The magnitude of the spin doubling and the origin of the bands remain to be found. The electronic splitting of the $^2\Pi$ state may be found from the extrapolation to $K=0$ of the differences $T_2(K)-T_1(K)$. Theoretically this should give

$$\Delta T_{12}(0)=A-B+B^2/A-o-\tfrac{1}{2}p,$$

where

$$o=-\frac{1}{4}\frac{A^2l(l+1)}{\nu(\Pi,\Sigma)}=-2.34\ \text{cm}^{-1}.$$

This gives $A=-378.9$ cm^{-1}. A better value of A and of ν_0 may be found from the following considerations. If $T(J)=T_0+F(J)$ then

$$Q_1=T_0'+F_1'(J)-[T_0''+F_{1c}''(J)],$$
$$P_1=T_0'+F_1'(J-1)-[T_0''+F_{1d}''(J)],$$
$$\nu_0=\tfrac{1}{2}\{Q_1-F_1'(J)+F_{1c}''J+P_1-F_1'(J-1)+F_{1d}''J\},$$
$$Q_2=T_0'+F_2'(J)-[T_0''+F_{2c}''(J)],$$
$$\nu_0=Q_2-F_2'(J)+F_{2c}''(J).$$

From these relations A is found to be -378.6 cm^{-1} and $\nu_0=30659.1$ cm^{-1}. The data are not sufficiently accurate or complete to permit any estimate of the system origin ν_e.

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On the Magnetic Scattering of Neutrons

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In this paper there is contained a full elaboration of two previously published short notes on the subject of magnetic scattering of neutrons together with a comprehensive treatment of certain sides of this problem which have already received some attention from other authors. After presenting the state of the problem in the introduction and discussing in detail our reasons for the choice of an interaction function between neutrons and electrons, and the nonmagnetic interaction between neutrons and nuclei, the various possible cases of coherent and incoherent scattering and depolarization phenomena are treated. Later applications to the theory of ferromagnetic scattering are kept in mind. The general expression for the cross section due to magnetic interaction is obtained and applied to various classes of phenomena (scattering by free, rigidly aligned, and coupled magnetic ions). The influence of the elastic form-factor is treated quantitatively with the aid of a simple model for the current distribution in the ion. Finally a series of performed or suggested experiments is discussed mainly from the point of view whether they will permit theoretical interpretation. Arrangements are described which will allow one to obtain a reliable value for the neutron's magnetic moment and also give insight into the magnetic constitution of the scatterer (ion or crystal) which will exceed the knowledge obtainable from macroscopic magnetic experiments.

I. Introduction

Sometime ago it was suggested[1] that a magnetic moment of the neutron should manifest itself in the scattering of slow neutrons from paramagnetic substances. The magnetic scattering should in some instances be several times as great as the total nuclear scattering if, for the neutron, a magnetic moment of two nuclear magnetons is assumed. This magnetic scattering could therefore be easily isolated by comparing the scattering cross section of the same atoms in different chemical combinations which show a varying magnetic susceptibility. It could also be separated from nuclear scattering by studying the angular distribution of the particles scattered from a single paramagnetic compound. This is due to the fact that magnetic scattering is, under practical conditions, strongly favored in the forward direction, whereas the nuclear scattering is isotropic. If the neutron moment is of the order of magnitude of a nuclear magneton the paramagnetic scattering provides a direct and simple method for its quantitative determination.

[1] O. Halpern and M. H. Johnson, Phys. Rev. **51**, 992; **52**, 52 (1937).

Our suggestions were subsequent to a letter by Bloch[2] in which it was pointed out that the interaction between an assumed magnetic moment of the neutron and that of a ferromagnetic ion should produce observable effects. Bloch supposed in his analysis that the scattering arose from a magnetic dipole distribution within the atom which was unaffected by the scattering process. He confined his attention to the production and detection of polarized neutron beams which are produced by the interference between nuclear and magnetic scattering in a saturated ferromagnetic medium. The appearance of polarization phenomena should make it possible to differentiate between nuclear and magnetic scattering and thereby to determine the magnetic moment of the neutron. Later Schwinger[3] gave a more detailed mathematical treatment of Bloch's idea. He, too, assumed that the scattering system remained unchanged during the scattering process and likewise limited himself to polarization effects in ferromagnetic bodies.

As pointed out in Section IV the physical basis of Bloch's ideas must be enlarged to include changes produced in the scattering system by the scattering process. Since the occurrence of such changes is the rule rather than the exception, results based on stricter assumptions have to be considerably modified. Especially in the case of scattering by a ferromagnetic material several new features appear which greatly increase the difficulty of determining the magnetic moment of the neutron from polarization experiments alone. Such uncertainties which are due to our ignorance of the exact state of the medium do not enter to any large degree into certain cases of paramagnetic scattering.

We intend to present in this paper a comprehensive treatment of the magnetic scattering of neutrons. The various phenomena are analyzed both in their dependence on the magnetic properties of the neutron and on the characteristics of the scatterer. The subject has been divided as follows: In Section II the possible forms of interaction between a neutron and an atomic electron are studied and their essential equivalence for low energy neutrons is demonstrated. In Section III we review the scattering of slow neutrons by nuclei whose spins are free; special emphasis is put on questions of coherence in preparation for the discussion of simultaneous nuclear and magnetic scattering. Section IV contains the deduction of a general formula describing the wave which is magnetically scattered from an isolated atom. The result is applied in Section V to the case of an isolated paramagnetic ion, in Section VI to the case of a ferromagnetic medium which is treated under the assumption that the spin of each ion is rigidly aligned along the magnetic axis of the microcrystals. Some other assumptions are also briefly discussed. Section VII deals with the nature of atomic form factors. Section VIII is devoted to a discussion of coupled scattering systems which are treated as a many-body problem. Here we also consider the influence of external forces acting on the nuclear spin and the modifications in the scattering of para- and ferromagnetic materials due to the forces between the spin vectors of the ions. The final section, IX, treats various experimental arrangements connected with the subject.

We begin with a few general remarks which will aid in fixing a suitable nomenclature. Let the beam of incident neutrons be represented by the wave function

$$\psi_{\text{in}}=(2\pi M_0/hk)^{\frac{1}{2}}\exp\,(i\mathbf{k}\cdot\mathbf{r})\chi_s\Phi_A. \quad (1.0)$$

Here $\mathbf{k}=2\pi\mathbf{P}/h$ is the propagation vector, $\mathbf{r}_n$ the position vector and $\mathbf{P}$ the momentum vector of the incident neutron with a mass M_0. χ_s describes the neutron spin state and Φ_A is the complete wave function for the initial state of the scattering system. We shall use the notation (quantity)$_{\text{av}}$ to denote integration and summation of the quantity in parenthesis over all coordinates of the scattering system. For example, the density, I, of incident neutrons is given by

$$I_{\text{in}}=(\psi_{\text{in}}^*\psi_{\text{in}})_{\text{av}}=(2\pi M_0/hk)|\chi_s|^2. \quad (1.01)$$

Denoting by $\mathbf{F}$ the differential operator $(h/4\pi M_0 i)\,\mathbf{grad}_n$ we have correspondingly for the flux of the incident neutrons

$$\mathbf{F}_{\text{in}}=(\psi_{\text{in}}^*\mathbf{F}\psi_{\text{in}}-\psi_{\text{in}}F\psi_{\text{in}}^*)=\mathbf{k}/k|\chi_s|^2. \quad (1.02)$$

Summation over the neutrons spin coordinate shall be denoted by $\sum_{\text{spin}}$. Furthermore, by use of

[2] F. Bloch, Phys. Rev. 50, 259 (1936).
[3] J. Schwinger, Phys. Rev. 51, 544 (1937).

normalized spin functions, $\sum_{\text{spin}}|\chi_s|^2=1$, we have $\sum_{\text{spin}}\mathbf{F}_{\text{in}}=\mathbf{k}/k$; this means that the incident flux is also normalized to unity. Correspondingly we have for the flux of the scattered wave

$$\mathbf{F}_{sc}=(\psi_{sc}^*\mathbf{F}\psi_{sc}-\psi_{sc}\mathbf{F}\psi_{sc}^*)_{\text{av}}. \tag{1.03}$$

The differential cross section $d\Phi$ for scattering into the solid angle subtended by the element of area dA which is perpendicular to $\mathbf{F}_{sc}$ is given by

$$d\Phi=F_{sc}dA. \tag{1.1}$$

Total scattering cross sections are obtained from (1.1) by summing over the neutron spin coordinate and integrating over all scattering angles.

In considering inelastic scattering the neutron density will be used in preference to the flux. This is done in consideration of the fact that all detectors of slow neutrons have cross sections which are inversely proportional to the velocity of the neutrons. The activity of a given detector depends, therefore, apart from geometrical factors, only on the volume density of the neutrons and not directly on the flux. This distinction becomes irrelevant for purely elastic scattering.

If the scattering system consists of two independent parts a and b and if multiple scattering can be neglected, the whole scattered wave

$$\psi_{sc}=\psi_a+\psi_b$$

can be subdivided by writing

$$\psi_a=\psi_a^1+\psi_a^2,$$
$$\psi_b=\psi_b^1+\psi_b^2.$$

In ψ_a^1 and ψ_b^1 the scattering system has remained in its initial state while in ψ_a^2 one of the parts has been changed by the scattering process. Obviously expressions like $(\psi_a^*\mathbf{F}\psi_b^2)_{\text{av}}$ vanish because of the orthogonality of the wave functions for the two different states of the scatterer. Substitution into (1.03) gives

$$\mathbf{F}_{sc}=\mathbf{F}_a+\mathbf{F}_b+\mathbf{F}_{ab}+\mathbf{F}_{ba} \tag{1.11}$$

with the simple abbreviations,

$$\mathbf{F}_a=(\psi_a^*\mathbf{F}\psi_a-\psi_a\mathbf{F}\psi_a^*)_{\text{av}}, \tag{1.111}$$

$$\mathbf{F}_b=(\psi_b^*\mathrm{F}\psi_b-\psi_b\mathbf{F}\psi_b^*)_{\text{av}}, \tag{1.112}$$

$$\mathbf{F}_{ab}=(\psi_a^{1*}\mathbf{F}\psi_b^1-\psi_b^1\mathbf{F}\psi_a^{1*})_{\text{av}}, \tag{1.113}$$

$$\mathbf{F}_{ba}=(\psi_b'^*\mathbf{F}\psi_a'-\psi_a'\mathbf{F}\psi_b'^*)_{\text{av}}. \tag{1.114}$$

The first two terms of (1.11) give the flux produced by each part of the system in the absence of the other part whereas the last two terms describe the interference between the waves scattered by the two parts. The interference terms (1.113) and (1.114) contain only the wave functions ψ^1 which describe an unchanged state of the scatterer. Already at this point we want to emphasize that the amplitudes enter linearly into (1.113) and (1.114). As a consequence, in many cases in which a parameter describing the state of the scattering system is unknown, the interference terms vanish on averaging over the parameter while the terms in (1.111) and (1.112) which involve the squares of the amplitudes do not disappear.

These remarks are easily extended to a scattering system composed of many particles.

II. The Hamiltonian for the Magnetic Interaction

The choice of a proper Hamiltonian representing the magnetic interaction between neutron and electron cannot be made unambiguously since we do not yet possess a satisfactory theory of the structure of the neutron. The question has been the subject of considerable discussion and disagreement.[4] We present in this paragraph our view on the subject which will determine the choice of the Hamiltonian.

By the statement that the neutron carries a magnetic moment we mean to describe the fact that it produces in its neighborhood a field analogous to that of a classical magnetic dipole which owes its existence to a stationary current distribution. Fields that do not satisfy the relation

$$\operatorname{div}\mathbf{H}=0 \tag{2.0}$$

may be present inside of the neutron; however, we refuse to recognize them as magnetic fields and prefer to designate them as some sort of spin dependent, short range forces. We know nothing about their existence, which could be detected only experimentally.

The rather trivial remark may be added that real singularities (discontinuities) which cannot be approximated by continuous functions sufficiently well for all physical purposes cannot claim any place in a physical theory.

[4] F. Bloch, Phys. Rev. 51, 994 (1937).

We finally recall the definition of a stationary magnetic moment $\mathbf{m}$ in classical physics as given by the relation

$$\mathbf{i}=c\,\mathbf{curl}\,\mathbf{m}. \tag{2.01}$$

As the divergence of $\mathbf{m}$ is still arbitrary, we can impose another condition on $\mathbf{m}$. This condition is usually chosen to make $\mathbf{m}$ disappear in a domain free of current. Since

$$\mathbf{curl}\,\mathbf{H}=4\pi\mathbf{i}/c, \tag{2.02}$$

it follows that $\mathbf{m}$ and $\mathbf{H}/4\pi$ have the same curl. Therefore a scalar function ϕ exists such that

$$\mathbf{H}=4\pi\mathbf{m}+\mathbf{grad}\,\phi. \tag{2.03}$$

The classical interaction energy can now be written in one of the following equivalent forms

$$W_1=(1/4\pi)\int\mathbf{H}_n\cdot\mathbf{H}_e d\tau, \tag{2.10}$$

$$W_2=\int\mathbf{H}_n\cdot\mathbf{m}_e d\tau, \tag{2.11}$$

$$W_3=\int\mathbf{H}_e\cdot\mathbf{m}_n d\tau. \tag{2.12}$$

These forms follow from each other since

$$\int\mathbf{H}\cdot\mathbf{grad}\,\phi d\tau=\int\mathrm{div}\,(\phi\mathbf{H})d\tau=0. \tag{2.13}$$

Two other forms can be obtained by introducing the vector potential $\mathbf{A}$ and making use of the well-known relation

$$\mathbf{H}\cdot\mathbf{m}=\mathbf{m}\cdot\mathbf{curl}\,\mathbf{A}=\mathbf{A}\cdot\mathbf{curl}\,\mathbf{m}+\mathrm{div}\,(\mathbf{A}\times\mathbf{m}). \tag{2.14}$$

Thus we obtain

$$W_4=(1/c)\int\mathbf{A}_n\cdot\mathbf{i}_e d\tau, \tag{2.20}$$

$$W_5=(1/c)\int\mathbf{A}_e\cdot\mathbf{i}_n d\tau. \tag{2.21}$$

In (2.20) and (2.21) we have dropped the term containing the surface integral which we are always entitled to do if, as mentioned before, we look upon real singularities, etc., as physically meaningless. *It will appear that we thereby do not lose any physical possibilities.*

The terms in the various forms for the interaction energy are determined by the Dirac equation so far as the electron is concerned. The external field of the neutron is given through

$$\mathbf{H}_n(\mathbf{r})=\frac{-\boldsymbol{\mu}_n}{|\mathbf{r}-\mathbf{r}_n|^3}+3(\mathbf{r}-\mathbf{r}_n)\frac{\boldsymbol{\mu}_n\cdot(\mathbf{r}-\mathbf{r}_n)}{|\mathbf{r}-\mathbf{r}_n|^5}, \tag{2.30}$$

where $\boldsymbol{\mu}_n=\int\mathbf{m}_n d\tau$ is the neutron's magnetic moment. Obviously this expression cannot be continued up to the origin; it would lead to an indeterminate value for the interaction energy of the form $0\cdot\log 0$. The forms (2.10) and (2.11) do not give us any hint as to the evaluation of the interaction integral over the interior of the neutron, but as soon as we decide to describe the neutron in the well-known way by a wave equation of the type

$$\left\{\frac{h^2}{8\pi^2M_0}\nabla^2+\gamma\frac{eh}{2\pi M_0c}\mathbf{s}\cdot\mathbf{H}-E\right\}\psi=0, \tag{2.40}$$

we are relieved of all difficulties. For we now have for $\mathbf{m}_n$ the expression

$$\mathbf{m}_n=\gamma(eh/2\pi M_0c)\psi^*\mathbf{s}\psi, \tag{2.50}$$

which, just as $\mathbf{m}_e$ in virtue of the Dirac equation, is not only everywhere regular but for neutrons of sufficiently large wave-length nowhere very large. Quantitatively speaking the integral over the interior of the neutron

$$\int\psi^*\mathbf{s}\psi d\tau$$

can be seen to go to zero for small neutron radius. This determines the previously undetermined form of W and allows us to use any of the expressions (2.10)–(2.12), (2.20), (2.21) according to our convenience.

