A Field Theory of Elementary Particles

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Previous attempts at constructing a classical field theory of the electron are reviewed and it is shown that hitherto it has not been possible to combine the two conditions: gauge-invariance and freedom from singularities. It is pointed out that this can be done by the introduction of new matter functions similar to the wave functions of quantum theory. A simple possibility of this type is considered and is found to lead to equations admitting electron-like solutions. The electron turns out to have a negative mass (but perhaps this difficulty will disappear when the equations are quantized). The theory, although classical, offers in principle a possibility of accounting for the Sommerfeld fine-structure constant. There are also solutions corresponding to excited states of the electron and to heavier particles, including particles with zero charge.

AS IS well known, the Maxwell equations of classical electrodynamics lead to the description of an elementary charged particle as a singular point in the electromagnetic field. Such a representation ("point-electron") is, of course, unsatisfactory (1) because it makes the particle have infinite self-energy, and (2) because the presence of the singularity means that the field equations are not valid at the point in question, so that it is necessary to have equations of motion for the charge in addition to the equations for the field.

In order to overcome these difficulties, Mie[1] proposed a modification of the Maxwell equations which would admit a solution for an electron as a charge, free from singularities, occupying a finite volume. However, there were serious objections to this theory and it was abandoned for a time.

When quantum electrodynamics was developed, it was found that the equations of the new theory, obtained by quantizing the classical equations, failed to get rid of the difficulties of the older theory in connection with the point-electron. It was therefore natural to conclude that one ought to try to remove these difficulties from the classical theory before going over to the quantum theory. This led Born and Infeld[2] to propose a nonlinear system of equations of electrodynamics to replace the Maxwell equations The new equations have a solution for the electron with the field everywhere finite and with finite self-energy. However the field in this case still contains a singularity and hence it is still necessary to supplement the equations for the field with equations of motion for the electron.[3] Efforts at modifying this theory have been only partially successful.

One must therefore conclude that the problem of the existence of the electron in the classical theory has not yet been satisfactorily solved. In the present paper is described another attempt at a solution to this problem.

I. THE EQUATIONS OF MAXWELL, MIE, AND BORN AND INFELD

The Maxwell equations can be written in tensor notation

$$F^{\mu\nu};_{\nu}=0, \tag{1}$$

where the electromagnetic field tensor $F^{\mu\nu}$ is derived from a potential vector φ_μ:

$$F_{\mu\nu}=(\partial\varphi_\nu/\partial x_\mu)-(\partial\varphi_\mu/\partial x_\nu) \tag{2}$$

and (;) denotes covariant differentiation.

Equations (1) can be obtained from a variational principle. If we define the Lagrangian

$$L_f=-F_{\mu\nu}F^{\mu\nu}/8\pi, \tag{3}$$

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[1] G. Mie, Ann. d. Physik **37**, 511 (1912); **39**, 1 (1912); **40**, 1 (1913).

[2] Born and Infeld, Proc. Roy. Soc. **A144**, 425 (1934); **147**, 522 (1934); **150**, 141 (1935).

[3] J. Frenkel, Proc. Roy. Soc. Lond. **A146**, 933 (1934); E. Feenberg, Phys. Rev. **47**, 148 (1935).

then the condition

$$\delta\int L_f(-g)^{\frac{1}{2}}d\tau=0 \quad (d\tau=dx_1dx_2dx_3dx_4) \quad (4)$$

leads to the Maxwell equations, provided the components of the potential are given infinitesimal variations, arbitrary except for the restriction that they vanish on the boundary of the (four-dimensional) region of integration.

A characteristic of the Eqs. (1) and the Lagrangian (3) is that they depend on the potential φ_μ only through the field tensor $F_{\mu\nu}$ and not directly. This has the well-known consequence that a gauge-transformation, whereby the potential φ_μ is replaced by

$$\varphi_\mu'=\varphi_\mu+\partial\lambda/\partial x_\mu \quad (5)$$

with λ an arbitrary function of the coordinates, does not change the function L_f or the form of Eqs. (1), since the field tensor remains unchanged. That is to say, the theory is gauge-invariant.

