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Influence of Atomic Electrons on Radiation and Pair Production*

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A fast electron passing through matter will radiate at a greater rate than would be expected if the atoms traversed presented to the electron simply a static field of force. This follows because of the additional possibility of collisions in which an atom is excited. In a similar way energetic quanta will produce pairs at a greater rate. Consequently the unit of length characteristic of the multiplicative production of showers will be decreased by a factor which in the case of air amounts to 17 percent. The probabilities of radiation and pair formation are not proportional to the square of the nuclear charge but are supplemented by terms linear in atomic number (mass absorption law).

INTRODUCTION

THE probability for the deflection of a fast electron by a nucleus of charge Ze can be calculated in the Born approximation in terms of the scattering of a plane wave by a medium whose scattering power at each point is proportional to the potential at that point. Thus the amplitude of the wave scattered at an angle θ to the original direction will be determined by

$$\int d\tau (Ze/r) \exp\{i(\mathbf{p}_1-\mathbf{p}_2)\cdot\mathbf{r}\} = 4\pi Ze/q^2, \quad (1)$$

where $q=|p_1-p_2|=2p_1 \sin\theta/2$ is the momentum transferred to the nucleus in the collision.[1]

At a sufficiently great speed, the electron will in addition have a large probability of radiation. When a particle of energy E_1mc^2 emits a quantum of energy νmc^2, the nucleus will take up a momentum qmc, where q lies between the limits $q_{\min}$ and $q_{\max}$. For large values of E_1 and ν we have approximately

$$q_{\min}=\nu/2E_1(E_1-\nu); \quad q_{\max}=2E_0. \quad (2)$$

The probability of such a radiative process will again contain the factor $(4\pi Ze/q^2)^2$. The total probability of radiating quanta of energy νmc^2 in such collisions will arise principally from processes in which the momentum transfer is small. The probability will be finite for a given value of the primary energy but for increasing E_1 it will rise indefinitely as $a \ln E_1+b$, where a and b are constants. This follows because of the growing contribution of small momentum transfers.

The orbital electrons, however, will screen the field of the nucleus and reduce the probability of small momentum transfers. With Z electrons located at points $\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_z$, we shall have for the probability amplitude for a given momentum transfer the expression

$$\int \exp(i\mathbf{q}\cdot\mathbf{r})\left\{\frac{Ze}{r}-\sum_i \frac{e}{|\mathbf{r}-\mathbf{r}_i|}\right\}d\tau = (4\pi e/q^2)\{Z-\sum_i \exp(i\mathbf{q}\cdot\mathbf{r}_i)\}. \quad (3)$$

This must now be averaged over all positions of the electrons. The result is

$$(4\pi e/q^2)\int \Psi_0^*(\mathbf{r}_1, \cdots \mathbf{r}_z)\{Z-\sum_i \exp(i\mathbf{q}\cdot\mathbf{r}_i)\} \times \Psi_0(\mathbf{r}_1, \cdots \mathbf{r}_z)d\tau_1\cdots d\tau_z = (4\pi Ze/q^2)\{1-F(q)\}, \quad (4)$$

where $ZF(q)$ is the same atomic form factor as that which describes the elastic scattering of x-rays. For small values of q, $1-F(q)$ approaches zero as q^2. Thus the probability of small momentum transfers is bounded and the probability for the radiation of a quantum with energy νmc^2 approaches a finite upper limit with increasing energy of the primary electron. This upper limit

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[1] We are using relativistic units $\hbar/mc$, mc^2, mc for length, energy and momentum.

has been calculated by Bethe and Heitler[2] by adopting the form factor for a Fermi-Thomas model atom. They obtain for the differential cross section the expression

$$\Phi(\nu)d\nu=(Z^2/137)(e^2/mc^2)^2(d\nu/\nu)(4/E_1^2)\times\{(E_1^2+E_2^2-2E_1E_2/3)\times\ln(183/Z^{\frac{1}{3}})+E_1E_2/9\},\quad(5)$$

where $E_2=E_1-\nu$. This follows directly by integrating over all momentum transfers q an expression of the form

$$(4\pi Ze/q^2)^2\{1-F(q)\}^2G(q,\nu,E_0)dq,\quad(6)$$

in which the factor[3] $G(q,\nu,E_1)$ contains all effects arising from the light quantum and the electron, while the other factors express the entire influence of the system which takes up the impulse.