It will be realized from the preceding discussion that we cannot *a priori* exclude the possibility of short range forces of the type

$$f(\mathbf{r}_e-\mathbf{r}_n,\,\mathbf{s}_n,\,\boldsymbol{\sigma}_e,\,\boldsymbol{\alpha}_e)$$

or of such a current distribution inside the neutron that the integration over the interior gives a considerable contribution to the whole interaction energy while the outside field is still determined by (2.30). To the first possibility we have to remark that there does not yet exist any evidence for such forces; as to the second, that it will only occur if the interaction between neutron and electron cannot be described by a Hamiltonian in a two-body problem. For this certainly possible complication we also lack at present any physical evidence. We shall therefore proceed with the most convenient classical analog given by (2.10).

III. Scattering by Independent Nuclei

We discuss in this paragraph the scattering of slow neutrons by independent nuclei and place special emphasis upon questions of coherence and polarization. Though quite a few of our results can probably be found in the literature it still seems advisable to offer a comprehensive discussion which will be of great help in the treatment of simultaneous scattering due to nuclear and magnetic forces.

As far as coherence and interference questions are concerned the neutron scattering differs from the analogous x-ray scattering by a wide variability in the intensity of the pattern. The amplitude of the scattered wave varies from one nucleus to another not only as to magnitude but also, unlike most cases of x-ray scattering, may change sign. The points in an interference pattern may show striking difference in intensity depending upon the phase of the waves scattered by the different nuclei. For essentially the same reason the spin degeneracy of nuclear states may cause a much larger diminution of coherence than atomic degeneracies in optical cases do.

In the case of slow neutron scattering only the s wave has to be considered; we can also exclude inelastic collisions since the energy is always too small to excite higher nuclear states. We denote the nuclear spin by $\mathbf{i}$ and the eigenvalues of its component along some arbitrary axis by m_i; the corresponding wave functions shall be Ω_{m_i}. Let m_s denote the eigenvalues of the neutron's spin projection along the same axis. Since we are dealing with s waves, the total spin alone must be conserved in the collision. If the scattering is described by an intermediate state of the system neutron plus nucleus then the magnitude of its total angular momentum, $\mathbf{K}$, must be equal to $i+\frac{1}{2}$ or $i-\frac{1}{2}$.

The incident wave can always be analyzed into wave functions corresponding to total angular momenta $i+\frac{1}{2}$ and $i-\frac{1}{2}$. The probability amplitudes of the scattered wave with the above total angular moments shall be denoted by a_1 and a_0, respectively. Except in the simplest cases we lack complete knowledge even as to the relative sign of a_1 and a_0. Both probability amplitudes are independent of the actual sub level m_k since no distinguished axis has entered into our problem. Denoting the two wave functions of the incident wave by $\psi_{i+\frac{1}{2}}$ and $\psi_{i-\frac{1}{2}}$ we have for the complete wave the expression

$$\chi_s\Omega_N = c_0\psi_{i-\frac{1}{2}} + c_1\psi_{i+\frac{1}{2}} \tag{3.0}$$

with arbitrary complex constants. The scattered wave can then be written in the form

$$\psi_n \cdot \left(\frac{2\pi M_0}{hk}\right)^{\frac{1}{2}} \frac{e^{ikr}}{r}(a_0c_0\psi_{i-\frac{1}{2}} + a_1c_1\psi_{i+\frac{1}{2}}). \tag{3.1}$$

To analyze the initial state into functions with definite total angular momentum it is convenient to use two operators η_0 and η_1 defined as follows:

$$\eta_0\psi_{i-\frac{1}{2}} = \psi_{i-\frac{1}{2}}, \tag{3.11}$$

$$\eta_0\psi_{i+\frac{1}{2}} = 0, \tag{3.12}$$

$$\eta_1\psi_{i+\frac{1}{2}} = \psi_{i+\frac{1}{2}}, \tag{3.13}$$

$$\eta_1\psi_{i-\frac{1}{2}} = 0. \tag{3.14}$$

From these definitions we now have

$$\eta_0\chi_s\Omega_N = c_0\psi_{i-\frac{1}{2}}, \tag{3.15}$$

$$\eta_1\chi_s\Omega_N = c_1\psi_{i+\frac{1}{2}}, \tag{3.16}$$

and for the scattered wave

$$\psi_n = \left(\frac{2\pi M_0}{hk}\right)^{\frac{1}{2}} \frac{e^{ikr}}{r}(a_0\eta_0 + a_1\eta_1)\chi_s\Omega_N. \tag{3.2}$$

Operators satisfying (3.11) to (3.14) are easily found to be given by

$$\eta_0 = (2i+1)^{-1}(i - 2\mathbf{i}\cdot\mathbf{s}), \tag{3.21}$$

$$\eta_1 = (2i+1)^{-1}(i+1+2\mathbf{i}\cdot\mathbf{s}). \tag{3.22}$$

We shall hereafter use the abbreviation

$$\eta = a_0\eta_0 + a_1\eta_1.$$

If several isotopes are present, our knowledge of the scattering nucleus is described by a wave function which corresponds to the simultaneous presence of all the isotopes weighted in the ratio of their abundance. If $\Omega_N{}^P$ describes the spin state of the Pth isotope the total spin wave function is given by

$$\Omega_N=\sum_P \Omega_N{}^P b_P,\quad \sum |b_P|^2=1. \tag{3.23}$$

The term $|b_P|^2$ is determined by the relative abundance of the Pth isotope, while b_P contains an undetermined phase. Quantities obtained from (3.23) must be averaged over the arbitrary phase occurring in each b_P. The previous result for the scattered wave is easily generalized. The scattered wave is still given by (3.2) if we understand by $\eta\Omega_N$ the expression

$$\eta\Omega_N=\sum_P b_P\eta_P\Omega_N{}^P, \tag{3.24}$$

where $\eta_P=a_0{}^P\eta_0{}^P+a_1{}^P\eta_1{}^P$,

$$\eta_0{}^P=(2i_P+1)^{-1}(i_P-2\mathbf{i}_P\cdot\mathbf{s}),$$

$$\eta_1{}^P=(2i_P+1)^{-1}(i_P+1+2\mathbf{i}_P\cdot\mathbf{s}).$$

To illustrate coherence properties by a simple example, the interference between the scattering of two identical nuclei a and b shall be discussed. The positions of the two nuclei are assumed to be precisely known and multiple scattering shall be neglected. If the total scattered wave is analyzed according to different spin states of the nucleus, we find for that part of the wave in which the nuclear state is unchanged the expression

$$\psi_{ma}{}^1=\left(\frac{2\pi M_0}{hk}\right)^{\frac{1}{2}}\frac{\exp\,(ik|\mathbf{r}-\mathbf{r}_a|+i\mathbf{k}\cdot\mathbf{r}_a)}{|\mathbf{r}-\mathbf{r}_a|}\times(N|\eta|N)_a\chi_s. \tag{3.30}$$

A similar expression holds for the nucleus b. This part of the wave suffices for the discussion of coherent properties; we have from (1.11) for the interference terms.

$$F_{ab}+F_{ba}=\frac{1}{r^2}\cos\,\{k|\mathbf{r}-\mathbf{r}|-k|\mathbf{r}-\mathbf{r}_b| +\mathbf{k}\cdot(\mathbf{r}_a-\mathbf{r}_b)\}\{[(N|\eta|N)_a{}^*\chi_s{}^*]\times[(N|\eta|N)_b\chi_s]+\text{conj.}\}. \tag{3.31}$$

Through η the vector $\mathbf{i}$ enters into (3.30); the amount of coherent scattering therefore depends upon the nuclear spin state through the matrix element $(N|\mathbf{i}|N)_a\cdot\mathbf{s}\chi_s{}^*$ etc. In practical cases the nuclear spin has no preferred axis which makes $(N|\mathbf{i}|N)=0$. Eq. (3.31) simplifies thereby to

$$F_{ab}+F_{ba}=2/r^2\cos\,\{k|\mathbf{r}-\mathbf{r}_a|-k|\mathbf{r}-\mathbf{r}_b| +\mathbf{k}\cdot(\mathbf{r}_a-\mathbf{r}_b)\}\left\{\frac{ia_0+(i+1)a_1}{2i+1}\right\}^2\chi_s{}^*\chi_s. \tag{3.32}$$

The first factor in (3.32) arises from the path difference between the two nuclei. The last factor $\chi_s{}^*\chi_s$ shows that the spin of the neutron is unchanged in the interference term. The middle factor makes it clear that the coherence depends upon a linear combination of a_0 and a_1 which even may be of opposite sign. We shall see that the total scattering depends only upon $a_0{}^2$ and $a_1{}^2$ so that even for large total scattering the spin degeneracy may completely destroy all interference effects. Such a case is approximately realized in ${}_1H^1$; it may be expected that hydrogen in a crystal lattice will not appreciably contribute to the formation of a diffraction pattern.

In the presence of several isotopes the results are easily generalized. By use of (3.23) and (3.24) and analogous reasoning, one obtains, after averaging over the arbitrary phases in the coefficients b_P, the expression

$$F_{ab}+F_{ba}=2/r^2\cos\,[k|\mathbf{r}-\mathbf{r}_a| -k|\mathbf{r}-\mathbf{r}_b|+\mathbf{k}\cdot(\mathbf{r}_a-\mathbf{r}_b)] \times\left\{\sum_P|b_P|^2\frac{i_Pa_0{}^P+(i_P+1)a_1{}^P}{2i_P+1}\right\}^2\chi_s{}^*\chi_s. \tag{3.33}$$

The bracket { } indicates that the amplitudes of all the isotopes enter a linear combination. Hence coherence may be reduced not only by opposite signs of a_0 and a_1 for a single isotope, but also by the cancellation of the amplitude for one isotope against that of another. Whereas a combination of two measurements (total scattering and interference) is sufficient for experimental determination of the amplitudes a_0 and a_1 in the case of a single isotope, this is no longer true in the case of several isotopes. Only in the case of a single isotope with no spin ($a_1=a_0$) does a scattering measurement allow predictions of interference phenomena.

In practice the thermal agitation prohibits a precise knowledge of the nuclear position in a crystal. Its position is then described by a spatial wave function $\theta(\mathbf{r}_N)$ whose extension is determined by the amplitude of the thermal oscillations. For the discussion of elastic scattering each point of the distribution $|\theta(\mathbf{r}_N)|^2$ can be considered the origin of a secondary wavelet. Thus the elastically scattered wave becomes

$$\psi_m=\left(\frac{2\pi M_0}{hk}\right)^{\frac{1}{2}}\frac{e^{ikr}}{r}F_N{}^{\frac{1}{2}}\eta\chi_s\Omega_N, \qquad (3.4)$$

$$F_N{}^{\frac{1}{2}}=\int \exp\left(i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_N\right)|\theta(\mathbf{r}_N)|^2 d\tau_N, \qquad (3.41)$$

where $\mathbf{k}'$ is the propagation vector of the scattered neutron. The coherent scattering is reduced by the factor F_N, which is called the form factor of the nuclear distribution.

To obtain the scattering from a single nucleus one has to insert the expression for the scattered wave (3.30) into (1.03) which gives the outgoing flux. One finds

$$F_N=r^{-2}[(\eta^*\chi_s^*\Omega_N^*)(\eta\chi_s\Omega_N)]_{\text{av}}, \qquad (3.5)$$

which can be transformed on account of the Hermitian character of η into

$$F_N=r^{-2}(N|(\eta'\chi_s)(\eta\chi_s)|N),$$
$$\eta'=(2i+1)^{-1}[ia_0+(i+1)a_1+2(a_1-a_0)\mathbf{i}\cdot\mathbf{s}^*]. \qquad (3.51)$$

After expansion of the product, the terms linear in the spin component can again be dropped since the nuclear spin has no preferred direction. We have for the same reason the relation

$$(N|i_xi_x|N)=(N|i_yi_y|N)=(N|i_zi_z|N)=\tfrac{1}{3}i(i+1). \qquad (3.6)$$

It is also clear that terms like $(N|i_xi_y|N)$ have to disappear. This can be seen as follows; if $\mathbf{A}$ is a real vector the average value of $(\mathbf{A}\cdot\mathbf{i})^2$ i.e., $(N|(\mathbf{i}\cdot\mathbf{A})(\mathbf{i}\cdot\mathbf{A})|N)$ must be expressible in terms of the scalar A^2 when $\mathbf{i}$ has no preferred direction. This requires that terms like $(N|i_xi_y|N)$ vanish.