For deriving other systems of equations in place of those of Maxwell, the variational principle provides a convenient starting point. Thus, one can choose a scalar function of the field variables, L, and by taking as the law for the field

$$\delta\int L(-g)^{\frac{1}{2}}d\tau=0, \quad (6)$$

where the potentials are varied as in the case of (4), one obtains a set of equations. If the equations are written in the form

$$F^{\mu\nu};_\nu=4\pi s^\mu, \quad (7)$$

the vector s^μ, depending on the field variables, can be considered as the charge-current density vector, in the sense of the Lorentz electron theory. If one can find a static spherically-symmetric solution of Eqs. (7), free from singularities, such that the charge is confined to a small volume, the solution might be considered as representing an electron at rest in the absence of an external field.

The question arises as to what form one is to choose for the function L. To agree with experiment the function L in the case of weak fields must go over into the Maxwell function L_f. But this in itself is, of course, far from sufficient to determine the function.

If L is allowed to depend explicitly on the potential φ_μ in addition to its dependence on $F_{\mu\nu}$, then in general L and hence the field equations must change under a gauge-transformation (5). However, it is believed that only the field $F_{\mu\nu}$ has physical significance and not the potential, so that the transformation (5), which does not change the field, should not bring about any essential changes in the predictions of the theory.

In Mie's theory of the electron L depends directly on φ_μ, and the resulting lack of gauge-invariance is one of the chief defects of the theory. A solution of Mie's equations for the electron might be expected to change if the external potential were changed by a constant, although from the physical point of view such a change of potential appears to have no significance. A further difficulty in Mie's theory is the lack of a criterion for the choice of the Lagrangian L.

To satisfy the requirement of gauge-invariance, Born and Infeld assume from the very beginning that L depends only on the field tensor and not explicitly on the potential. However, this too has its difficulties, as can be seen from the following discussion.

Let us assume that L is a function only of $F_{\mu\nu}$. Then Eq. (4) can be written in the form

$$\int D^{\mu\nu}\delta F_{\mu\nu}(-g)^{\frac{1}{2}}d\tau=0, \quad (4a)$$

where $D^{\mu\nu}$ is an antisymmetric tensor defined by

$$D^{\mu\nu}=-4\pi\frac{\partial L}{\partial F_{\mu\nu}}. \quad (8)$$

It follows that

$$\int D^{\mu\nu}\delta F_{\mu\nu}(-g)^{\frac{1}{2}}d\tau=2\int D^{\mu\nu}\delta\frac{\partial\varphi_\mu}{\partial x_\nu}(-g)^{\frac{1}{2}}d\tau,$$

$$=-2\int D^{\mu\nu};_\nu\delta\varphi_\mu(-g)^{\frac{1}{2}}d\tau, \quad (9)$$

after integrating by parts and discarding the (vanishing) surface integral. For the integral (9) to vanish with $\delta\varphi_\mu$ arbitrary, one must have

$$D^{\mu\nu};_\nu=0, \quad (10)$$

which represents the set of field equations and is therefore equivalent to (7).

In view of the antisymmetry of $D^{\mu\nu}$, one can rewrite (10) in the form

$$\frac{\partial}{\partial x_\nu}[D^{\mu\nu}(-g)^{\frac{1}{2}}]=0, \tag{11}$$

that is, as the vanishing of an ordinary divergence. In the case of a static spherically-symmetric solution, expressed in polar coordinates, this reduces to the single relationship:

$$d(r^2D_r)/dr=0, \tag{12}$$

where D_r is the radial component of the three-dimensional vector $\mathbf{D}=(D^{41}, D^{42}, D^{43})$. The solution of (12) is

$$D_r=e/r^2, \tag{13}$$

where e is a constant of integration. If L has been chosen so that for weak fields the equations go over into those of Maxwell, then by comparing (8) and (3) one sees that for large values of r (13) goes over into

$$E_r=e/r^2,$$

where E_r is the radial component of the three-dimensional electric field vector $\mathbf{E}=(F^{41}, F^{42}, F^{43})$. From this it is clear that e in (13) is the charge of the electron and therefore cannot be put equal to zero. Consequently D_r is infinite for $r=0$. If D_r is a well-behaved function of E_r, as defined by Eq. (8), this makes E_r infinite for $r=0$ and one has a singular point.[4]

To avoid this difficulty one can try choosing L such that D_r as a function of E_r becomes infinite for some finite value E_0 of the latter. In this case, at the origin $r=0$ where D_r becomes infinite, E_r takes on this value E_0, the limiting value of the field, and thus remains finite. This is the situation in the theory of Born and Infeld. At first sight it appears that the singularity has been avoided, but a closer inspection shows that the vector $\mathbf{E}$ has a discontinuity in direction at the origin, which is therefore still a singular point. In consequence the field equations do not completely determine the motion of the electron and additional conditions have to be added.

