleus and electron in the hydrogen atom practically impossible.

### Conclusion

On the whole as a final summation of this review the present author feels inclined to conclude that the most likely source of the discrepancy lies in the "low" continuous x-ray spectrum value of  $h/e$  with a fair chance that some unsuspected source of systematic experimental error may still be discovered by careful measurements with high spectral resolving power over a wider range of voltages. The possibility of revealing an important modification of theory in this way is also present. A glance at Fig. 3 shows that a revision of the results of this one experiment would go very far toward clearing up all difficulties.

### Acknowledgments

The author wishes especially to thank Professors W. V. Houston, Ira S. Bowen, R. T. Birge and F. G. Dunnington for much helpful discussion relative to the contents of this paper. He had the privilege of examining a manuscript by Dunnington in the course of the preparation of this paper in its later stages, and the author owes this invention of the new chart to an effort to clarify a point under discussion with Dunnington.

<sup>29</sup> A. E. Shaw, Phys. Rev. **44**, 1009 (1933); **54**, 193 (1938); R. L. Stewart, Phys. Rev. **45**, 488 (1934).

<sup>30</sup> H. A. Kramers, K. Akad. Amsterdam Proc. **40**, 814 (1937).