We thus arrive at the expression

$$F_n=r^{-2}(2i+1)^{-2}\{[ia_0+(i+1)a_1]^2|\chi_s|^2+(4/3)i(i+1)(a_1-a_0)^2(\mathbf{s}^*\chi_s^*)\cdot(\mathbf{s}\chi_s)\}. \qquad (3.7)$$

The total scattering is found by summing (3.7) over the spin coordinate of the neutron. Again on account of the Hermitian character of $\mathbf{s}$ we can write

$$\sum_{\text{spin}}\cdot(\mathbf{s}^*\chi_s^*)\cdot s\chi_s=\sum_{\text{spin}}\chi_s^*\mathbf{s}\cdot\mathbf{s}\chi_s=\tfrac{3}{4}\sum_{\text{spin}}|\chi_s|^2=\tfrac{3}{4} \qquad (3.7)$$

and so obtain finally for the total outgoing flux

$$\sum_{\text{spin}}F_n=r^{-2}(2i+1)^{-1}[ia_0^2+(i+1)a_1^2]. \qquad (3.80)$$

From (3.80) there follows for the differential cross section for scattering into the solid angle $d\Omega$ the expression

$$d\Phi_n=(2i+1)^{-1}[ia_0^2+(i+1)a_1^2]d\Omega, \qquad (3.81)$$

and since the scattered wave is an s wave we have a total cross section

$$\sigma_n=4\pi(2i+1)^{-1}[ia_0^2+(i+1)a_1^2]. \qquad (3.82)$$

Equation (3.82) contains the important result that the total scattering is independent of the initial spin state of the neutron and involves only the absolute magnitude of a_0 and a_1. In the presence of isotopes the Eqs. (3.81) and (3.82) must be averaged over the isotopes; each isotope is weighted in the degree of its abundance. It is important to realize the totally different behavior that the expressions for the interference terms (representing the coherent scattering) and that for the total scattering exhibit as far as the dependence on sign and relative magnitude of the probability amplitudes is concerned.

The relation (3.5) also determines the change produced by the scattering in the spin state of the neutron. Let α and β be two orthogonal and normalized spin functions representing the spin parallel and antiparallel to the z axis, respectively. The spin coordinate of the neutron also taken along the z axis can have the values $\pm\frac{1}{2}$. We have with these coordinates

$$\alpha(\tfrac{1}{2})=1, \quad \beta(\tfrac{1}{2})=0, \quad \alpha(-\tfrac{1}{2})=0, \quad \beta(-\tfrac{1}{2})=1. \qquad (3.90)$$

The spin vector is then represented by the well-known relations

$$S_z\alpha=(\tfrac{1}{2})\alpha,\quad S_y\alpha=(i/2)\beta,$$
$$S_z\beta=-(\tfrac{1}{2})\beta,\quad S_y\beta=-(i/2)\alpha,$$
$$S_x\alpha=(\tfrac{1}{2})\beta,\qquad (3.901)$$
$$S_y\beta=(\tfrac{1}{2})\alpha.$$

The initial spin state of the neutron can be written as follows:

$$\chi_s=a\alpha+b\beta,\quad |a|^2+|b|^2=1,\quad a^*-a=0,\qquad (3.91)$$

where the last relation is used only to eliminate a trivial phase factor. The initial state is polarized in a direction whose polar angles are given by

$$\cot(\theta/2)=|a/b|,\qquad (3.911)$$
$$\tan\phi=(b-b^*)/i(b+b^*).\qquad (3.912)$$

As a measure of the polarization relative to the z axis, we take a quantity P_0 defined by

$$P_0=|a|^2-|b|^2.\qquad (3.92)$$

If the initial state is unpolarized, all results which contain χ_s must be averaged over the angles θ and ϕ. The relation (3.7) now becomes

$$F_n=r^{-2}(2i+1)^{-2}\{[(ia_0+(i+1)a_1)^2$$
$$+\tfrac{1}{3}(a_1-a_0)^2i(i+1)](|a|^2\alpha^*\alpha+|b|^2\beta^*\beta)$$
$$+\tfrac{2}{3}i(i+1)(a_1-a_0)^2(|b|^2\alpha^*\alpha+|a|^2\beta^*\beta\}.\qquad (3.93)$$

With the abbreviation

$$Q=\tfrac{2}{3}(2i+1)^{-1}i(i+1)\frac{(a_1-a_0)^2}{ia_0^2+(i+1)a_1^2}\qquad (3.94)$$

we can now write (3.93) in the form

$$F_n=r^{-2}\{[(1-Q)|a|^2+Q|b|^2]\alpha^*\alpha$$
$$+[(1-Q)|b|^2+Q|a|^2]\beta^*\beta\}$$
$$\times[ia_0^2+(i+1)a_1^2](2i+1)^{-1}.\qquad (3.95)$$

Constructing a quantity P for the scattered wave as we did in (3.92) for the incident wave, we find for the polarization of the scattered wave the expression

$$P=(1-2Q)(|a|^2-|b|^2)=(1-2Q)P_0.\qquad (3.96)$$

Since the z axis is arbitrary, we learn from (3.96) that the polarization of the incident beam relative to any axis is changed through the scattering process by a factor $1-2Q$. Large polarization changes will occur if a_0 and a_1 show opposite signs; in that case Eq. (3.33) leads to small interference effects.

In the presence of isotopes the quantity Q must be averaged as before.

IV. Magnetic Scattering of an Isolated Ion

To place in evidence the uniqueness of result as discussed in Section II, we shall use the fundamental form (2.00) for the interaction between electron and neutron, which must be summed over all the electrons of the atom. By taking the nucleus as the origin of coordinates and designating by Ψ_A the wave function of the electrons in the state A, the Born method in first approximation gives for the scattered wave at large distances from the ion, the expression

$$\psi_m=(2\pi M_0/h^2)(2\pi M_0/hk)^{\frac{1}{2}}\sum_{s'A'}r^{-1}e^{ik'r}$$
$$\times\chi_{s'}\Psi_{A'}(\mathbf{k}',s',A'|V|\mathbf{k},s,A),\qquad (4.00)$$

where the primes indicate final states. The propagation vector of the neutron, $\mathbf{k}$, satisfied the energy condition

$$(h^2/8\pi^2M_0)(k^2-k'^2)=E_{A'}-E_A,\qquad (4.01)$$

where E refers to the energy of the atomic electrons. The matrix elements in (4.00) when written out in full are

$$(\mathbf{k}'s'A'|V|\mathbf{k}sA)=-(4\pi)^{-1}\int\sum_l\mathbf{H}(\mathbf{r}_l)\cdot\mathbf{curl}_l$$
$$\times\left[\int\exp\left(i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}\right)\frac{(s'|\boldsymbol{\mu}|s)\times(\mathbf{r}-\mathbf{r}_l)}{|\mathbf{r}-\mathbf{r}_l|^3}d\tau\right]d\tau_{\text{atom}}\qquad (4.02)$$

when $d\tau_{\text{atom}}$ is the volume element for all the internal coordinates of the atom and $d\tau$ is the volume element for the space coordinate of the neutron. $\mathbf{H}(\mathbf{r}_l)$ is the magnetic field due to the current distribution of the l'th electron while $\mathbf{curl}_l$ is the indicated operation for the coordinates of the lth electron. $(s'|\boldsymbol{\mu}|s)$ are the matrix elements of the neutron's magnetic moment referred to the initial and final spin states, s and s'. Inasmuch as the magnetic field of the electrons is nowhere discontinuous or singular, we may integrate by parts over the electron coordinates and safely discard the surface integrals that result. The expression $\mathbf{curl}_l\,\mathbf{H}(\mathbf{r}_l)$ which results may then be replaced by $4\pi\mathbf{i}_l/c$ where $\mathbf{i}_l$ is the current density of the l'th electron. Thus we find

$$(\mathbf{k}'s'A'|V|\mathbf{k}sA)=\int\sum_l \Psi_{A'}{}^*c\alpha_l\Psi_A \cdot\frac{(s'|\mathbf{\mu}|s)\times(\mathbf{r}-\mathbf{r}_l)}{|\mathbf{r}-\mathbf{r}_l|^3}\cdot\exp\left(i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}\right)d\tau_{\text{atom}}d\tau. \tag{4.03}$$

The current density of the electrons[5] can be written up to the order v^2/c^2 in the following form

$$\mathbf{i}_l=\mathbf{i}_l{}^{(0)}+\mathbf{i}_l{}^{(1)}, \tag{4.04}$$

$$\mathbf{i}_l{}^{(0)}=\frac{eh}{4\pi mi}[\Psi^*\,\mathbf{grad}_l\,\Psi-\Psi\,\mathbf{grad}_l\,\Psi^*], \tag{4.041}$$

$$i_l{}^{(1)}=\frac{eh}{2\pi mc}\,\mathbf{curl}_l\,(\Psi^*\mathbf{s}_l\Psi), \tag{4.042}$$

where $\mathbf{s}_l$ is the spin of the l'th electron, $\mathbf{i}_l{}^{(0)}$ its orbital current, and $\mathbf{i}_l{}^{(1)}$ its spin current. In all cases of practical interest (ion in crystals) the orbital current is either absent (s state) or quenched. The magnetic properties of the ion are almost wholly due to the electron spins. We shall therefore in what follows use $\mathbf{i}_l{}^{(1)}$ for the electron current. Inserting it into (4.03), changing the origin of coordinates for the integration over the space of the neutron to $\mathbf{r}_l$, and integrating by parts over the electron coordinates we obtain

$$(\mathbf{k}'s'A'|V|\mathbf{k}sA)=-i\left(\frac{eh}{2\pi mc}\right)\int\exp\left(i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}\right)\times\frac{(s'|\mathbf{\mu}|s)\times\mathbf{r}}{|\mathbf{r}|^3}\cdot\left[(\mathbf{k}-\mathbf{k}')\times\sum_l\int\Psi_{A'}{}^*\exp\left(i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_l\right)\mathbf{s}_l\Psi_A d\tau_{\text{atom}}\right]d\tau. \tag{4.05}$$

The integration over $\mathbf{r}$ is now easily carried out with the result

$$(\mathbf{k}'s'A'|V|ksA)=4\pi\left(\frac{eh}{2\pi mc}\right)\left(\frac{eh\gamma}{2\pi M_0c}\right)\times[(s'|\mathbf{s}|s)\times\mathbf{e}]\cdot[\mathbf{e}\times(A'|\mathbf{P}|A)], \tag{4.06}$$

$$\mathbf{e}=\frac{\mathbf{k}-\mathbf{k}'}{|\mathbf{k}-\mathbf{k}'|}, \tag{4.07}$$

$$\mathbf{P}=\sum_l\exp\left(i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_l\right)\mathbf{s}_l, \tag{4.08}$$

where $\mathbf{s}$ stands for the spin of the neutron and γ for its moment in nuclear magnetons. In the limit of long wave-lengths the exponential in (4.08) can be replaced by unity. $\mathbf{P}$ then becomes the total spin of the ion. Substitution of the matrix element into the wave function (4.00) gives the result

$$\psi_m=\left(\frac{2\pi M_0}{hk}\right)^{\frac{1}{2}}\left(\frac{2e^2\gamma}{mc^2}\right)\sum_{s'A'}\frac{e^{ik'r}}{r}\chi_{s'}\Psi_{A'}\times(s'A'|(\mathbf{e}\cdot\mathbf{s})(\mathbf{e}\cdot\mathbf{P})-\mathbf{P}\cdot\mathbf{s}|sA). \tag{4.1}$$

If the ground state of the ion is alone involved in the collision the expression (4.1) may be further simplified. Let Ω_M be the spin wave function describing the orientation of the spin $\mathbf{S}$ of the ground state and let B stand for the remaining atomic quantum numbers so that M and B replace the totality of quantum numbers A in (4.1). We then have by a well-known result in the algebra of vector coupling[6]

$$(M'B|\mathbf{P}|MB)=(M'|\mathbf{S}|M)\cdot F^{\frac{1}{2}},$$

$$F^{\frac{1}{2}}=\left(B\left|\frac{\mathbf{P}\cdot\mathbf{S}}{S(S+1)}\right|B\right)=\left(B\left|\sum_l\exp\left(i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_l\right)\frac{\mathbf{s}_l\cdot\mathbf{S}}{S(S+1)}\right|B\right). \tag{4.2}$$

We have with the aid of (4.2) obtained an expression for $(A|\mathbf{P}|A)$ as a product of two factors, one of which refers to the spin coordinates only, while the other deals with the remaining coordinates. The quantity F which now refers to all the electrons in the atom can be

[5] W. Pauli, article on quantum mechanics in Springer's *Handbuch der Physik*, Vol. 24-A, p. 238.

[6] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, 1935), p. 59.

called the atomic form factor. It reduces to unity if $\lambda/2\pi$ is large compared to the linear size of the electron distribution. This condition is in practical cases not even satisfied by liquid-air neutrons. For elastic transitions $k'=k$ and F then becomes unity in the forward direction and decreases to its smallest value for scattering under 180° ($\mathbf{k}'=-\mathbf{k}$). Since, under the last made assumptions, the atomic system can only change its spin state M during collision, we may omit the part of the wave function referring to the other internal coordinates and can write (4.1) in the form

$$\psi_m=\left(\frac{2\pi M_0}{hk}\right)^{\frac{1}{2}}\left(\frac{2e^2\gamma}{mc^2}\right)\sum_{s'M'}\frac{e^{ik'r}}{r}\chi_{s'}\Omega_{M'}\times(s'M'|\mathfrak{M}|sM)F^{\frac{1}{2}} \quad (4.3)$$

$$\mathfrak{M}=(\mathbf{e}\cdot\mathbf{s})(\mathbf{e}\cdot\mathbf{S})-\mathbf{s}\cdot\mathbf{S}. \quad (4.4)$$

If all collisions are elastic so that $F^{\frac{1}{2}}$ is the same for every term under the summation sign, (4.3) can be written

$$\psi_m=\left(\frac{2\pi M_0}{hk}\right)^{\frac{1}{2}}\left(\frac{2e^2\gamma F^{\frac{1}{2}}}{mc^2}\right)\frac{e^{ikr}}{r}\mathfrak{M}\chi_s\Omega_M. \quad (4.5)$$

V. Scattering From Independent Paramagnetic Ions

The formulae for magnetic scattering as derived in the last paragraph are far too general to permit a direct physical discussion. We shall investigate in this and the following sections a number of special cases which allow a more explicit evaluation as well as a direct physical interpretation. These special cases will also form the theoretical basis for various groups of experiments. Unfortunately we can proceed quantitatively only in selected circumstances and have to limit ourselves frequently to qualitative discussions.