[4] For the preceding argument I am indebted to Professor A. Einstein.

Hoffmann and Infeld[5] discuss this difficulty and proceed to remedy it. In order to avoid the discontinuity in $\mathbf{E}$ at the origin it is necessary to make $E_r=0$ there; that is, one must choose a Lagrangian such that $E_r=0$ for $D_r=\infty$. But this also leads to a difficulty, for at large distances from the origin both E_r and D_r must tend to zero, and thus we have the situation that D_r as a function of E_r must be double-valued: for $E_r=0$, $D_r=0$ and $D_r=\infty$. This means that the Lagrangian L must be a double-valued function of the field $F_{\mu\nu}$. That is essentially the case with the function proposed by Hoffmann and Infeld. One can give it the appearance of single-valuedness by writing it as a function of both $F_{\mu\nu}$ and $D_{\mu\nu}$, but the tensor components $D_{\mu\nu}$ cannot be considered as independent physical variables since, as we have seen from (13), they have singularities. In general, a theory with a double-valued action function does not seem satisfactory; in particular, the derivatives of the latter with respect to the field will have a singularity on the surface where the two values join. Incidentally, Hoffmann and Infeld obtain for E_r an *even function* of r in the static spherically-symmetric case, and this might also be interpreted to mean that there is a singularity at $r=0$ (in the higher derivatives).

Thus one must conclude that choosing L to depend only on $F_{\mu\nu}$ as has been done in the more recent papers, is unsatisfactory.

II. Proposed Lagrangian

From the preceding it appears that, in a classical field theory of the type considered here, on the one hand it is necessary for the Lagrangian function to depend explicitly on the potentials to avoid singularities, and on the other hand the theory ought to be gauge-invariant. The question arises as to how one can reconcile these two requirements with each other. One way of doing this that suggests itself is to choose L an explicit function of the potentials, but to introduce, in addition to the potentials, new variables which will make L gauge-invariant.

A familiar example of this situation is the Lagrangian function from which the Schrödinger wave equation is derived. In this case, the

[5] Hoffmann and Infeld, Phys. Rev. **51**, 765 (1937).

Lagrangian depends on the potential and on the wave function. Under a gauge transformation both the potential and the wave function transform in such a way as to leave the Lagrangian invariant.

In accordance with this point of view we take

$$L=L_f+L_m, \tag{14}$$

where L_f is given by Eq. (3) and describes the electromagnetic field, while L_m is a function of the potentials and of the new variables (maintaining gauge-invariance) and can be considered as describing matter.

The simplest nontrivial possibility is to introduce as additional variable (in analogy to the example mentioned) a scalar function ψ such that when φ_μ is transformed according to (5), ψ is transformed to

$$\psi'=\psi e^{i\epsilon\lambda}, \tag{15}$$

where ϵ is a (real) constant of the dimensions (charge)$^{-1}$. From ψ and φ_μ one can form two simple invariants suitable for use in the variational principle:[6] $\psi\psi^*$ and $\chi^\mu\chi_\mu^*$, where

$$\chi_\mu=\partial\psi/\partial x_\mu-i\epsilon\varphi_\mu\psi. \tag{16}$$

One can take for L_m a linear combination of these scalars with arbitrary numerical coefficients. One of the coefficients can be absorbed by the function ψ, so that there is left only one arbitrary constant. The signs of the terms, however, cannot be changed by any transformation of ψ. It is therefore necessary to consider all possible combinations of signs of the two terms in L_m. It turns out that of the four possibilities only one combination allows "electron" solutions.