In addition to radiative processes in which the atom is left unexcited, there will, however, occur collision[4] in which the momentum transfer to an atomic electron excites the atom to a state Ψ_n. Radiative processes of this kind, analogous to the inelastic or Compton scattering of x-rays, may be divided into two groups, as follows.

Small Momentum Transfers

The probability of a momentum transfer qmc, where q is small compared with unity, will be determined by squaring the absolute value of

$$(4\pi Ze/q^2)\int\Psi_0^*(\mathbf{r}_1,\cdots,\mathbf{r}_z)\{Z-\sum\exp(i\mathbf{q}\cdot\mathbf{r}_i)\}\times\Psi_n(\mathbf{r}_1,\cdots,\mathbf{r}_z)d\tau_1\cdots d\tau_z,\quad(7)$$

multiplying by $G(q,\nu,E_0)$, and summing over all excited states Ψ_n. A free electron would receive an energy $(qmc)^2/2m$ in the encounter, and therefore only those states Ψ_n would give a contribution whose energies are exactly equal to this value. When we allow for the binding of the electrons, it will still be true that the only appreciable effect will come from excitation energies small compared with mc^2. Therefore retardation and other possible relativistic corrections will not arise from the motion of the atomic electron, and we shall be justified in considering $G(q,\nu,E_1)$ in the summation as a factor independent of Ψ_n. Adding to the sum the contribution of the elastic radiation process considered by Bethe and Heitler, and using the completeness theorem for the orthogonal states Ψ_n, we obtain

$$(4\pi e/q^2)^2\int\Psi_0^*(\mathbf{r}_1,\cdots\mathbf{r}_z)|Z-\sum\exp(i\mathbf{q}\cdot\mathbf{r}_i)|^2\times\Psi_0(\mathbf{r}_1,\cdots\mathbf{r}_z)d\tau_1\cdots d\tau_z\quad(8)$$

for the factor which determines the total probability (elastic plus inelastic) for the atom to take up a recoil of magnitude qmc.

The above result may be understood as follows: The time required for the primary electron to traverse the atom is short in comparison with the period of the motion of the atomic electrons. Therefore we may visualize the process in terms of an instantaneous photograph, in which we find the Z electrons respectively at positions $\mathbf{r}_1,\cdots,\mathbf{r}_z$. The probability that this field of force will take up a recoil qmc is determined by the quantity

$$(4\pi e/q^2)^2|Z-\sum\exp(i\mathbf{q}\cdot\mathbf{r}_i)|^2.\quad(9)$$

In a repetition of the experiment, the position of the electrons will be different; averaging over many experiments, we obtain just the expression (8). From this point of view, the validity of the result depends upon the energy of the primary being sufficiently high and the recoil momentum being sufficiently small so that the motion of the atomic electron during the passage of the primary can be neglected. This condition will be satisfied to a good approximation for primary energies large in comparison with mc^2.

The production of pairs by energetic gamma-radiation is closely related to the radiative effect just considered. All the arguments referring to the relative probability of momentum transfers of various magnitudes can be taken over the simple changes, as will be seen in more detail below.

[2] H. Bethe and W. Heitler, Proc. Roy. Soc. **A146**, 83 (1934).

[3] G is given by H. Bethe, Proc. Camb. Phil. Soc. **30**, 524 (1934).

[4] The possibility of pair production as well as radiation in the field of atomic electrons was first pointed out by F. Perrin, Comptes rendus **197**, 1100 (1933).

Large Momentum Transfers

In radiative and pair production processes where the recoil momentum is comparable with or larger than mc, the atomic electron may be treated as free (no screening). The total cross section (including all momentum transfers) for the production of pairs in the field of a free electron can be obtained by inverting a type of argument due to Williams. Racah[5] has calculated the cross section Q according to quantum electrodynamics for the production of pairs in the impact of an electron of energy E_0mc^2 on a heavy nucleus of charge Z_1:

$$Q(E_0)=(Z_1e^2/137mc^2)^2(1/\pi)\cdot\{(28/27)\ln^3 E_0 + [(28/9)\ln 2-178/27]\ln^2 E_0+d\ln E_0+f\}. \quad (10)$$

Here d and f are numerical constants, given by Racah, the values of which are slightly different in the two cases where the incident particle is an electron or a heavy particle. If we now consider the process in the frame of reference in which the electron is initially at rest, Q will represent the cross section for the production of pairs in the impact of a nucleus of charge Z_1 moving with the original velocity $(1-1/E_0^2)^{\frac{1}{2}}c$ toward the electron, considered to be initially at rest. The field of the passing nucleus can be replaced by an equivalent radiation field, in which the number of quanta in the spectral range $\nu mc^2/h$ to $(\nu+d\nu)mc^2/h$, after summing over all impact parameters, is[6]

$$N(\nu, E_0)d\nu=(2Z_1^2/137\pi)(d\nu/\nu)\ln kE_0/\nu, \quad (11)$$

provided ν is less than $E_0(\gg 1)$. The limitations of the semi-classical argument involved make it impossible to give a precise value to the numerical constant k, which is of the order of magnitude unity.