Beginning with the treatment of paramagnetic substances especially salts containing magnetically active ions, we remember that, in the solid state, the levels of the ions are as a rule strongly affected by interionic forces. If e.g. the ground state of an ion forms a multiplet having an orbital angular momentum $\mathbf{L}$ and a spin angular momentum $\mathbf{S}$, then the crystalline field will, as a rule, decouple $\mathbf{L}$ and $\mathbf{S}$ and quench the orbital currents. This is the reason why we retained in (4.05) the spin current only. In most cases the states associated with different orientations of $\mathbf{L}$ are split and differ from each other by energies large compared to kT. The ion then will be normally found in the lowest of these levels. Since energetic considerations make it impossible for a thermal neutron to transfer the ion to higher states, only the ground state will be active during the collision. We shall in this section assume throughout that the ionic spin is uninfluenced by forces due to the neighboring atoms; also we shall, at first, suppose that the spin is completely free from internal forces. The ground state of the magnetic ion is then degenerate and Eq. (4.4) can be applied directly.

To illustrate coherence we treat the interference between the magnetic scattering by two ions a and b which are situated at the points $\mathbf{r}_a$ and $\mathbf{r}_b$. According to (3.30) it is for the interference questions only necessary to consider that part of the scattered wave $\psi_m{}^1$ in which the ionic spin is unchanged. By expanding $\psi_m{}^1$ in terms of the spin states M of the ion, we find for the part of the wave in which the spin state is unchanged by the scattering process the expression

$$\psi_{ma}'=\left(\frac{2\pi M_0}{kk}\right)^{\frac{1}{2}}\left(\frac{2e^2\gamma F^{\frac{1}{2}}}{mc^2}\right)|r-r_a|^{-1}\times\exp\left(ik|\mathbf{r}-\mathbf{r}_a|+i\mathbf{k}\cdot\mathbf{r}_a\right)\times(M|\mathfrak{M}|M)_a\chi_s\Omega_M{}^a. \quad (5.0)$$

Since the ionic spin has no preferred direction in space, we obtain

$$(M|\mathbf{S}|M)_a=0, \quad (5.01)$$

$$\psi_{ma}{}^1=0. \quad (5.02)$$

The result (5.01) is valid as long as the medium is far from magnetic saturation. In accordance with (5.02) the interference terms are zero and all the magnetic scattering is incoherent. There is no interference between the scattering from two ions in a lattice or between the magnetic and nuclear scattering of the same ion. The consequent additivity of nuclear and magnetic scattering from paramagnetic substances forms the basis for the method of separating the two effects as mentioned in the introduction.

The scattering by a single ion is determined by (4.05) and (1.03). We obtain

$$F_m=r^{-2}\left(\frac{2c^2\gamma F^{\frac{1}{2}}}{mc^2}\right)^2[(\mathfrak{M}\chi_s\Omega_M)^*(\mathfrak{M}\chi_s\Omega_M)]_{\text{av}}. \quad (5.1)$$

By use of the Hermitian property of $\mathbf{S}$, (5.1) becomes

$$F_m=r^{-2}\left(\frac{2e^2\gamma F^{\frac{1}{2}}}{mc^2}\right)^2(M|(\mathfrak{M}'\chi_s)^*(\mathfrak{M}\chi_s)|M), \quad (5.2)$$

$$\mathfrak{M}'=(\mathbf{e}\cdot\mathbf{s})(\mathbf{e}\cdot\mathbf{S})^*-\mathbf{s}\cdot\mathbf{S}^*. \quad (5.21)$$

Just as in the case of the nuclear spin, the random orientation of $\mathbf{S}$ implies

$$(M|S_xS_x|M)=\tfrac{1}{3}S(S+1), \quad (5.22)$$

$$(M|S_xS_y|M)=0, \text{ etc.} \quad (5.23)$$

Eq. (5.2) now becomes

$$F_n=\tfrac{1}{3}S(S+1)r^{-2}\left(\frac{2c^2\gamma F^{\frac{1}{2}}}{mc^2}\right)^2$$
$$\times[\mathbf{s}^*\chi_s^*\cdot\mathbf{s}\chi_s-(\mathbf{e}\cdot\mathbf{s})^*\chi_s^*(\mathbf{e}\cdot\mathbf{s})\chi_s]. \quad (5.3)$$

The total scattering is found from (5.3) by summing over the spin coordinates of the neutron. Again the Hermitian character of $\mathbf{s}$ gives

$$\sum_{\text{spin}}\mathbf{s}^*\chi_s\cdot\mathbf{s}\chi_s=\tfrac{3}{4}, \quad (5.31)$$

$$\sum_{\text{spin}}(\mathbf{e}\cdot\mathbf{s})^*\chi_s^*(\mathbf{e}\cdot\mathbf{s})\chi_s=\sum_{\text{spin}}\chi_s^*(\mathbf{e}\cdot\mathbf{s})(\mathbf{e}\cdot\mathbf{s})\chi_s=\tfrac{1}{4}. \quad (5.32)$$

Eq. (5.3) then becomes

$$\sum_{\text{spin}}F_m=r^{-2}\left(\frac{e^2\gamma F^{\frac{1}{2}}}{mc^2}\right)\tfrac{2}{3}S(S+1). \quad (5.4)$$

We finally have for the differential cross section for scattering into the solid angle $d\Omega$ the expression

$$d\Phi_m=\tfrac{2}{3}S(S+1)\left(\frac{e^2\gamma F^{\frac{1}{2}}}{mc^2}\right)^2 d\Omega. \quad (5.41)$$

This cross section depends on the angle of scattering only in virtue of the form factor and is independent of the initial spin state of the neutron. In the limit of long wave-lengths the form factor F approaches one and we obtain for the integrated cross section the simple expression

$$\sigma_m=\frac{8\pi}{3}S(S+1)\left(\frac{e^2\gamma}{mc^2}\right)^2, \quad (5.5)$$

which may be large in comparison with nuclear cross sections.

Equation (5.3) also allows a study of the change in polarization produced by the scattering process. Placing (3.91) for the spin state of the incident neutron into the expression (5.3) we find, for the scattered flux

$$F_m=\tfrac{1}{3}S(S+1)\left(\frac{e^2\gamma F^{\frac{1}{2}}}{mc^2}\right)^2\{[(1-e_z^2)|a|^2$$
$$+(1+e_z^2)|b|^2-K]\alpha^*\alpha+[(1-e_z^2)|b|^2$$
$$+(1+e_z^2)|a|^2+K]\beta^*\beta\}, \quad (5.60)$$

$$K=e_z[a^*b(e_x-ie_y)+ab^*(e_x+ie_y)]. \quad (5.61)$$

If we now express a and b in terms of the angles θ, ϕ, we have

$$K=2\cos\theta/2\sin(\theta/2)e_z(e_x\cos\varphi+e_y\sin\varphi)$$
$$=(\mathbf{e}\cdot\boldsymbol{\lambda}-e_z\lambda_z)e_z. \quad (5.62)$$

Here $\boldsymbol{\lambda}$ is a unit vector in the direction of polarization of the incident neutrons. By remembering the relations

$$|b|^2-|a|^2=\sin^2\theta/2-\cos^2\theta/2$$
$$=-\cos\theta=-\lambda_z, \quad (5.63)$$

we can write (5.60) in the form

$$F_m=\tfrac{1}{3}S(S+1)r^{-2}\left(\frac{e^2\gamma F^{\frac{1}{2}}}{mc^2}\right)^2$$
$$\times\{[1-e_z(\mathbf{e}\cdot\boldsymbol{\lambda})]\alpha^*\alpha+[1+e_z(\mathbf{e}\cdot\boldsymbol{\lambda})]\beta^*\beta\}. \quad (5.7)$$

If we designate by $\boldsymbol{\lambda}'$ the direction along which the scattered beam is analyzed, we find for its polarization P the value

$$P=-(\mathbf{e}\cdot\boldsymbol{\lambda})(\mathbf{e}\cdot\boldsymbol{\lambda}'). \quad (5.8)$$

Eq. (5.8) allows a simple interpretation: the part of the original polarization which is parallel to $\mathbf{e}$ is reversed in the scattered beam while the part which is perpendicular to $\mathbf{e}$ is completely depolarized.

Let us next examine in a general manner the modification of these results required by the presence of internal forces acting on the ionic spin. Such forces are present for all ions of the ion group except Fe^{+++} and Mn^{++} which are in an S state; they arise from the incomplete decoupling of the spin $\mathbf{S}$ and the orbital angular momentum; their interaction normally causes the separation of the multiplet states formed from $\mathbf{L}$ and $\mathbf{S}$. This incomplete decoupling can produce an energy separation of the orientation states of the spin which may result in a more or less complete quenching of the spin currents also. Since anomalies in the magnetic susceptibility frequently begin around 70°K, the separations are of the order of magnitude of $50k$. Consequently a part of the neutron scattering will be inelastic and involve energy changes of a similar amount.

For the reason indicated in Section I we use the neutron density I in dealing with inelastic collisions. The density of scattered neutrons is constructed by summing $|\psi_m|^2$ over the spin variables of the ion and the neutron and dividing by the density of incident neutrons, i.e., $2\pi M_0/kk$. We find

$$\frac{I_{sc}}{I_{in}}=r^{-2}\left(\frac{2e^2\gamma}{mc^2}\right)^2\sum_{s'M'}|(s'M'|\mathfrak{M}|sM)F^{\frac{1}{2}}|^2. \quad (5.90)$$

The form factor F has been included under the summation sign as it depends on the energy change in each transition and is to be taken zero for those conditions which are energetically impossible. The quantity (5.90) must be averaged over the orientation states of the ionic spin in accordance with the Boltzmann distribution law. If the wave-length of the neutron suffers a small change on scattering, F is nearly the same for all transitions and the summation yields our former result for the total scattering. If the scattering becomes inelastic, the influence of this loss of energy varies with the scattering angle. The forward scattering is diminished from that given by (5.41) because $\mathbf{k}-\mathbf{k}'$ is no longer zero when $\mathbf{k}$ and $\mathbf{k}'$ are parallel, and this leads to a decrease in the value for F. Transitions in which the neutron gains energy $k'>k$ decrease in frequency for all scattering angles while transitions in which the neutron loses energy, $k'<k$ are more frequent in the case of backward scattering.

It is of interest to see how much of the scattering given by (5.90) is elastic. Summing over the spin variable of the neutron in that term of (5.90) for which $M=M'$, one obtains

$$I_{sc}(\text{elastic})/I_{in}$$
$$=r^{-2}\left(\frac{2e^2\gamma F}{mc^2}\right)^2\tfrac{3}{4}[(M|\mathbf{S}|M)\cdot(M|\mathbf{S}|M)-(M|\mathbf{e}\cdot\mathbf{S}|M)]. \quad (5.91)$$

The expression (5.91) again must be averaged over the temperature distribution of the ions among their orientation states. If the spins are fully quenched in all these states, we have

$$(M|\mathbf{S}|M)=0, \quad (M|S_x|M)^2=0, \quad (5.92)$$

which means that all the scattering is inelastic. The greatest decrease will be shown for scattering through small angles and particularly so if the neutron energy becomes smaller than the energy difference between the orientation states. In this latter case only hyperelastic collisions could occur which are made less frequent if these higher orientation states are less populated because of a lowering of the temperature. If the quenching of the spin current is incomplete, intermediary results have to be expected. This brief discussion is sufficient to show the great complexity which even in absence of a coupling between the spins of different ions must be expected in the scattering of very slow neutrons unless the ion is originally in an S state.

VI. Scattering by an Individual Ion in a Ferromagnetic Material

A ferromagnetic material below the Curie point consists of microcrystals each of which shows an alignment of the spins of its ions along a definite but arbitrary axis. We shall assume as basis of our discussions that each spin in the microcrystal is parallel to the axis, the direction of which shall be designated by the unit vector $\boldsymbol{\kappa}$. In addition, it shall be supposed that the spins are rigidly aligned so that the initial kinetic energy of the neutron is insufficient to change the orientation of an ionic spin. Elastic collisions in which the ionic spin state is unchanged are then alone possible for energetic reasons. The condition that the scattering system remains unchanged during the scattering process underlies the previous discussions by Bloch and Schwinger.

If the ferromagnetic medium is magnetized to saturation the vectors, $\boldsymbol{\kappa}$, of all microcrystals become parallel while in the unmagnetized state they are oriented at random. The discussion of a change in the spin orientation of a whole microcrystal will be taken up later.

Let us designate by C the amplitude of the coherent nuclear scattering

$$C=\sum_P |b_P|^2(2i_P+1)^{-1}[i_Pa_0{}^P+(i_P+1)a_1{}^P]. \quad (6.0)$$

If E^2 be the quantity

$$E^2=\sum_P |b_P|^2(2i_P+1)^{-1} \times i_P(i_P+1)(a_1{}^P-a_0{}^P)^2, \quad (6.01)$$

which is proportional to the intensity of incoherent scattering, then the total nuclear scattering is given by E^2+C^2. Here we have neglected the possible effect of the nuclear form factor F_N of (3.4) as it is probably not very large, at least for the forward scattering and sufficiently heavy nuclei.

For the magnetic scattering the relation (4.5) is here applicable. The spin function Ω_M of the initial state of the ion must represent the ionic spin parallel to the axis $\boldsymbol{\kappa}$. Since transitions to other ionic spin states do not enter, we may write

$$\mathbf{S}\Omega_M=S\boldsymbol{\kappa}\Omega_M. \quad (6.10)$$

The spin function Ω_M may therefore be omitted in the incident and scattered wave. The magnetically scattered wave is now given by

$$\psi_m=\left(\frac{2\pi M_0}{h\mathbf{k}}\right)^{\frac{1}{2}} 2Dr^{-1}e^{ikr}\mathbf{q}\cdot\mathbf{s}\chi_s \quad (6.20)$$

with the abbreviations

$$D=\frac{e^2\gamma S}{mc^2}F, \quad (6.21)$$

$$\mathbf{q}=\mathbf{e}(\mathbf{e}\cdot\boldsymbol{\kappa})-\boldsymbol{\kappa}. \quad (6.22)$$

The sum of the coherent nuclear and magnetically scattered wave now becomes

$$\psi_{sc}=\left(\frac{2\pi M_0}{hk}\right)^{\frac{1}{2}} r^{-1}e^{ikr}(C+2D\mathbf{q}\cdot\mathbf{s})\chi_s. \quad (6.30)$$

The vector $\mathbf{q}$ is *not* a unit vector but has the magnitude

$$q=[1-(\mathbf{e}\cdot\boldsymbol{\kappa})^2]^{\frac{1}{2}}. \quad (6.31)$$

It is perpendicular to $\mathbf{e}$ and has a projection on $\boldsymbol{\kappa}$ equal to $q^2\leqq 1$.