Let us restrict ourselves hereafter to the case of special relativity and let us take the metric in the form

$$ds^2=-dx_1^2-dx_2^2-dx_3^2+dx_4^2 \tag{17}$$

with $\quad x_1=x, \quad x_2=y, \quad x_3=z_1, \quad x_4=t$

(the unit of time is chosen so that the velocity of light is equal to unity). Then the proper linear combination for L_m is given by

$$L_m=-\chi^\mu\chi_\mu^*+\sigma^2\psi\psi^*, \tag{18}$$

where σ is a real constant of the dimensions (length)$^{-1}$. The complete Lagrangian is then

$$L=-F_{\mu\nu}F^{\mu\nu}/8\pi-\chi^\mu\chi_\mu^*+\sigma^2\psi\psi^*, \tag{19}$$

where $\quad(\varphi^1, \varphi^2, \varphi^3, \varphi^4)=(A_1, A_2, A_3, \varphi),$

$(F_{12}, F_{23}, F_{31}, F_{14}, F_{24}, F_{34})=(H_3, H_1, H_2, E_1, E_2, E_3).$

If one takes, as the law for the field, Eq. (6) where φ_μ and ψ are varied independently, one obtains the following system of differential equations

$$\chi^\nu{}_{;\nu}-i\epsilon\varphi_\nu\chi^\nu+\sigma^2\psi=0, \tag{20}$$

and

$$F^{\mu\nu}{}_{;\nu}=4\pi s^\mu, \tag{21}$$

with

$$s^\mu=\tfrac{1}{2}i\epsilon(\psi^*\chi^\mu-\psi\chi^{\mu*}). \tag{22}$$

The connection with the usual notation is given by

$$(s^1, s^2, s^3, s^4)=(j_1, j_2, j_3, \rho),$$

so that

$$\rho=\frac{i\epsilon}{2}\left(\psi^*\frac{\partial\psi}{\partial t}-\psi\frac{\partial\psi^*}{\partial t}\right)+\epsilon^2\varphi\psi\psi^*. \tag{23}$$

Equation (20) written in terms of ψ has practically the same form as the Gordon-Klein relativistic wave equation in quantum mechanics. An essential difference, however, is that in the Gordon-Klein equation the potentials which occur are those of the "outer" field, i.e., they do not include the field due to the charge and current determined by ψ, whereas in the present equation the potentials include the latter field as well. The result is that here the system of Eqs. (20) and (21) is nonlinear in ψ, which therefore may not be normalized arbitrarily as is done in wave mechanics. Furthermore the constants occurring in (20) will have values different from the corresponding ones in the Gordon-Klein equation.

The constants ϵ and σ in the present case are dimensional constants and hence their values cannot be predicted from the theory since these depend on the system of units used. They must be chosen so as to agree with experiment, that is,

[6] There are, of course, other invariants. If we accept only terms of the second or lower order of differentiation and bilinear in ψ and ψ^* there remains $F_{\mu\nu}F^{\mu\nu}\psi\psi^*$. Since this is the product of two invariants already present, it was considered as being of a more complicated type and was discarded. It could be added with another arbitrary constant.

give the correct values for the charge and mass of the electron. One can predict in advance their orders of magnitude:

$$\epsilon\sim 1/e, \quad \sigma\sim 1/a=mc^2/e^2,$$

where e, m, and a are the charge, mass and "radius" of the electron, respectively.

From the Lagrangian L one can calculate the energy-impulse density tensor $T_{\mu\nu}$ by means of the relation

$$T_{\mu\nu}=\partial L/\partial g_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}L. \tag{24}$$

One finds

$$T_{\mu\nu}=(T_{\mu\nu})_f+(T_{\mu\nu})_m, \tag{25}$$

where

$$(T_{\mu\nu})_f=(g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}-2F_{\mu\alpha}F_{\nu}{}^{\alpha})/16\pi, \tag{26}$$

$$(T_{\mu\nu})_m=\tfrac{1}{2}[g_{\mu\nu}(\chi^{\alpha}\chi_{\alpha}{}^{*}-\sigma^2\psi\psi^{*}) -\chi_{\mu}\chi_{\nu}{}^{*}-\chi_{\mu}{}^{*}\chi_{\nu}]. \tag{27}$$

