DECEMBER 15, 1939 PHYSICAL REVIEW VOLUME 56

Wave Reflections from Diffuse Boundaries

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(Received September 18, 1939)

The intensity of the waves reflected from various types of diffuse boundaries in which the dielectric constant changes only slightly is considered. The constant inside the reflection layer is assumed to vary as some power of the distance. It is further assumed that above and below this transition layer the media are uniform. The calculations show that the rate of decrease of intensity of the reflected wave with increasing layer thickness is more rapid for large values of μ the index of refraction, than for values close to unity. Fresnel's equations are therefore applicable to layers of considerable thickness if μ is very close to unity.

IT is customary to consider reflections from a boundary to be of considerable magnitude only if the boundary is very sharp compared to the wave-length of the incident wave. This is the case of importance in optics where the reflected light is usually near the order of magnitude of the incident ray. In the reflection of radio waves, however, the sensitivity of the receiver permits the detection of comparatively feeble waves, and it has been shown that detectable reflections may be produced by non-ionic layers if the boundaries are considered sharp.[1] It does not seem probable, on the other hand, that the layers could have extremely sharp boundaries, relative even to radio wave-lengths. It is therefore desirable to consider whether or not one is justified in using Fresnel's reflection laws here, as has sometimes been done. The important difference between these radio reflections and those of light is that the radio reflections are from regions where the relative refractive index is very close to unity so that the coefficient must be very small even though the boundary is truly sharp.

For the purpose of testing whether or not Fresnel's laws are a good approximation for thicker transition layers as the index approaches unity, a one-dimensional problem will be considered, i.e., plane waves progressing in the direction of variation of the dielectric constant. The region near the origin will be taken as uniform up to some point x_1, above this the dielectric constant will be assumed to vary as some monotonic function of x up to x_2, above this it is to be again uniform.

The wave equation, after removing the time dependent part, becomes

$$d^2X/dx^2 = -\nu^2\epsilon X/c^2, \tag{1}$$

where X is the space dependent part of the electric vector, c is the velocity of light, and ϵ is the dielectric constant which is itself a function of x. The general solution of this equation is

$$X = Af(x) + Bg(x). \tag{2}$$

The boundary conditions which must be satisfied by the solution are that x and its first derivative for the lower region must equal those of the transition layer at its lower boundary, and a similar set relating the upper region to the upper boundary of the layer. Thus there are four conditions for the six constants A, B (A_0, B_0 in the lower space; A_1, B_1 in the layer; A_2, B_2 in the upper space). However, it is only the ratio A_0/B_0 that is required if $f(x)$ and $g(x)$ are made the reflected and incident waves, respectively. Moreover, in the upper space one may assume that the backward component of the wave is zero ($A_2=0$). Hence, there are left only four constants (A_0/B_0, A_1, B_1, B_2), and there are four boundary conditions for determining these. The solutions of (1) in the three regions are now:

$$\begin{aligned} X_0 &= A_0' \exp(ik\epsilon_1^{\frac{1}{2}}x) + B_0' \exp(-ik\epsilon_1^{\frac{1}{2}}x), \\ X_1 &= A_1 f(x) + B_1 g(x), \\ X_2 &= B_2' \exp(-ik\epsilon_2^{\frac{1}{2}}x), \end{aligned} \tag{3}$$

where $k=\nu/c=2\pi$ frequency$/c$. For convenience,

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[1] C. R. Englund, A. B. Crawford and W. W. Mumford, Bell Sys. Tech. J. **14**, 369–387 (1935).

let

$$A_0' \exp\,(ik\epsilon_1^{\frac{1}{2}}x)+B_0' \exp\,(-ik\epsilon_1^{\frac{1}{2}}x) = A_0 \exp\,[ik\epsilon_1^{\frac{1}{2}}(x-x_1)] + B \exp\,[-ik\epsilon_1^{\frac{1}{2}}(x-x)]. \tag{4}$$

The boundary conditions then become

$$\begin{aligned} A_0+B_0&=A_1f(x_1)+B_1g(x_1),\\ ik\epsilon_1^{\frac{1}{2}}(A_0-B_0)&=A_1f'(x_1)+B_1g'(x_1). \end{aligned} \tag{5}$$

And similarly

$$\begin{aligned} B_2&=A_1f(x_2)+B_1g(x_2),\\ -ik\epsilon_2^{\frac{1}{2}}B_2&=A_1f'(x_2)+B_1g'(x_2). \end{aligned} \tag{6}$$

The primes on f and g of (5) and (6) indicate differentiation with respect to x before evaluation at x_1 and x_2.