The scattering by a single ion is now described by considering the flux associated with the wave function. Thus

$$\mathbf{F}_{sc}=r^{-2}[(C+2D\mathbf{q}\cdot\mathbf{s})^*\chi_s{}^*] \times[(C+2D\mathbf{q}\cdot\mathbf{s})\chi_s]. \quad (6.32)$$

To obtain the total scattering cross section, (6.32) must be summed over the neutron spin coordinate. Using the hermitian property of $\mathbf{s}$ and the relation

$$(\mathbf{q}\cdot\mathbf{s})(\mathbf{q}\cdot\mathbf{s})=\tfrac{1}{4}q^2 \quad (6.321)$$

we find

$$\sum_{\text{spin}} \mathbf{F}_m=r^{-2}[C^2+q^2D^2+\sum_{\text{spin}} \chi_s{}^*4CD\mathbf{q}\cdot\mathbf{s}\chi_s]. \quad (6.33)$$

The corresponding differential cross section for scattering into a solid angle $d\Omega$ is then

$$d\Phi=[C^2+D^2q^2+\sum_{\text{spin}} 4CD\chi_s{}^*\mathbf{q}\cdot\mathbf{s}\chi_s]d\Omega. \quad (6.34)$$

Here, for the first time, the scattering in a given direction depends upon the initial spin state of a neutron. It is remarkable that the distinguished direction is not characterized by the direction of the ionic spin alone, but rather by the vector $\mathbf{q}$ as defined by (6.22).

For an unpolarized incident beam the spin of the neutron has no preferred axis. The last term on the right side of (6.34) which involves the matrix elements $(s|\mathbf{s}|s)$ linearly, therefore vanishes and the differential cross section becomes

$$d\Phi=(C^2+D^2q^2)d\Omega. \quad (6.40)$$

Eq. (6.40) expresses the fact that as far as the total scattering is concerned nuclear and magnetic scattering of an unpolarized beam superpose their intensity.

For an *unmagnetized* medium we must average (6.32) and (6.34) over all directions of $\boldsymbol{\kappa}$ which gives

$$q_{\mathrm{av}}^2=\tfrac{2}{3}. \qquad (6.41)$$

If we pass from an unmagnetized to a completely magnetized state of the medium, the differential cross section $d\Phi$ given by (6.34) will increase or decrease from its average value

$$(C^2+\tfrac{2}{3}D^2)d\Omega, \qquad (6.42)$$

depending upon the direction of scattering. It will become a maximum for $q=1$, i.e., $\mathbf{e}$ perpendicular to $\boldsymbol{\kappa}$. It will be a minimum for $q=0$, i.e., $\mathbf{e}$ parallel to $\boldsymbol{\kappa}$. The vector $\mathbf{e}$ lies in the plane defined by the directions of incidence and scattering, $\mathbf{k}$ and $\mathbf{k}'$, and is perpendicular to $\mathbf{k}$ for scattering through small angles. Thus, the first case can be realized for small angle scattering if the medium is magnetized transversely, perpendicular to the plane defined by $\mathbf{k}'$ and $\mathbf{k}$; the second case if the medium is transversely magnetized in the plane of $\mathbf{k}'$ and $\mathbf{k}$.

If the incident beam is completely polarized in an arbitrary direction defined by a unit vector $\boldsymbol{\lambda}$ we have to proceed as follows. We use the spin wave functions which characterize states with a neutron spin parallel to $\boldsymbol{\lambda}$ and obtain

$$2\sum_{\text{spin}}\chi_s^*(\mathbf{s})\chi_s=2(s|\mathbf{s}|s)=\boldsymbol{\lambda}. \qquad (6.43)$$

This result is, of course, independent of the direction along which we analyze the spin. Eq. (6.34) now takes on the form

$$d\Phi=[C^2+D^2q^2+2CD\mathbf{q}\cdot\boldsymbol{\lambda}]d\Omega. \qquad (6.5)$$

When $\mathbf{q}$ is kept fixed, the cross section obviously becomes a maximum for $\boldsymbol{\lambda}$ parallel to $\mathbf{q}$ if the amplitudes C and D have the same sign, and a minimum if $\boldsymbol{\lambda}$ is antiparallel to $\mathbf{q}$. If C and D have opposite signs, the behavior is just reversed. In the first case discussed in the previous paragraph, $\mathbf{e}\cdot\boldsymbol{\kappa}=0$ $\mathbf{q}=-\boldsymbol{\kappa}$ and so is antiparallel to $\boldsymbol{\kappa}$ and of absolute magnitude unity. In this case the greatest difference of scattering from the parallel and antiparallel polarization states is obtained. The second case mentioned above $\mathbf{e}\cdot\boldsymbol{\kappa}=1$, leads to $q=0$ and therefore to the absence of any magnetic scattering. All polarization states of the neutron are then of course scattered alike. It must be kept in mind that according to its definition $\mathbf{q}$ depends upon the scattering angle and *the azimuth*. It does not therefore represent a fixed direction even for a constant scattering angle. Relations between $\mathbf{q}$ and $\boldsymbol{\kappa}$ therefore hold true only for special scattering angle and azimuth. If, for example the direction of polarization lies along $\mathbf{q}$, $d\Phi$ becomes a perfect square, $d\Phi=(C\pm Dq)^2d\Omega$ in agreement with Bloch's result. But this special expression for the differential cross section cannot be integrated over all angles of scattering for which purpose recourse must be had to the generally valid relation (6.34).

It is interesting to observe that the form factor which becomes important when $d\Phi$ is integrated over all angles has a double influence. On one hand it reduces the total scattering, and on the other it changes the influence of magnetization on the total scattering. Taking for example, as a crude approximation for illustrative purposes, $F=1$ inside a cone of scattering directions making an angle θ_0 with the incident direction, and $F=0$ outside this cone, one obtains for the integrated cross section for an unpolarized beam the expression

$$\sigma=4\pi C^2+\pi D^2\eta[2+\eta-\kappa_z^2(3\eta-2)]. \qquad (6.51)$$

Here the direction of incidence is taken as the z axis and η stands for $\sin^2(\theta_0/2)$. The cross section for an unmagnetized medium is obtained by averaging over $\boldsymbol{\kappa}$

$$\sigma_{\mathrm{unmag}}=4\pi C^2+8\pi D^2\eta/3. \qquad (6.52)$$

The change in scattering, due to magnetization of the medium is given by

$$\sigma-\sigma_{\mathrm{unmag}}=\pi D^2\eta(3\eta-2)(\tfrac{1}{3}-\kappa_z^2). \qquad (6.53)$$

Equation (6.32) allows us to determine the polarization of the scattered neutrons. Using the spin functions of Eq. (3.91) along an arbitrarily chosen z axis we find by substitution into (6.32) that

$$\begin{aligned}\mathbf{F}_{\mathrm{sc}}=r^{-2}\{&[(C+Dq_z)^2|a|^2\\&+D^2(q^2-q_z^2)|b|^2+D(C+Dq_z)\\&\times(a^*b(q_x-iq_y)+ab^*(q_x+iq_y))]\alpha^*\alpha\\&+[(C-Dq_z)^2|b|^2+D^2(q^2-q_z^2)|a|^2\\&+D(C-Dq_z)(a^*b(q_x-iq_y)\\&\qquad+ab^*(q_x+iq_y)]\beta^*\beta\}.\end{aligned} \qquad (6.60)$$

If we express a and b in terms of the polarization angles θ, ϕ and let $\boldsymbol{\lambda}$ be a unit vector along the direction of polarization, we find

$$a^*b(q_x-iq_y)+ab^*(q_x+iq_y)$$
$$=2\cos\theta/2\sin\theta/2(q_x\cos\phi+q_y\sin\phi)$$
$$=\boldsymbol{\lambda}\cdot\mathbf{q}-q_z(|a|^2-|b|^2). \quad (6.61)$$

The coefficients of $\alpha^*\alpha$ and $\beta^*\beta$ become

$$N_\alpha=r^{-2}[C^2|a|^2+D^2q^2|b|^2+CDq_z+D(C+Dq_z)\boldsymbol{\lambda}\cdot\mathbf{q}], \quad (6.62)$$

$$N_\beta=r^{-2}[C^2|b|^2+D^2q^2|a|^2-CDq_z+D(C-Dq_z)\boldsymbol{\lambda}\cdot\mathbf{q}]. \quad (6.63)$$

The sum $N_\alpha+N_\beta$ leads at once to (6.5). The polarization of a scattered beam defined by

$$P=(N_\alpha-N_\beta)(N_\alpha+N_\beta)^{-1} \quad (6.64)$$

now becomes equal to

$$P_{\lambda'}=\frac{(C^2-D^2q^2)\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}'+2CD\boldsymbol{\lambda}'\cdot\mathbf{q}+2D^2(\boldsymbol{\lambda}\cdot\mathbf{q})(\boldsymbol{\lambda}'\cdot\mathbf{q})}{C^2+D^2q^2+2CD\boldsymbol{\lambda}\cdot\mathbf{q}}, \quad (6.65)$$

where $\boldsymbol{\lambda}'$ denotes the axis along which the spin is analyzed. If the original polarization is parallel to $\mathbf{q}$, $P_{\lambda'}$ becomes equal to $\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}'$ which shows that the scattering *in this exceptional case* leaves the polarization unaltered. In all other cases the direction and amount of polarization are altered. Of special interest is the case of an initially unpolarized beam. Here it is necessary to average N_α and N_β over the incident polarization direction

$$(N_\alpha)_{\text{av}}=r^{-2}[\tfrac{1}{2}(C^2+D^2q^2)+CD\boldsymbol{\lambda}'\cdot\mathbf{q}],$$
$$(N_\beta)_{\text{av}}=r^{-2}[\tfrac{1}{2}(C^2+D^2q^2)-CD\boldsymbol{\lambda}'\cdot\mathbf{q}]. \quad (6.66)$$

Thus

$$(P_{\lambda'})_{\text{av}}=\frac{2CD\boldsymbol{\lambda}'\cdot q}{C^2+D^2q^2}=\frac{2\boldsymbol{\lambda}'\cdot\mathbf{q}}{q}\left[\frac{C}{Dq}+\frac{Dq}{C}\right]^{-1}. \quad (6.67)$$

The polarization can be complete only if $C=Dq$. It is a symmetrical function of the ratios C/Dq and Dq/C and becomes small whenever either of these quantities is large.

If we want to study intensity and polarization of a transmitted beam, we must average (6.60) and (6.67) over all angles of scattering. The results would consist of so lengthy formulae that we limit ourselves to the discussion of the cross section in the special case of a transverse field, i.e., $\mathbf{k}\cdot\boldsymbol{\kappa}=0$. From the previously mentioned crude approximation for $F^{\frac{1}{2}}$, we find

$$\sigma_{\lambda'}=\pi\{\tfrac{1}{2}[4C^2+D^2\eta(2+\eta)]+\tfrac{1}{2}[4C^2-D^2\eta(2+\eta)]\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}'-CD\eta(2+\eta)(\boldsymbol{\lambda}+\boldsymbol{\lambda}')\cdot\boldsymbol{\kappa}+D^2\eta[(1+\eta+\tfrac{1}{3}\eta^2)(\boldsymbol{\lambda}\cdot\boldsymbol{\kappa})(\boldsymbol{\lambda}'\cdot\boldsymbol{\kappa})+\tfrac{1}{2}(-1+\eta+\tfrac{1}{3}\eta^2)\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}'+\tfrac{1}{2}(-1+3\eta-5\eta^2/3)\times(\boldsymbol{\lambda}\cdot\mathbf{k}/k)(\boldsymbol{\lambda}'\cdot\mathbf{k}/k)]\}. \quad (6.71)$$

The total cross section for scattering into both polarization states λ' and $-\lambda'$ is obtained by adding $\sigma_{\lambda'}$ and $\sigma_{-\lambda'}$

$$\sigma=\sigma_{\lambda'}+\sigma_{-\lambda'}=\pi[4C^2+D^2\eta(2+\eta)-2CD\eta(2+\eta)\boldsymbol{\lambda}\cdot\boldsymbol{\kappa}]. \quad (6.72)$$

The result (6.72) can also be obtained by integrating (6.5) over all angles of scattering.

So far our attention has been confined to single scattering processes. The treatment of double (or multiple) scattering is only possible for very restricted cases. It is easily seen that large polarization effects may be expected in two successive single scattering processes. If the first scattering occurs under the small forward angle and such an azimuth that $\mathbf{e}\cdot\boldsymbol{\kappa}=0$, the scattered beam according to (6.65) is almost completely polarized if we assume for illustrative purposes and probably as an approximation to the case of Fe that $C=2D$. If $\mathbf{e}\cdot\boldsymbol{\kappa}=0$ for the second scattering also and if the polarization is parallel or antiparallel to the magnetic field at the second scatterer, the cross sections become

$$d\Phi=(C+D)^2d\Omega \quad (6.731)$$

or

$$d\Phi=(C-D)^2d\Omega, \quad (6.732)$$

respectively; consequently reversal of the polarization state relative to the magnetic field of the second scatterer can cause the intensity to change by a factor of 9 in our assumed case.

To obtain a precise estimate it is necessary to treat the double scattering *without averaging* over the initial polarization state until the end ("interference of amplitudes"). By omitting factors referring to the space coordinates the spin wave function of a twice scattered neutron becomes

$$\psi_{sc}=(C+2D\mathbf{q}_1\cdot\mathbf{s})(C+2D\mathbf{q}_2\cdot\mathbf{s})\chi_s. \quad (6.74)$$

Constructing then the flux as before, we find for an initially unpolarized beam

$$\sum_{\text{spin}} F_{sc}=\text{constant}\times[C^4+D^4q_1^2q_2^2 +C^2D^2(2\mathbf{q}_1\cdot\mathbf{q}_2+|\mathbf{q}_1+\mathbf{q}_2|^2)]. \quad (6.75)$$

For the conditions mentioned above $q_1=q_2=1$ and q_1 is parallel or antiparallel to q_2. The ratio of intensities then becomes 41 : 9.

In his treatment of double scattering Schwinger[3] used the conditions $\mathbf{e}_1\cdot\boldsymbol{\kappa}_1=\mathbf{e}_2\cdot\boldsymbol{\kappa}_2=0$ to simplify his formulae and added the remark that they describe a convenient set of experimental conditions. It deserves mentioning that these conditions give rise to the maximum effect as the magnetic scattering is then greatest. Furthermore it should be noted that the theoretical treatment is strictly correct only under these same conditions. In all other cases the scattered neutrons are not polarized parallel to the magnetic field, and there results a precession of the spin about the direction of the field as the neutron passes from the first to the second scatterer. Only the component of the spin parallel to the field is preserved; the perpendicular components average to zero because of the varying neutron velocities and different distances between points in the two scatterers. The polarization at the second scatterer is thereby reduced. In our discussion of polarization and double scattering these precessions must be taken into account, a task to which we now address ourselves.