The energy density is given by T_{44}. From (26) and (27) one finds, using the values of the metric-tensor components in (17), that

$$(T_{44})_f=\frac{1}{16\pi}\sum_{\alpha,\beta}F_{\alpha\beta}F_{\alpha\beta}, \tag{28}$$

$$(T_{44})_m=-\tfrac{1}{2}(\sum_{\alpha}\chi_{\alpha}\chi_{\alpha}{}^{*}+\sigma^2\psi\psi^{*}), \tag{29}$$

from which it follows that

$$(T_{44})_f\geqslant 0, \quad (T_{44})_m\leqslant 0,$$

so that the sign of T_{44} is not definite. It appears that this is closely connected with the existence of discrete charges. Thus, one can reverse the sign of L_m and in that way make T_{44} positive definite, but one then finds that there are no electron-like solutions of the resulting field equations. We shall have occasion later on to return to this question.

The tensor $T_{\mu\nu}$, on the basis of the field equations, satisfies the divergence equation

$$\partial T^{\mu\nu}/\partial x_{\nu}=0. \tag{30}$$

From this it follows that if there exists a solution, for which the energy is localized in a finite volume (or tends rapidly enough to zero at infinity), then one can define an energy-impulse vector[7]

$$P^{\mu}=\int T^{\mu 4}dv \tag{31}$$

(where the integral is taken over all three-space), such that

$$dP^{\mu}/dt=0. \tag{32}$$

The energy is given by

$$W=P^4=\int T^{44}dv=\int T_{44}dv. \tag{33}$$

If one integrates (30) over a finite three-dimensional volume V bounded by a surface S one gets

$$\frac{d}{dt}\int_V T^{\mu 4}dv=-\int_S T^{\mu n}df, \tag{34}$$

where df is a surface element and the index n denotes the component in the direction of the outward normal to the surface. One can apply this to the case of a solution corresponding to a particle moving slowly in a weak, slowly-varying, external field. One takes the surface S in the form of a sphere with center at the particle and radius large compared to the particle radius, so that on the surface one can neglect $(T^{\mu\nu})_m$ and take $(T^{\mu\nu})_f$ corresponding to the external field + the electromagnetic field of the particle (considered as a point charge). One can then show that (34) gives the equations of motion of the particle corresponding to the action of the Lorentz force.

III. Spherically-Symmetric Case

We now investigate the spherically-symmetric static solution of the field equations. A static solution means one in which the electromagnetic field, and hence the charge distribution, does not change with time. For this it is sufficient for ψ to be of the form:

$$\psi=\theta e^{-i\epsilon\mu t}, \tag{35}$$

where θ is a function of (x, y, z) and μ an arbitrary constant. Furthermore, from the condition of spherical symmetry it follows that

$$s_1=s_2=s_3=0 \quad s_4=\rho(r), \tag{36a}$$

where r is the distance from the origin of coordinates. One can also take

$$\varphi_1=\varphi_2=\varphi_3=0 \quad \varphi_4=\phi(r) \tag{36b}$$

and, on the basis of (36a), one can assume θ to

[7] Cf. H. Weyl, *Raum, Zeit Materie* (Berlin, 1923), p. 204.

be real, without loss of generality. One then has that

$$\rho=\epsilon^2(\phi+\mu)\theta^2. \tag{37}$$

With these simplifications the field equations (20) and (21) take on the form

$$\nabla^2\theta+[\epsilon^2(\phi+\mu)^2-\sigma^2]\theta=0, \tag{38}$$

$$\nabla^2\phi+4\pi\epsilon^2(\phi+\mu)\theta^2=0. \tag{39}$$

The boundary conditions for a particle solution are

$$r=0: \quad d\theta/dr=d\phi/dr=0, \tag{40a}$$

$$r\to\infty: \quad \theta\to 0, \quad \phi\sim\text{const.}/r. \tag{40b}$$

The conditions (40a) are necessary in order that there be no discontinuities in the gradients of θ and ϕ at the origin.