Upon solving Eqs. (5) and (6) for the reflection coefficient when the thickness is t

$$R_t=\frac{A_0}{B_0}=\frac{[f'(x_1)+ik\epsilon_1^{\frac{1}{2}}f(x_1)][g'(x_2)+ik\epsilon_2^{\frac{1}{2}}g(x_2)]-[f'(x_2)+ik\epsilon_2^{\frac{1}{2}}f(x_2)][g'(x_1)+ik\epsilon_1^{\frac{1}{2}}g(x_1)]}{-[f'(x_1)-ik\epsilon_1^{\frac{1}{2}}f(x_1)][g'(x_2)+ik\epsilon_2^{\frac{1}{2}}g(x_2)]+[f'(x_2)+ik\epsilon_2^{\frac{1}{2}}f(x_2)][g'(x_1)-ik\epsilon_1^{\frac{1}{2}}g(x_1)]}. \tag{7}$$

This is applicable to any sort of transition layer of any thickness and for any value of the relative index $\mu=(\epsilon_2/\epsilon_1)^{\frac{1}{2}}$.

More specifically suppose that Eq. (1) inside the layer is

$$\frac{d^2X}{dx^2}+F(x)X=0, \tag{8}$$

where

$$[F(x)]^{\frac{1}{2}}=\frac{\nu}{c}\epsilon_1^{\frac{1}{2}}\frac{x^n}{x_1^n}=\frac{\nu}{c}\epsilon_2^{\frac{1}{2}}\frac{x^n}{x_2^n}. \tag{9}$$

This is not a very serious restriction on $F(x)$ when it is recalled that the layer is not very thick and that n may have any value whatsoever either positive or negative. Since x lies between x_1 and x_2, it is evident that if $(\epsilon_2/\epsilon_1)^{\frac{1}{2}}$ is greater than unity, n must be positive, while if $(\epsilon_2/\epsilon_1)^{\frac{1}{2}}$ is less than unity, n must be negative. In general from Eq. (9)

$$\mu=(\epsilon_2/\epsilon_1)^{\frac{1}{2}}=(x_2/x_1)^n \tag{10}$$

or using

$$x_2-x_1=t, \quad x_1=t/(\mu^{1/n}-1), \tag{11}$$

where t is the thickness of the transition layer.

In the cases under consideration μ is greater than one by a very small amount so that in Eq. (11), x_1 and x_2 are large except for layers which are extremely thin. Under these circumstances, it is allowable to put

$$f=e^{i\Delta+\delta}, \quad g=e^{-i\Delta+\delta}, \tag{12}$$

where

g

$$\Delta=\frac{k\epsilon_1^{\frac{1}{2}}}{n+1}\frac{x^{n+1}}{(x_1)^n}, \quad \delta=-\frac{n}{2}\log x_1, \tag{13}$$

as may be verified by substitution into Eq. (8) with the aid of (9).

If these values of f and g are put into Eq. (7), the coefficient R_t takes the form

$$R_t=\frac{-in\left\{\exp\left[\dfrac{ik}{n+1}(\epsilon_2^{\frac{1}{2}}x_2-\epsilon_1^{\frac{1}{2}}x_1)\right]-\dfrac{1}{\mu}\dfrac{x_1}{x_2}\exp\left[\dfrac{-ik}{n+1}(\epsilon_2^{\frac{1}{2}}x_2-\epsilon_1^{\frac{1}{2}}x_1)\right]\right\}}{4k\epsilon_1^{\frac{1}{2}}x_1\exp\left[\dfrac{ik}{n+1}(\epsilon_2^{\frac{1}{2}}x_2-\epsilon_1^{\frac{1}{2}}x_1)\right]}. \tag{14}$$

If use is now made of the value of x_1 and x_2 from Eq. (11) along with the restriction that μ is always near unity,

$$R_t^2=\frac{n^2(\mu^{1/n}-1)^2\sin^2\left[k\epsilon_1^{\frac{1}{2}}t\left(\dfrac{\mu+n}{n+1}\right)\right]}{4k^2\epsilon_1t^2}. \tag{15}$$