The wave equation of a neutron moving in a constant magnetic field is

$$\left\{-\frac{h^2}{8\pi^2M_0}\nabla^2+\gamma\frac{eh}{2\pi M_0c}\mathbf{s}\cdot\mathbf{H}-E\right\}\psi=0. \quad (6.80)$$

By use of spin functions α and β representing states of spin parallel and antiparallel $\mathbf{H}$, this equation has the general solution

$$\psi=\exp(i\mathbf{k}\cdot\mathbf{r})\left\{a\alpha\exp\left[\frac{i\omega}{2v}\left(\frac{\mathbf{k}\cdot\mathbf{r}}{k}\right)\right] +b\beta\exp\left[-\frac{i\omega}{2v}\left(\frac{\mathbf{k}\cdot\mathbf{r}}{k}\right)\right]\right\}, \quad (6.81)$$

$$\omega=\gamma(eH/M_0c). \quad (6.811)$$

Here $\mathbf{H}$ is the homogeneous magnetic field in the ferromagnet which we have omitted from our previous considerations since the scattering is caused entirely by the atomic fields. We assume the field to be either constant or to change so abruptly that the passage of the neutrons through the regions of inhomogeneity can be treated as a quasi-instantaneous process.

According to (6.81) the two components of the spin function of the neutron

$$\chi_s=a\alpha+b\beta \quad (6.812)$$

have phase factors which vary from point to point along the trajectory of the neutron. This result can be put into a more convenient form by means of the operators $\frac{1}{2}(1+2\boldsymbol{\kappa}\cdot\mathbf{s})$ where $\boldsymbol{\kappa}$ is, as usual, a unit vector parallel to $\mathbf{H}$. These operators obviously have the following properties

$$\begin{aligned}\tfrac{1}{2}(1+2\boldsymbol{\kappa}\cdot\mathbf{s})\alpha&=\alpha, & \tfrac{1}{2}(1-2\boldsymbol{\kappa}\cdot\mathbf{s})\beta&=\beta,\\ \tfrac{1}{2}(1+2\boldsymbol{\kappa}\cdot\mathbf{s})\beta&=0, & \tfrac{1}{2}(1-2\boldsymbol{\kappa}\cdot\mathbf{s})\alpha&=0.\end{aligned} \quad (6.82)$$

Hence

$$\psi=\exp(i\mathbf{k}\cdot\mathbf{r})\left\{\tfrac{1}{2}(1+2\boldsymbol{\kappa}\cdot\mathbf{s})\exp\left[\frac{i\omega}{2v}\left(\frac{\mathbf{k}\cdot\mathbf{r}}{k}\right)\right] \times\tfrac{1}{2}(1-2\boldsymbol{\kappa}\cdot\mathbf{s})\exp\left[-\frac{i\omega}{2v}\left(\frac{\mathbf{k}\cdot\mathbf{r}}{k}\right)\right]\right\}(a\alpha+b\beta). \quad (6.83)$$

If χ_s' is the spin function of the neutron at the point $\mathbf{r}_1$ of its trajectory and χ_s its spin function in another point $\mathbf{r}_2$ then

$$\chi_s'=\left[\cos\frac{\omega r}{2v}+2i\boldsymbol{\kappa}\cdot\mathbf{s}\sin\frac{\omega r}{2v}\right]\chi_s, \quad (6.84)$$

where r denotes the distance along the trajectory between the points $\mathbf{r}_1$ and $\mathbf{r}_2$.

If now χ_s denotes the spin function of the incident neutron at the location of the scatterer, the scattered wave is given by

$$\psi_{sc}=\left(\frac{2\pi M_0}{hk}\right)^{\frac{1}{2}}\frac{e^{ikr}}{r}\left[\cos\frac{\omega r}{2v}+2i\boldsymbol{\kappa}\cdot\mathbf{s}\sin\frac{\omega r}{2v}\right]\times(C+2D\mathbf{q}\cdot\mathbf{s})\chi_s. \quad (6.85)$$

The polarization of the scattered wave can be studied exactly as before; the general result is of course considerably more complicated than (6.65). Because the neutrons have traveled over varying distances with varying velocities, it is permissible to put

$$\cos^2\omega r/2v=\sin^2\omega r/2v=\tfrac{1}{2}, \quad (6.851)$$

$$\cos\omega r/2v\cdot\sin\omega r/2v=0. \quad (6.852)$$

By taking the important special case $\mathbf{e}\cdot\boldsymbol{\kappa}=0$, the general equation for the polarization reduces to

$$P_{\lambda'}=\boldsymbol{\lambda}'\cdot\boldsymbol{\kappa}\frac{(C^2+D^2)\boldsymbol{\lambda}\cdot\boldsymbol{\kappa}-2CD}{C^2+D^2+2CD\boldsymbol{\lambda}\cdot\boldsymbol{\kappa}}. \quad (6.86)$$

The polarization $(P_{\lambda'})_{\mathrm{av}}$ for an initially unpolarized beam is then still given by (6.67) with $\mathbf{q}=-\boldsymbol{\kappa}$; this agrees with our expectation that the polarization is unaffected if it is parallel to **H**.

In the double-scattering problem only the effect of the magnetic field on the neutron as it traverses the distance r between the scatterers is of importance because we are interested in the total flux originating from the second scatterer. Hence we may write for the spin function of the doubly scattered wave,

$$\psi_{sc}=\text{const.}\times[C+2D\mathbf{q}_2\cdot\mathbf{s}]\times[\cos(\omega r/2v)+2i\boldsymbol{\kappa}\cdot\mathbf{s}\sin(\omega r/2v)\times[C+2D\mathbf{q}_1\cdot\mathbf{s}]\chi_s. \quad (6.90)$$

The flux is now computed in the familiar way. Again we find that the result in the important case described by $\mathbf{e}_1\cdot\boldsymbol{\kappa}_1=\mathbf{e}_2\cdot\boldsymbol{\kappa}_2=0$ reduces to that previously given.

Only a few words need to be said about the modifications which are introduced if the ionic spin can change its orientation during the collision. Such transitions are very unlikely in Fe, Co, or Ni since the work required to change the spin state is approximately equal to zJ where J is the well-known exchange integral and z the number of neighbors of every ion. The value of zJ lies for the substances mentioned in the neighborhood of a fraction of a volt. Only in a medium with a very low Curie temperature can such transitions be of importance. To treat this problem we construct the density of scattered neutrons $|\psi_m|^2$ from (4.4). Dividing the result by the density of incident neutrons and omitting the term corresponding to elastic scattering, $M'=M$, we have

$$I(\text{inelastic})/I_{\text{in}}=(2e^2\gamma/mc^2)^2\times\sum_{s'}|(M's'|\mathfrak{M}|Ms)F'|^2. \quad (6.91)$$

As the ion is initially parallel to field, $M=S$ and $M'=S-1$. The matrix element of **S** for this transition is given by the well-known formulae

$$\begin{aligned}(S-1|S_x+iS_y|S)&=0,\\(S-1|S_x-iS_y|S)&=(2S)^{\frac{1}{2}},\\(S-1|S_z|S)&=0.\end{aligned} \quad (6.92)$$

By use of (4.4), and substitution of (6.92) into (6.91) we obtain

$$\begin{aligned}I(\text{inelastic})/I_{\text{in}}&=(e^2\gamma F'/mc^2)^2 2S\sum_{s'}|(s'|(\mathbf{e}\cdot\mathbf{s})(e_x+ie_y)-s_x-is_y|s)|^2\\&=(e^2\gamma F'/mc^2)2S\sum_{\text{spin}}\chi_s^*|(\mathbf{e}\cdot\mathbf{s})(e_x+ie_y)-s_x-is_y|^2\chi_s.\end{aligned} \quad (6.93)$$

By using the familiar commutation relations, $\mathbf{s}\times\mathbf{s}=i/2\mathbf{s}$, and the fact $s_x^2=s_y^2=s_z^2=\frac{1}{4}$, this reduces to

$$I(\text{inelastic})/I_{\text{in}}=\left(\frac{e^2\gamma F'}{mc^2}\right)^2\frac{S}{2}\{1+(\mathbf{e}\cdot\boldsymbol{\kappa})^2-2\sum_{\text{spin}}\chi_s^*(\mathbf{e}\cdot\boldsymbol{\kappa})(\mathbf{e}\cdot\mathbf{s})\chi_s\}. \quad (6.94)$$

If the incident beam is unpolarized, the sum in (6.94) vanishes. The intensity becomes a maximum for $\mathbf{e}\cdot\boldsymbol{\kappa}=1$; this condition implies that the elastic magnetic scattering treated previously becomes zero (cf. p. 911). If the incident beam is completely polarized along the direction $\boldsymbol{\lambda}$, the term under summation becomes $-(\mathbf{e}\cdot\boldsymbol{\kappa})(\mathbf{e}\cdot\boldsymbol{\lambda})$.

Consequently states which are polarized antiparallel to **e** will be scattered most; the scattered neutrons will therefore be partially polarized in this direction. The details of the state of polarization can be determined analogously to the cases discussed before.

In dealing with ferromagnetic media, it should finally be remembered that we do not possess a satisfactory theory of the magnetic moment of the metals below the Curie point. The magnetic moments obtained through saturation at low temperatures do not agree at all with the magnetic moments above the Curie point or with the spin which has to be ascribed to the ion for the purpose of accounting theoretically for the saturation curve. These discrepancies have led to the well-known attempts to explain the behavior of the ferromagnetic metal by ascribing several states to the atom (ion) in the metal. The discrepancy becomes the more important since, in no case, does the magnetic moment of metallic Fe approach that of the ions Fe^{+++} or Fe^{++} as observed in salts. It appears therefore that the model for a ferromagnetic medium so far used in the case of neutron scattering may well require drastic modifications. This can affect the neutron scattering principally in two ways. The atomic domain giving rise to the magnetic moment may not be confined to the d shell which would reduce the scattering on account of the form factor. There may also be several ionic levels of different spins involved which would permit inelastic scattering of the neutron. Finally there may occur a considerable amount of quenching of the spin current which is indicated by the experimental fact that the gyromagnetic ratio lies only close to two but differs from it measurably.

VII. The Form Factor for Elastic Collisions

We have seen in IV that the matrix element which determines the transition in the scattering process is essentially defined

$$\mathbf{P}=\sum_e \exp\,[i\mathbf{r}\cdot(\mathbf{k}-\mathbf{k}')]\mathbf{s}_e. \tag{4.08}$$

To evaluate (4.08) we have to know the energy change in the transition as well as the spin density distribution in the initial and final state.

We shall be concerned in this paragraph with elastic scattering only. The numerous difficulties arising in the case of inelastic transitions are qualitatively discussed in Section VIII.

Assuming therefore the same spin distribution for the initial and final state we have to make more or less arbitrary assumptions about the spin density. It is reasonable to take the spin density proportional to the charge density of the outer electrons (cf. 4.042). The charge density on the other hand can be estimated either from semi-theoretical expressions given by the Hartree distribution or with the aid of simple assumptions treating the problem as quasi-hydrogenic.

In the case of greatest interest (Mn^{++}, Fe^{+++}; $j=S=5/2$) we know that the five extra electrons are in a $3d$-state and that the whole distribution is spherically symmetrical. It therefore does not seem to be without interest to represent the charge distribution with the aid of the hydrogenic wave function

$$\psi=Cr^2e^{-\alpha r}, \tag{7.0}$$

where α is in a simple way connected with the effective nuclear charge Z_{eff} or the ionization potential. The most probable radius r_0 is defined by

$$(d/dr)(r^6e^{-2\alpha r})=0, \tag{7.01}$$

$$\alpha=3/r_0. \tag{7.02}$$

It can also be expressed in the form

$$\alpha=Z_{\text{eff}}/3a_0, \tag{7.03}$$

where $a_0=0.52$A stands for the Bohr radius of the ground state of hydrogen. Since for room-temperature neutrons the wave-length λ is approximately 1.45A, the influence of the form factor will be considerable since Z_{eff} will lie between 3 and 6 and $2\pi r_0/\lambda$ therefore will be of the order of magnitude of one.

To find the differential form factor, i.e., the factor with which the cross section for scattering under the angle ϑ has to be multiplied we have to evaluate the integral

$$F^{\frac{1}{2}}_{\text{diff}}=A\int\int \exp\,[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}-2\alpha r] \times r^6\sin\theta dr d\theta \tag{7.1}$$

and to normalize the constant A in such a form that F_{diff} approaches unity for very small values of k. F_{diff} is a function of the scattering angle ϑ alone. Choosing the polar axis along the direction of $\mathbf{k}'-\mathbf{k}$ and introducing spherical coordinates we find for F_{diff} the form

$$F^{\frac{1}{2}}{}_{\mathrm{diff}}=\int_0^\infty r^6 e^{-2\alpha r}dr \times\int_0^\pi \exp\left[2ik\sin\tfrac{1}{2}\vartheta r\cos\theta\right]\sin\theta d\theta, \quad (7.2)$$

which can be immediately evaluated. We so obtain

$$F_{\mathrm{diff}}(\vartheta)=\frac{\left[1-\frac{10}{3}\left(\frac{k\sin\frac{1}{2}\vartheta}{\alpha}\right)^2+\left(\frac{k\sin\frac{1}{2}\vartheta}{\alpha}\right)^4\right]^2}{\left[1+\left(\frac{k\sin\frac{1}{2}\vartheta}{\alpha}\right)^2\right]^{12}}. \quad (7.3)$$

The integral form factor F_{int} follows from (7.3) by integration over ϑ. A trivial but lengthy procedure leads to

$$F_{\mathrm{int}}=I_0+I_1+I_2+I_3+I_4+I_5, \quad (7.40)$$

$$I_0=\frac{4}{77}\left(\frac{\alpha}{k}\right)^2, \quad (7.41)$$

$$I_1=-\left(\frac{\alpha}{k}\right)^2\frac{1-\frac{20}{3}\left(\frac{k}{\alpha}\right)^2+\left(\frac{118}{9}\right)\left(\frac{h}{\alpha}\right)^4-\frac{20}{3}\left(\frac{k}{\alpha}\right)^6+\left(\frac{k}{\alpha}\right)^8}{11\cdot\left[1+\frac{k^2}{\alpha^2}\right]^{11}}, \quad (7.42)$$

$$I_2=\frac{\alpha^2}{k^2}\frac{\frac{20}{3}-\frac{236}{9}\left(\frac{k}{\alpha}\right)^2+20\left(\frac{k}{\alpha}\right)^4-4\left(\frac{k}{\alpha}\right)^6}{11\cdot10\left[1+\left(\frac{h}{\alpha}\right)^2\right]^{10}}, \quad (7.43)$$

$$I_3=-\frac{\alpha^2}{h^2}\frac{\frac{236}{9}-40\left(\frac{k}{\alpha}\right)^2+12\left(\frac{h}{\alpha}\right)^4}{11\cdot10\cdot9\left[1+\left(\frac{k}{\alpha}\right)^2\right]^9}, \quad (7.44)$$

$$I_4=\frac{\alpha^2}{k^2}\frac{40-24(k/\alpha)^2}{11\cdot10\cdot9\cdot8[1+(k/\alpha)^2]^8}, \quad (7.45)$$

$$I_5=-\frac{\alpha^2}{k^2}\frac{24}{11\cdot10\cdot9\cdot8\cdot7[1+(k/\alpha)^2]^7}. \quad (7.46)$$

It seems noteworthy that the differential form factor F_{diff} is not a monotonic function of the angle; it has two zeros which are due to the diffraction from the spherical spin distribution and which are located at the two angles

$$\sin\tfrac{1}{2}\vartheta_1=\alpha/k3^{\frac{1}{2}}, \quad (7.51)$$

$$\sin\tfrac{1}{2}\vartheta_2=3^{\frac{1}{2}}\alpha/k. \quad (7.52)$$

The first zero ϑ_1 makes the expression

$$k\sin\tfrac{1}{2}\vartheta/\alpha$$

equal to $3^{\frac{1}{2}}/3$. The denominator in (7.3) has not yet grown too large so that this zero is of physical interest. The second zero if reached at all, is without significance since the form factor is already unobservably small. The zeros shift with increasing wave-length towards larger angles; their presence is of particular importance for comparatively short wave-length since it then diminishes the scattering under small angles which otherwise is of greatest importance. It is also clear by inspection that for any value of Z_{eff} between 3 and 6, the scattering of room temperature neutrons, is negligible for angles larger than say, 60°.