It is convenient to introduce dimensionless variables by means of the substitution

$$\theta=\sigma y/\epsilon(4\pi)^{\frac{1}{2}}, \quad \phi+\mu=\sigma z/\epsilon, \quad r=x/\sigma. \tag{41}$$

The equations then become

$$d^2(xy)/dx^2=x(1-z^2)y, \tag{42}$$

$$d^2(xz)/dx^2=-xy^2z. \tag{43}$$

If one makes a further change of variables

$$xy=\eta, \quad xz=\zeta \tag{41a}$$

one gets for the equations

$$\eta''=(1-\zeta^2/x^2)\eta, \tag{44}$$

$$\zeta''=-\eta^2\zeta/x^2, \tag{45}$$

where primes denote differentiation with respect to x. The boundary conditions now become

$$\eta(0)=\zeta(0)=\eta(\infty)=0, \tag{46a}$$

$$(\zeta/x)_\infty=\epsilon\mu/\sigma(\equiv\beta). \tag{46b}$$

But since μ is not known, (46b) is really not a condition but an equation which determines μ (or β) after a solution has been found.

If one has obtained a solution of the field equations, corresponding to a particle, the charge of the latter e is connected with the constant ϵ by the relation

$$e=4\pi\int_0^\infty \rho r^2dr=\alpha/\epsilon, \tag{47}$$

where

$$\alpha=\int_0^\infty y^2zx^2dx. \tag{48}$$

For the energy density one finds

$$T_{44}=\frac{1}{8\pi}\left(\frac{d\phi}{dr}\right)^2-\tfrac{1}{2}\epsilon^2(\phi+\mu)^2\theta^2-\tfrac{1}{2}\left(\frac{d\theta}{dr}\right)^2-\tfrac{1}{2}\sigma^2\theta^2.$$

The total energy is then given by

$$W=4\pi\int_0^\infty T_{44}r^2dr.$$

Integrating by parts and making use of the field equations, one obtains

$$W=-\sigma w/2\epsilon^2, \tag{49}$$

where

$$w=\alpha\beta+\gamma, \tag{50a}$$

and

$$\gamma=\int_0^\infty y^2z^2x^2dx. \tag{50b}$$

IV. Electron Solution

The field equations, in the form (44) and (45), for example, can be integrated starting from $x=0$. If one takes $\eta(0)=\zeta(0)=0$, the solution is determined by the parameters

$$\eta'(0)=y(0)=a, \quad \zeta'(0)=z(0)=b. \tag{51}$$

In order for the solution to have the proper behavior at infinity one must choose suitable values for these parameters. It is seen from Eq. (44) that, for a solution representing a particle, η must fall off exponentially at large distances. This will be the case if $z^2<1$ at infinity, i.e., if $\beta^2<1$. It can be easily shown that to obtain a solution of this type it is necessary to have $b^2>1$. In the region where $z^2>1$, η will have an oscillatory behavior.

If one has found a solution for which η has the proper behavior at infinity, then it follows from Eq. (45) that z or ζ will also behave properly at large distances and one has there

$$z\sim\alpha/x+\beta, \quad \zeta\sim\alpha+\beta x. \tag{52}$$

This means that, if the value of β is not prescribed, the parameters a and b of (51) have to be chosen so as to satisfy the boundary conditions at infinity for only one function, η. In general this can be done by choosing one of the parameters arbitrarily and then suitably determining the other.

The simplest solution is one for which both θ and ϕ are without nodes in the finite part of space. It is reasonable to expect that this might describe the simplest known charged particle, the electron (or, if one changes the signs of ϕ and μ, the positron). Hence this solution was investigated in some detail by means of numerical integration.

As has been pointed out, one can obtain solutions of this type for a finite range of values of one of the parameters a and b if one determines the other accordingly. In other words there exists a one-parameter family of solutions of the nodeless type. For fixed values of the constants ϵ and σ these solutions differ from one another with respect to charge and energy. The question then arises: if this type of solution describes an electron, how is one to account for the fact that all ordinary electrons have the same charge and mass? To be sure, the charge and mass of a single isolated particle are conserved; but if several particles interact what is to prevent one of them from acquiring part of the charge and mass of another?

A possible explanation appears to be that, of the family of solutions, there is one solution for which the energy is a *minimum;* and that this represents the normal state of the isolated electron. Calculations were therefore carried out to determine whether there exists a minimum for the energy. Several values of a were taken and in each case b was determined by trial so as to give a solution of the desired type. For each solution the quantity w was calculated according to (50a). It was found that w has a maximum. The position of the latter has not been determined very accurately but is in the neighborhood of $a=1.6$ ($b=2.19$, $\alpha=1.91$, $\beta=0.015$), where it has the value 2.83.