Let us suppose for the moment that we know the cross section $\sigma(\nu)$ for the production of pairs by a light quantum of energy νmc^2 in the field of a free electron. Then, subject to the limitations of the method of virtual light quanta, we shall have for Q, the expression

$$Q(E_0)=\int_0^{E_0}\sigma(\nu)N(\nu, E_0)d\nu =(2Z_1^2/137\pi)\int_0^{E_0}\sigma(\nu)\ln(kE_0/\nu)d\nu/\nu. \quad (12)$$

Conversely, knowing Q, we may find $\sigma(\nu)$ by differentiating (12) twice with respect to $\ln E_0$ and integrating the resultant first-order differential equation:

$$\sigma(\nu)=(137\pi/2Z_1^2)\exp(-\ln\nu/\ln k)(1/\ln k)\cdot\int_{-\infty}^{\ln\nu}\exp(\ln E_0/\ln k)\frac{\partial^2 Q}{\partial(\ln E_0)^2}d(\ln E_0). \quad (13)$$

Upon insertion of Racah's result in (13), we obtain for $\sigma(\nu)$ a value which is independent of the mass of the particle in whose field the gamma-ray is absorbed, since the constants d and f in (10) fall out on the differentiation:

$$\sigma(\nu)=(1/137)(e^2/mc^2)^2\{(28/9)\ln\nu +(28/9)\ln(2/k)-178/27\}. \quad (14)$$

Except for the difference in charge and the indeterminacy in the constant term of (14) due to the lack of definition of k, the cross section obtained agrees with the formula of Bethe and Heitler for pair production in the field of an unscreened heavy nucleus. It also agrees with what our calculations give just from a consideration of momentum transfers small compared with mc, but large in comparison with the momentum of an atomic electron in its orbit. Therefore we conclude that momentum transfers large compared with mc give a contribution to the total cross section, either for pair production or for radiation, which is small (since k is of the order of magnitude unity) and enters only as a part of the constant term in (14).

Pair production processes involving large recoils of the atomic electron ("triplet production") and the analogous radiation effects can only be treated by a systematic application of quantum electrodynamics. Since there is a clear cut distinction between them and the processes discussed in this paper, we shall not consider them further here. We shall therefore in the following calculate the probabilities for pair production and radiation by integrating only over momentum transfers up to mc.

[5] G. Racah, Nuovo Cimento **14**, 112 (1937).
[6] E. J. Williams, Kgl. Dansk. Videnskab. Selskab. **13**, No. 4 (1935).

TABLE I. *Incoherent scattering factor, defined by Eq. (17), as calculated from the Fermi-Thomas atom model. The values for $v>0.04$ are due to Bewilogua. For small values of v, $S(v)=13.8v-55.4v^{\frac{3}{2}}$. The momentum transfer qmc (which for the incoherent scattering of x-rays would be represented by $(2h/\lambda)\sin\theta/2$) is reduced to Fermi-Thomas units by the substitution $v=(6\pi)^{\frac{1}{3}}8^{-1}137qZ^{-\frac{2}{3}}$.*

v	$q/Z^{\frac{2}{3}}$	S	v	$q/Z^{\frac{2}{3}}$	S
0.00	0.00000	13.8v	0.3	0.0066	0.776
0.01	0.00022	0.097	0.4	0.0088	0.839
0.02	0.00044	0.169	0.5	0.0110	0.880
0.03	0.00066	0.227	0.6	0.0132	0.909
0.04	0.00088	0.277	0.7	0.0154	0.929
0.05	0.00110	0.319	0.8	0.0176	0.944
0.10	0.00219	0.486	0.9	0.0197	0.954
0.20	0.00440	0.674	1.0	0.0219	0.963