In the expression for the integral form factor only the term I_0 which does not contain denominators contributes in most cases of practical interest. We see from (7.40) that the total scattering is inversely proportional to the absolute neutron temperature provided that α/k does not become so large that the terms with the denominators in (7.42–7.46) must be taken into account. But even for $\alpha/k\sim2$ these terms would give only a small correction. Furthermore, we see that in our model the integral form factor becomes proportional to Z_{eff}^2. We assume for the purpose of illustration that Z_{eff} equals 6 which probably overrates the screening effect of equivalent electrons and obtain for F_{int} with $\lambda\sim1.45$A

$$F_{\mathrm{int}}\sim1/23.$$

The exact numerical value of the integral form factor as well as the occurrence of diffraction zeros in the differential form factor are of course dependent upon the special model we have chosen; it seems plausible on the other hand that the general features of the angular dependence as well as the big reduction in total scattering is not too badly represented by these calculations. They also allow us to understand why in the case of backwards scattering of neutrons from MnS Whitaker[7] was unable to find any indication of additional magnetic scattering.

VIII. Coupled Scattering Systems

The treatment of neutron-scattering by coupled systems can be carried out quantitatively only in a few special cases. The solution of the more general problem appears to be extremely laborious due to great mathematical difficulties.

Coupling may occur between the spins of the individual ions, between the electronic and nuclear spins of an atom or between spins of the nuclei in a homonuclear molecule. Since the nuclear coupling is most easily considered, we discuss it first.

The coherent part of nuclear scattering (cf. Section III) arises from the terms in (3.51) which do not contain the nuclear spin. It is therefore always present and remains coherent independently of the coupling between the nuclear spin states. Furthermore, in all attainable experimental arrangements the nuclear spin will have no preferred axis; our previous arguments therefore show that this term will give the entire coherent scattering. The additional scattering will be inelastic because of the energy separations introduced by the coupling. If the coupling is due to atomic electrons (hyperfine structure), then the energy separations are, in every case, very small as compared with the neutron energy. The energy change will therefore have no appreciable effect on the atomic form factor. For the same reason, the different states of the scattering system will be equally populated. An obvious application of the principle of spectroscopic stability shows us that the total scattering is under these conditions unaltered by the coupling. Similarly in the homonuclear molecules, the coupling forces are in general too small to have any influence on the scattering. The only important exception to this rule can be found in the case of parahydrogen to which Teller[8] was the first to call attention. Here the coupling energy equals the energy separation of the neighboring rotational states of the molecule. If the neutron energy is less than the energy necessary to excite the first rotational state, then inelastic collisions with parahydrogen in its ground state become energetically impossible. Since the ground state is nondegenerate, only the coherent scattering remains. The striking transparency of parahydrogen for very slow neutrons thus illustrates the marked effect previously mentioned (cf. Section III) which the spin degeneracy may have upon coherent nuclear scattering.

The problem of the spin coupling of neighboring ions in a salt presents difficulties of a different order of magnitude essentially for two reasons: the energy changes due to the exchange forces are very much larger than those due to nuclear coupling; furthermore, the problem becomes a many-body problem since the spins of all ions have to be taken into account simultaneously. It is customary to write for the interaction function between the spins the expression[9]

$$H = -2J\sum \mathbf{s}_P \cdot \mathbf{s}_q, \tag{8.0}$$

where J is the Heisenberg exchange integral and $\mathbf{s}_P$ is the spin of the P'th ion. The summation extends as a rule, over all neighboring ion pairs. If J is positive, then states of large resultant spin of the crystal are most stable (ferromagnetic case) whereas a negative value of J makes states of small resultant spin energetically most stable (antiferromagnetic case). The resultant crystalline spin $\mathbf{S}'$ can be used as one quantum number to describe the stationary states of lattice since it commutes with the interaction function H.

The discussion branches now along two ways. The ferromagnetic case below the Curie point has been treated before. The ferromagnetic case above the Curie point and the antiferromagnetic case can be treated together for our present purposes.

The presence of the spin exchange forces had manifested itself previously in the deviation of the paramagnetic susceptibility of most salts of

[7] M. D. Whitaker, Phys. Rev. **52**, 384 (1937).

[8] E. Teller, Phys. Rev. **49**, 421 (1936).

[9] J. H. Van Vleck, *Theory of Electricity and Magnetic Susceptibilities*, p. 328.

the iron group from the simple Curie law. To determine theoretically the paramagnetic permeability it is necessary to know the energies of the $(2S+1)^f$ states into which the originally $(2S+1)^f$ times degenerate state of the crystal of f ions splits up in virtue of the spin coupling forces. It is, of course, still possible and will actually be the case that some of these states still retain their energy degeneracy. We encounter here a problem which is well known from the theory of ferromagnetism and is at present anything but solved. We have to be satisfied to know for the moment the number of states $\omega(S')$ which belong to a total resultant spin vector $\mathbf{S}'$ and to possess some information about the *average* energy values $\bar{W}(S')$ for levels belonging to a total resultant spin S. We quote the following well-known relations[9]

$$\bar{W}(S')=\frac{-zJ}{f-1}[S'(S'+1)-fS'(S'+1)], \quad (8.01)$$

$$\langle[W(S')-\bar{W}(S')]^2\rangle_{\text{av}} = zJ^2\frac{(f^2-4S'^2)(3f^2-4S'^2)}{f^3} \quad (8.02)$$

(z = number of neighbors per atom.)

It is clear from (8.01) that the average energy for small values of the resultant spin is small and the difference between the average energies of two groups of levels with neighboring resultant spins is quite negligible for $W(S')-W(S'-1) = (2zJ/(f-1))S'$. On the other hand the "spread in energy" even of the levels belonging to $S'=0$ is considerable; namely, of the order of magnitude fJ.

For the purpose of calculating paramagnetic susceptibilities, it is customary and has been found to be a sufficient approximation, to ascribe to the levels of total spin S' a weight as determined by simple permutations and the energy $-(zJ/f)S'^2$. With these approximations one obtains for the magnetic susceptibility the following expressions

$$\chi=\left(\frac{eh}{4\pi mc}\right)^2\frac{4S(S+1)}{3k(T-T_c)}, \quad (8.11)$$

$$T_c=2zJS(S+1)/3k. \quad (8.12)$$

Here z denotes the number of neighbors possessed by each ion. It is clear that this statistical procedure is not at all rigorous; it seems to work for the determination of susceptibilities because in the presence of even weak magnetic fields the resultant spin is of the order αfS where α is a small number depending upon the external conditions. Still αfS is almost always very much larger than $f^{\frac{1}{2}}S$ which is a measure of the spin in the absence of an external magnetic field. We can sum up by stating that paramagnetic investigations do not lead very far in the study of the levels of a spin lattice.

Returning now to the scattering of neutrons in a spin lattice, we may write from equation

$$\psi_m{}^{(P)}=\left(\frac{2\pi M_0}{hk}\right)^{\frac{1}{2}}\left(\frac{2e^2\gamma}{mc^2}\right)\sum_{s'M'}\frac{e^{ik'r}}{r}\chi_{s'}\Omega_{M'}{}^{(P)} \times e^{i\delta_P}(s'M'|\mathfrak{M}|sM)_P F\times \Pi_l{}'\Omega_M{}^{(l)} \quad (8.20)$$

for the wave scattered from the P'th lattice point. Here the P'th atom is located at the position $\mathbf{r}_P$, which is contained in δ_P, the phase difference for waves scattered from this point. The $\Pi_l{}'$ means a product of the spin functions of all the atoms but the P'th. The entire wave scattered by the lattice is then

$$\psi_m{}^{(P)}=\left(\frac{2\pi M_0}{hk}\right)^{\frac{1}{2}}\left(\frac{2e^2\gamma}{mc^2}\right)\sum_{Ps'}\frac{e^{ik'r}}{r}\chi_{s'}e^{i\delta_P} \times\sum_{M_{P'}}(s'M'|\mathfrak{M}|sM)_P F^{\frac{1}{2}}\Pi\Omega_M{}^{(1)}\cdots \times\Omega^{(P-1)}_{M_{P-1}}\cdot\Omega^{(P)}_{M_{P'}}\cdot\Omega^{(P+1)}_{M_{P+1}}\cdots. \quad (8.21)$$

If we let L stand for the totality of lattice quantum numbers, thus replacing $M_1\cdots M_P\cdots$ by L and write Ω_L for the spin wave function of the entire lattice, we have

$$\psi_m=\left(\frac{2\pi M_0}{hk}\right)^{\frac{1}{2}}\left(\frac{2e^2\gamma}{mc^2}\right)\sum_{s'L'}\frac{e^{ik'r}}{r}\chi_{s'}\Omega_{L'} \times(s'L'/\sum_P e^{i\delta_P}\mathfrak{M}_P|sL)F^{\frac{1}{2}}. \quad (8.22)$$

Summing the density of scattered neutrons, $|\psi_m|^2$, over all spin variables and dividing the result by the incident density, we have

$$I/I_{\text{in}}=r^{-2}(2e^2\gamma/mc^2)^2 \sum_{s'L'}|(s'L'|\sum_P e^{i\delta_P}\mathfrak{M}_P|sL)F^{\frac{1}{2}}|^2. \quad (8.30)$$

To obtain the complete result, we must average (8.30) over the temperature distribution among the lattice states.

If now the energy changes in all transitions were small so that the atomic form factors were the same for all transitions, then expression (8.30) becomes

$$I/I_{\text{in}}=r^{-2}(2e^2\gamma F^{\frac{1}{2}}/mc^2) \times(sL|\sum_{PP'}e^{i(\delta_P-\delta_{P'})}\mathfrak{M}_{P'}{}^*\mathfrak{M}_P|sL). \quad (8.31)$$

Here the interference between different ions can be neglected. We can furthermore put $(L|s_{xP}s_{yP}|L)=0$ since the total crystalline spin S' is always small compared with fS in paramagnetic substances. Eq. (8.31) thereby reduces to our previous result (5.2). Changes in the scattering formula can therefore only occur if in virtue of the spin coupling the transitions in the lattice are accompanied by energy changes large enough to alter substantially the form factor F. It should be remembered that almost all of the scattering is, strictly speaking, inelastic; the elastic transitions, $L=L'$ in (8.31) are proportional to terms of the type $|(L|s_x|L)|^2$. This expression is very small since the total crystalline spin lies in the neighborhood of zero.

If we now employ the approximation which we describe as being useful for the treatment of susceptibilities, then it can easily be seen that the scattering is uninfluenced by the spin coupling. This is due to the fact that as we learn from

$$W(S')-W(S'-1)=(2zJ/(f-1))S'\sim(2zJS/f^{\frac{1}{2}})$$

whenever the crystalline spin is small. But only such transitions can occur in (8.31) because $\sum_P e^{i\delta_P}\mathfrak{M}_P$ has nonvanishing matrix components only between states for which $\Delta S'=\pm 1$, 0. The energy change during the scattering process would thus be completely unobservable; the scattering from the crystal would be therefore, the same as that from an equal number of free ions.

Unfortunately we are not justified *a priori* to use the approximation as sketched. The splitting of the levels of low spin is in general so wide that energy differences which are quite comparable with the neutron energy must be expected to occur. There does not exist therefore any direct connection between the theory of magnetic susceptibilities and that for neutron scattering.

Relation (8.31) allows a further simplification if we neglect for a moment the interference terms which are without interest for us at the present stage of the discussion. Dropping the phase factors $e^{i(\delta_P-\delta_{P'})}$ we find

$$I_{\text{sc}}/I_{\text{in}}=r^{-2}(2e^2\gamma/mc^2)^2 \times\sum_{Ps'L'}|(s'L'|\mathfrak{M}_P|sL)F^{\frac{1}{2}}|^2. \quad (8.40)$$

Since now the spins are no longer distinguished by their position, we can write instead of

$$I_{\text{sc}}/I_{\text{in}}=r^{-2}(2e^2\gamma/mc^2)^2 f \times\sum_{s'L'}|(s'L'|\mathfrak{M}_0|sL)F^{\frac{1}{2}}|^2. \quad (8.41)$$

Here $\mathfrak{M}_0$ refers to any of the ions in the lattice. If we now use the abbreviation

$$P(LL')=|(s'L'|\mathfrak{M}_0|sL)|^2 \div\sum_{L'}|(s'L'|\mathfrak{M}_0|sL)|^2, \quad (8.42)$$

then $P(LL')$ gives us, apart from the form factor, the probability that due to the collision, the lattice will change from the state L to the state L'. The problem of neutron scattering by lattice therefore consists in the determination of the matrix elements $P(LL')$ as a function of the energy difference between the initial and the final crystalline state. The form factor belonging to the transition $L\rightarrow L'$ may then be constructed and the distribution of energy changes for each collision determined. The simplification now achieved consists in the fact that (8.42) contains the spin coordinates of one ion only in the operator. Mathematical difficulties seem to us to prevent an immediate and direct attack even on the simplified problem.