From (49) it follows that the mass of such an electron is *negative*. This accounts for the stability of the electron (since energy would have to be added to it in order to explode it), but unfortunately it is in disagreement with our knowledge about the electron. However, it must be borne in mind that one is dealing here with a classical theory. Before comparing with experiment, one must find the quantum generalization. This should be of such a form that the equations will agree with the Dirac equations for the electron in the approximation in which one neglects the finite radius of the electron and considers it as a point charge. Now the Dirac equations give essentially the same solutions (with a different numbering) for a negative as for a positive mass. If one considers states corresponding to *positive energy*, the Dirac electron with negative mass is indistinguishable in its behavior from one with its mass positive. It may therefore be that in the quantum generalization of the present equations the difficulty with the sign of the mass will disappear.

In the neighborhood of $a=1.6$ it is found that β, as given by (46b) passes through zero. It may be that the minimum of the energy is located at the point where $\beta=0$. If this is *not* the case, there arises an interesting possibility. In this case the electron has both energy and frequency, and one can define for the ratio a constant

$$h'=|W/\nu|. \tag{53}$$

If one makes use of Eqs. (35), (46b), (47), (49) and (50a) and takes the customary unit for time, this relation can be written in the form

$$\alpha'\equiv 2\pi e^2/h'c=|2\alpha^2\beta/(\alpha\beta+\gamma)|.$$

Calculations of α' were attempted, but it was difficult to obtain any great accuracy. It was found that $\alpha'\lesssim 0.04$, It may be that it has the value 1/137 of the Sommerfeld fine-structure constant. Should this turn out to be the case, it would mean that there is a much closer relationship between the quantum theory and the classical theory than has hitherto been supposed. Even in the classical theory there would be associated with a moving electron something like a de Broglie wave.

It is to be noted that the electron here is similar in its structure to the "wave packet" electron considered during the early period of quantum mechanics. However one would expect that the mutual interaction between various parts of the electron considered here prevents it from spreading in the course of time (as occurs in wave mechanics) at least in the case of weak fields. To be sure, in strong fields the present electron will flow apart. Thus one sees that even a classical theory can explain why electrons, *as particles*, cannot be nuclear constituents.

V. Other Solutions

Besides the simple solution already considered there are also more complicated solutions. Among these are solutions for which θ is of the same form as in the previous case whereas z has a number of nodes. Such solutions correspond to particles in which the charge distribution consists of concentric layers alternating in sign. One would expect that a particle of this type, compared with the simple particle considered in the previous section having the same charge, will have a considerably greater mass, since the quantity γ, which essentially determines the energy, depends on z^2 and thus adds for the various layers, whereas their charges tend to cancel each other. It may be that such solutions are associated with heavy particles. In this connection it is worth pointing out that among these solutions there will also be some for which the net charge of the particle vanishes.

Then there are also solutions for which θ has nodes. It may be that these correspond to internal excited states of elementary particles (heavy electrons?).

VI. Conclusions

In the present paper is suggested a possibility for a classical theory of elementary particles. It has a shortcoming in that it leads to a negative mass for the electron. However it may be that this difficulty will be removed in the process of going over from the classical to the quantum theory. At any rate one has here for the first time a theory leading to equations which are gauge-invariant and at the same time give solutions which are free from singularities, in fact, which are analytic functions.

There appears to be a possibility of deriving the Sommerfeld fine-structure constant from the theory considered here in spite of the fact that it is a classical theory. Should this turn out to be an actuality, it would have an important bearing on the problem of the foundations of the quantum theory. If it is found that the numerical value obtained for the constant is not correct, it does not necessarily mean that the general idea considered here must be abandoned. The particular Lagrangian (19), which was used to derive the field equations, is only one—the simplest—of a number of possibilities. Perhaps some other one should have been chosen.

In conclusion, the writer wishes to express his deep indebtedness to Professor A. Einstein for many interesting discussions on the subject of field theories. He would also like to thank Mr. M. Feingold for assistance with the numerical calculations.