EVALUATION OF CROSS SECTION

The contribution of the inelastic radiative and pair production processes is best calculated by returning directly to the expression (7) which determines the probability of a recoil q in a transition from the state o to an excited state n. The term with Z may be omitted from the integrand because of the orthogonality of the two wave functions Ψ_0 and Ψ_n. With the help of the completeness relation, we obtain after summation the factor determining the intensity of the incoherent radiative processes:

$$(4\pi e/q^2)^2\left\{\sum_{i,j}\int\Psi_0^*\exp\left[i\mathbf{q}\cdot(\mathbf{r}_i-\mathbf{r}_j)\right]\Psi_0 d\tau -\left|\int\Psi_0^*\sum_i\exp(i\mathbf{q}\cdot\mathbf{r}_i)\Psi_0 d\tau\right|^2\right\}. \quad (15)$$

We write Ψ_0 as a determinant built from Z single particle wave functions $\psi_{i\alpha}(\mathbf{r})$ where $i=1, 2, \cdots, Z$, and $\alpha=1, 2$ gives the two spin components of each function. Expression (15) then reduces to

$$(4\pi e/q^2)^2 ZS(q), \quad (16)$$

where $S(q)$ is a quantity analogous to the atomic form factor, and is defined by

$$ZS(q)=Z-\sum_{i,j}\int\psi_{i\alpha}^*(\mathbf{r})\psi_{j\beta}(\mathbf{r}')\exp\left[iq\cdot(\mathbf{r}-\mathbf{r}')\right] \times\psi_{j\alpha}(\mathbf{r})\psi_{i\beta}(\mathbf{r}')d\mathbf{r}d\mathbf{r}'. \quad (17)$$

The values of $S(q)$ range from $S=0$ for small values of q (recoil not sufficient to excite the atom), to $S=1$ for $q\gg\hbar/$(atomic radius) (i.e., electrons effectively free, and random phase relations between the contributions of different electrons). Heisenberg[7] has shown how the incoherent scattering factor $S(q)$ can be computed on the basis of the Fermi-Thomas atom model. The numerical values are given by Bewilogua. We have extended his calculations to smaller values of q. The results of Bewilogua[8] thus supplemented are given in Table I.

For the cross section $\Phi(E_1, \nu)d\nu$ for emission of a quantum of energy between νmc^2 and $(\nu+d\nu)mc^2$ in an incoherent radiative process, we obtain the general expression

$$d\nu\int(4\pi e/q^2)^2 ZS(q)G(q, \nu, E_1)dq, \quad (18)$$

where $G(q, \nu, E_1)$ is already known from the theory of the coherent radiative processes. Evaluating (18) on the basis of the Fermi-Thomas model, we find in the limit of primary energies very great in comparison with mc^2 (complete screening)

$$\Phi_{\text{inelastic}}(E_1, \nu)d\nu=(Z/137)(e^2/mc^2)^2(d\nu/E_1^2\nu) \cdot\{(E_1^2+E_2^2)(29.1-8/3\ln Z) -(2E_1E_2/3)(28.4-8/3\ln Z)\}. \quad (19)$$

Here $E_2=E_1-\nu$.

The cross section for production of a pair of particles with energies E_1mc^2 and E_2mc^2 is obtained in a similar way in the case $\nu\gg1$:

$$\Phi_{\text{inelastic}}(\nu, E_1)dE_1=(Z/137)(e^2/mc^2)^2(dE_1/\nu^3) \cdot\{(E_1^2+E_2^2)(29.1-8/3\ln Z) +(2E_1E_2/3)(28.4-8/3\ln Z)\}. \quad (20)$$

In order to have a check on the accuracy of this application of the Fermi-Thomas model, we have also carried through the integration of (18) for nitrogen in the limiting case of high energies, by using atomic wave functions,[9] and for hydrogen, for all energies. The results are given in Fig. 1.

Radiative and pair production processes in the fields of atomic electrons will make themselves very noticeable in the multiplicative production

[7] W. Heisenberg, Physik. Zeits. **32**, 737 (1931).
[8] L. Bewilogua, Physik. Zeits. **32**, 740 (1931).
[9] Morse, Young and Haurwitz, Phys. Rev. **48**, 948 (1935). We are indebted to Professor Morse for supplying us with the tables employed in computing the wave functions for nitrogen.