The conditions in a saturated ferromagnetic medium which we have treated before are quite different. If the medium is close to saturation, the elastic term, $L' = L$, in (8.22) gives the coherent scattering treated in Section VI. Since this scattering equals the fraction $S/(S+1)$ of the total scattering which is to be expected from a free ion of equal spin, the remaining terms cannot contribute more than the fraction S^{-1} of the coherent scattering. It is possible that a considerable amount of this scattering appears in transitions between states of the lattice having the same resultant spin. It would probably exhibit polarization phenomena which are similar to those discussed in Section VI. Quantitative estimates of this scattering face exactly the same difficulties as those outlined in the previous paragraph.

If the spin coupling should actually lead to considerable splitting of crystalline levels having equal total spin, then a remarkable phenomenon should become observable at the Curie point of ferromagnetic bodies. Since below the Curie point the ferromagnet should show the scattering given by (6.40) which is large and coherent, a marked decrease is to be expected if the ferromagnet becomes paramagnetic. Since the Curie temperatures of all ferromagnets lie appreciable above room temperature (exchange integral large), the energy changes in the paramagnetic state would become very appreciable and the accompanying form factors would cut down the scattering to a large extent. Particularly the scattering into forward angles should be strongly diminished. Absence of the effect just described should be interpreted in our opinion as information that the transitions occur mostly between levels of small energy difference, and that the wide fluctuation in the energy of levels with equal spin as given by (8.02) is due to a comparatively small number of levels with energies vastly different from the average.

IX. Experiments

The considerations of the last sections obviously suggest a number of experiments in the field. The purpose of such experiments can essentially be twofold. They can be arranged with the aim of obtaining information about the magnetic moment of the neutron, or they can be used to explore the magnetic structure of the scattering system provided that the magnetic moment of the neutron is known. Quantitative information about it therefore seems to be of paramount importance.

The determination of the magnetic moment of the neutron can be carried out by scattering experiments alone or by a combination of scattering experiments with outside fields which act upon the polarization state of neutron beams. The last type of experiment is the one carried out by Frisch, Von Halban and Koch.[10] In these experiments a beam of neutrons was partially polarized by allowing it to pass through a magnetized piece of iron. Upon leaving the ferromagnet the neutrons were exposed to a magnetic field which could be varied in magnitude and direction, and which changed the state of polarization by an amount dependent on the neutron's magnetic moment during the passage from the first ferromagnet to a second. To determine the angle through which the individual spins rotate it is necessary to know the time of passage, i.e., the velocity of the neutrons. The scattering by, or the transmission through, the second ferromagnet will show a maximum (minimum) for definite directions of polarization of the incident beam. It is obvious that by assuming a certain knowledge about the velocity of the neutron, the variation of the strength of the magnetic field will permit a change from maximum to minimum transmission and thereby obtain information about the magnetic moment of the neutron. The authors gave as the most probable value two nuclear magnetons for the magnitude of the magnetic moment of the neutron.

Unfortunately the effect observed is very small, though, according to the authors, beyond the experimental error. It seems that iron is as poor a polarizer as it is an analyzer since the change in scattering (transmission intensity) remains a fraction of a percent. Furthermore, we do not seem to be able to determine the neutron velocity with too great an accuracy which also has a detrimental influence on the quantitative reliability of the method described.

Turning now to pure scattering and transmission experiments, it may, perhaps, be pointed

[10] Frisch, Von Halban and Koch, Phys. Rev. **53**, 719 (1938).

out that the treatment of the scattering of a neutron by a *free* paramagnetic ion involves no uncertainties with the possible exception that the interaction between neutron and electron may not be wholly magnetic in origin. Furthermore, the scattering effects which are to be expected if the magnetic moment of the neutron is of the order of magnitude of one nuclear magneton can be very large (cf. (5.5)). Therefore, as previously stated, the simplest and most direct method of determining the neutron's moment seems, in our opinion, to be a scattering experiment with an appropriate paramagnetic salt. The salt selected should have a large magnetic moment, a condition which is well satisfied by the salts of divalent manganese and trivalent iron, and to a lesser extent by salts of several other ions in the iron group. (The rare earths are not very suitable for experiments of that type since they cannot be freed from highly absorbing members of their group). A particular salt must then be chosen which approximates as closely as possible the condition of a free ion. The ion will be called free if there is no appreciable effect from quenched orbital currents, from the spin-orbit coupling and from the spin-spin coupling between different ions. There is convincing evidence that divalent manganese and trivalent iron ions are in an S state so that the first two conditions are well satisfied. If we finally choose a salt which shows the full molar magnetic susceptibility as calculated on the assumption of a free spin (T_c in (8.12) small), we may be fairly certain to have also avoided difficulties arising from spin-spin coupling. Several salts whose susceptibility has been studied over a wide range of temperature satisfy (8.11) with a constant T_c considerably less than 100°K (e.g. $MnSO_4$). After selection of the salts as described, scattering experiments with slow neutrons should be carried out under several angles up to a minimum deflection chosen as small as the experimental arrangement will permit. These scattering measurements should be made relative to some standard which is nonmagnetic and shows only isotropic nuclear scattering; thereby geometric uncertainties can be avoided. The scattering data can then be fitted to the curve corresponding to a form factor (cf. Section VII) by an appropriate choice of the size of the scattering domain. Even without such theoretical assistance the measurement of the scattering at several angles should permit a satisfactory extrapolation to very small forward angles where the form factor becomes unity. The experiments can then be evaluated by the use of (5.9).

The various formulae of Section VI offer in principle a basis for the determination of the magnetic moment of the neutron through experiments with ferromagnets. By far the simplest method can be deduced from (6.40). It consists in an observation of the beam scattered by a magnetized body under a constant angle of scattering θ and at two different azimuths so chosen that $\mathbf{e}\cdot\boldsymbol{\kappa}=1$ and $\mathbf{e}\cdot\boldsymbol{\kappa}=0$. The difference in scattering then amounts to the *total magnetic scattering* independent of the amplitude and phase of the nuclear scattering. The disadvantages of this arrangement are several: It is necessary to delimit the scattered beam to a fraction of the azimuthal circumference which weakens the intensity by a factor of 2π approximately. It is furthermore essential to have single scattering since otherwise additional neutrons will be scattered into the chosen angle and also the assumption of a primarily unpolarized beam would no longer be justified. Apart from these not unsurmountable technical difficulties there remain certain theoretical ambiguities of perhaps greater importance. It would of course also be necessary, as in the paramagnetic case discussed before, to carry out observations at different angles θ to obtain information about the form factor. Even for the case of elastic scattering the form factor for ferromagnetic bodies will probably be of greater importance than in the case of paramagnetic scattering from free ions since it is likely that the outer shells will also be somewhat coupled and that there has occurred a noncalculable but probably large amount of spin quenching. A considerable contribution from incoherent magnetic scattering must also be expected to be present which will probably strongly diminish the otherwise marked azimuthal effect.

It should also be mentioned that the relative effect, even in the absence of all the difficulties mentioned above is rather unfavorable in the case of iron, since the nuclear cross section is large, $d\Phi_n=10^{-24}d\Omega$, while the elastic magnetic scattering amounts to $d\Phi_n\sim 2.5\times 10^{-25}d\Omega$ if we use for γ the empirically determined magnetic moment ~ 2 Bohr magnetons.

Originally it had been suggested[2] and an attempt[11] made to determine the magnetic moment with the aid of polarization experiments based upon the interference between nuclear and magnetic scattering. The two types of experiments mostly discussed are the double scattering and transmission arrangements which we mentioned in Section VI. Here too there exist in our opinion grave difficulties if one attempts to evaluate the experiments quantitatively for the purpose of determining the neutron moment. Section III dealt with the problem how far the coherent scattering of a nucleus can be determined from observation of the total nuclear cross section. We found that in the presence of isotopes and the nuclear spin it is almost impossible to draw quantitatively reliable conclusions. It must be admitted that the case of iron is favorable insofar as the dominant isotope has an atomic weight of 56 and therefore, probably no spin. It nevertheless does not seem feasible to determine the amplitude of coherent scattering quantitatively; we must be satisfied to enclose it within certain not improbable limits. There must, in addition, be considered a background due to inelastic nuclear scattering which arises from the coupling with the lattice. Since this coupling need not be the same for nuclear and magnetic scattering there arise inaccuracies which cannot be removed at our present state of knowledge. The inelastic scattering which is due to the strong spin coupling forces will, in the case of a saturated ferromagnet also show polarization effects of an unpredictable magnitude. The difficulty of obtaining sufficient intensity in a neutron beam for the purpose of carrying out a double scattering experiment has already been pointed out in the literature.

That all these factors mentioned are of considerable influence on the actual state of polarization of a beam passing through iron follows in our opinion from the minuteness of the effect which has been observed in the experiments by Frisch, von Halban and Koch. A rough estimate on the basis of (6.30) etc. would indicate that iron is by far a better polarizer (and analyzer) than the actual observations indicate. We consider these experiments as offering direct support for our contention as to the complexity of scattering processes in ferromagnetic bodies.

Similar difficulties enter into the quantitative evaluation of experiments concerned with the attenuation of an incident unpolarized neutron beam. Since it is the *total* and not the *differential* cross section that here becomes of importance, we are not able to study independently the form factor as function of the scattering angle. This form factor enters (cf. e.g. (6.53)) in a decisive but complex manner into the absorption cross section; without knowing the form factor no conclusions can be drawn as to the magnitude of the magnetic moment. If there is an appreciable amount of incoherent nuclear scattering present, the beam will be depolarized as it passes through the ferromagnet. Care must also be taken that in the case of a highly attenuated primary beam no secondary neutrons shall be scattered into the forward direction. Previous evaluations have also neglected the purely magnetic coherent and incoherent scattering as well as the effect of the precession of the neutron spin during the passage through the ferromagnet.[12] The combined weight of all these variable factors seems, in our opinion, sufficient to make a quantitative evaluation of transmission experiments appear rather improbable.

Assuming a quantitative knowledge of the magnetic moment of the neutron which, as we have shown, can probably be obtained with fewest ambiguities from observations of paramagnetic scattering, we are able to outline a series of experiments from which much information can be gained as to the magnetic structure of the scatterer. As previously pointed out, we find ourselves now on less satisfactory ground as far as the theory is concerned, and we think that the most promising attack can be made from the experimental side. The difficulties to which we

[11] Cf. e.g. Hoffman, Livingstone and Bethe, Phys. Rev. **51**, 214 (1937); several notes by J. Dunning and collaborators in Phys. Rev. **51**, (1937).

[12] *Note added in proof.*—The problem of magnetic scattering of neutrons has been further treated together with related questions in two notes which will appear shortly. Abstracts have been presented to the meetings of the American Physical Society in Washington (cf. a paper by O. Halpern and Th. Holstein presented in December, 1938; Phys. Rev. **55**, 601 (1939); and a paper by Halpern, Hammermesh and Johnson presented in April, 1939; Abstract No. 73 in Am. Phys. Soc. Bull. **14**, No. 2, April 12, 1939).

are referring are mainly our lack of knowledge concerning the elastic and even more important, the inelastic form factors. The complications which we encounter in attempting to estimate the energy change per collision due to the presence of spin coupling forces have been discussed fully in Section VIII. We there realized the impossibility of drawing reliable conclusions from the mean square fluctuation of the energy which, in our opinion, does not at all determine the actual energy change per collision. Ceding this group of problems to experimental investigation, we have learned in the meantime by letter and personal communication that Professor Van Vleck has become interested in the theoretical side of this problem and is attempting to derive information about the energy changes from considerations about energy fluctuations. His results[13] show that from such considerations the average energy change is of the order of the exchange integral.

Leaving these theoretical considerations aside for the moment we may point out that there exists a very simple type of experiment which will show if it is justified to predict energy changes of the order of magnitude of the exchange integral J. For this purpose it is only necessary to determine the integral cross section of various salts containing the same paramagnetic ion (Mn^{++} or Fe^{+++}). Since these salts show widely varying susceptibilities and exchange integrals (Curie temperatures between 100° and 1000°K), we should expect large changes in the integral cross section. If, for example, we choose a salt like MnS which does not show an appreciable magnetic scattering in the backward direction, then we should expect according to the hypothesis mentioned a vanishingly small total magnetic cross section, since the inelastic form factor will almost completely eliminate the forward scattering also. If, on the other hand, the small backward scattering is due to the influence of the elastic form factor which (cf. (7.30)) is small enough to make it unobservable, then a sizable magnetic cross section should be measured which would be due to a large amount of forward scattering. All these discussions presuppose of course, the existence of a magnetic moment of the neutron which is not much smaller than one nuclear magneton.

Experimenting with quasi-free magnetic ions we can use the angular dependence of the magnetic scattering to obtain information on the magnetically active size of the ion. Similarly observations on the angular dependence of scattering can be used to obtain information about the probability $P(L, L')$. For this purpose it seems advisable to carry out observations at various neutron energies, and for a series of salts of variable "magnetic dilution." Since the magnitude of spin coupling is known from susceptibility data, one can thus obtain experimentally a relationship between spin coupling and neutron scattering. Again the salts of divalent manganese and trivalent iron are especially suitable because of the absence of orbital currents. It will also prove interesting to investigate the scattering from paramagnetic metals. Susceptibility data are here difficult to interpret because of the erratic temperature dependence which may well be connected with a large spin coupling in these magnetically concentrated materials. This spin coupling will in the antiferromagnetic case operate to quench the ionic spins in the presence of an external magnetic field. After one has determined the general effect of spin coupling with the aid of observations on ions of known magnetic structure, the neutron scattering from metals should present information as to the actual ionic moment and hence, the ionic state in the metal.

We have already pointed out that the presence of spin coupling forces could give rise to marked phenomena if the scattering by ferromagnets is observed above and below the Curie point. If the energy changes should really turn out to be of the order of magnitude of the exchange integral, then we would expect that the scattering above the Curie point is practically absent on account of the large inelastic form factor. An experiment of such a kind should give us information similar to that obtained from observations on the total cross sections of paramagnetic salts.

[13] See following paper.