The value of R_t^2 reaches zero at approximately

$$kt=\frac{\pi}{\epsilon_1^{\frac{1}{2}}}\frac{n+1}{\mu+n}. \tag{16}$$

It may easily be verified from this that the value of the thickness t for the first zero is greater for a value of μ nearer unity, since the factor

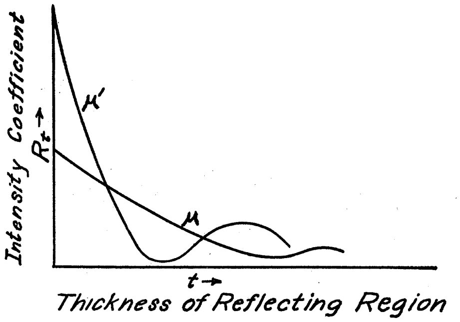

FIG. 1. The variation of the intensity coefficient with layer thickness for two sets of values of ϵ_1 and ϵ_2. The index of refraction $\mu=(\epsilon_2/\epsilon_1)^{\frac{1}{2}}$ is nearer unity than μ'.

$(n+1)/(\mu+n)$ in Eq. (16) increases for smaller values of μ. Also from the value of dR_t^2/dt it may be seen that R_t^2 decreases less rapidly for values of μ nearer unity.

The result on an exaggerated scale is illustrated in Fig. 1. Thus, although the reflection is much greater for truly sharp boundaries if μ differs considerably from one, the rate of decrease is also much greater. Hence, the Fresnel coefficient ($t=0$) becomes a better approximation for a given thickness if the value of μ is closer to one. In order to obtain information as to the exact way in which R_t decreases with t, it is necessary to make some assumptions as to the value of ϵ in the lower and upper regions and regarding the manner in which it changes in the transition layer (i.e., the nature of $F(x)$ of Eq. (8). These calculations will be published later.

The formulas developed here are applicable to the reflection of radio waves from different air masses in the troposphere. It is known from experiment that such reflections take place.[2]

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[2] R. C. Colwell and A. W. Friend, Nature **137**, 782 (1936).

DECEMBER 15, 1939 PHYSICAL REVIEW VOLUME 56

On the Interpretation of Atomic Distribution Curves for Liquids

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(Received July 24, 1939)

It is shown that if we approximate to a liquid by an Einstein model, in which each atom has a restricted region of motion, wherein it moves independently of its neighbors, and is surrounded by coordination shells of other atoms; and if we denote the density distribution of the atoms by $\rho(r)$, where r is the radial distance from any given atom; then the contribution to $\rho(r)$ made by any coordination shell, e.g., the ith is a function $\rho_i(r)$ for which $r\rho_i(r)$ is symmetrical about its corresponding maximum value. The complete distribution curve, $r\rho(r)$ against r, is the sum of peaks of equal width and similar shape. A semi-empirical application of this theory to liquid sodium (a reapplication of C. N. Wall's theory) gives a latent heat of melting in fair agreement with experiment. The model suggests a change of structure on melting, since agreement with the experimental distribution curve is impossible if the number of atoms in the first coordination shell is that of solid sodium. It is also shown that if the parameters in a partition function developed in this way are chosen to give agreement with any one physical property, then it is incorrect to add to the partition function terms representing "communal entropy."

§1. INTRODUCTION

SEVERAL experimental papers have appeared recently publishing atomic distribution curves for liquids.[1] These curves, found from Fourier analyses of the intensity distributions in x-ray scattering photographs, show $\rho(r)$, or $4\pi r^2\rho(r)$, as a function of r. $4\pi r^2\rho(r)dr$ is proportional to the probability that two unspecified atoms of the liquid are distant r to $r+dr$ apart.

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[1] (a) F. H. Trimble and N. S. Gingrich, Phys. Rev. **53**, 278 (1938); (b) C. D. Thomas and N. S. Gingrich, J. Chem. Phys. **6**, 411 (1938); (c) C. D. Thomas and N. S. Gingrich, J. Chem. Phys. **6**, 659 (1938); (d) J. Morgan and B. E. Warren, J. Chem. Phys. **6**, 666 (1938).