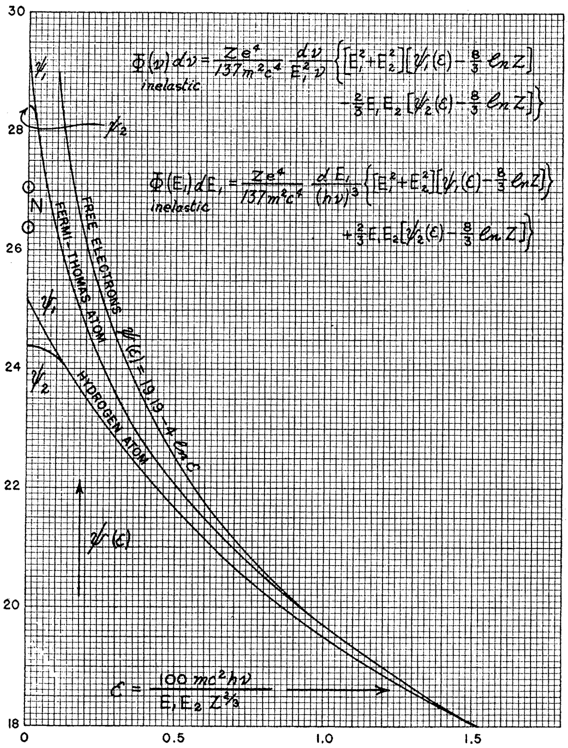

FIG. 1. Screening factors for inelastic pair production and radiative processes. The two points marked N give the factors ψ_1 and ψ_2 as calculated for nitrogen from atomic wave functions. For free electrons $\psi_1=\psi_2=\psi$.

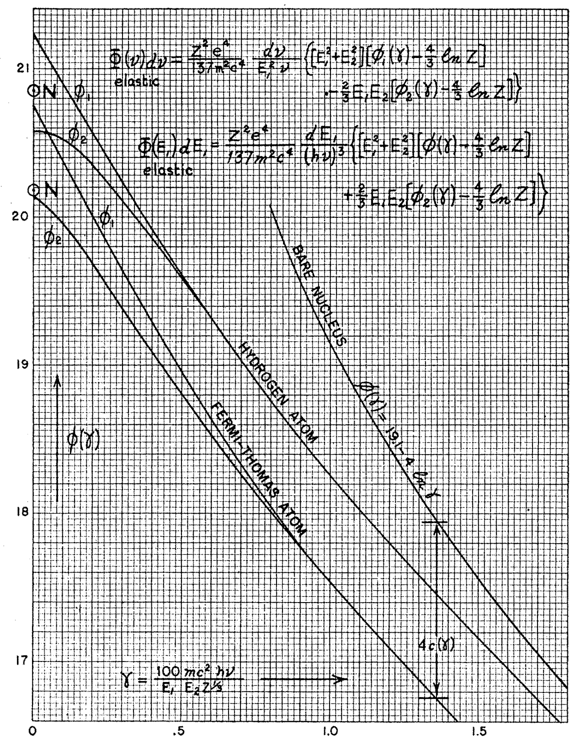

FIG. 2. Screening factors for pair production and radiative processes which leave the atom unexcited. The points N are for nitrogen. The two curves marked "Fermi-Thomas atom" are taken from the paper of Bethe and Heitler, where a table is also given for larger values of γ of the quantity $c(\gamma)$ which determines the difference between the curves for the Fermi-Thomas atom and a bare nucleus.

of cosmic-ray showers in substances of low atomic weight, such as air and water.[10] In order to make it possible to compare their importance for light elements with the corresponding elastic processes considered by Bethe and Heitler, we have supplemented the calculations of the latter authors, based on the Fermi-Thomas atom, with computations for nitrogen and hydrogen, by using atomic wave functions (Fig. 2).

Taking air as an example, and basing our choice of screening constants on the values obtained for nitrogen, we obtain for the integrated cross section for production of pairs by a very high energy quantum the result

$$\Phi_{\text{pairs}}=\Phi_{\text{elastic}}+\Phi_{\text{inelastic}}$$
$$=(41.9+6.94)\times 10^{-26}\ \text{cm}^2. \qquad (21)$$

[10] L. Landau and G. Rumer (Proc. Roy. Soc. **A166**, 213 (1938)) have taken this effect into account in a preliminary way by replacing Z^2 in the formula of Bethe and Heitler by $Z(Z+1)$.

Conclusion

Because of the possibility of collisions in which an atom is excited, a fast electron passing through matter will radiate at a greater rate than would be expected if the atoms traversed presented to the electron simply a static field of force. In a similar way, energetic quanta will produce pairs at a greater rate. Consequently the unit of length characteristic of the multiplicative production of cosmic-ray showers will be decreased by a factor which in the case of air amounts to 17 percent (Eq. (21)). In addition, the probabilities of radiation and pair formation are not entirely proportional to the square of the atomic number of the absorbing material, but are supplemented by terms linear in the atomic number (mass absorption law).