Nuclear Spins and Magnetic Moments by the α-Particle Model

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Some properties of the light nuclei of the type $n\alpha\pm1$ have been calculated on the basis of the α-particle model. By means of this description, spins and magnetic moments of the same nuclei are calculated here. The results for the magnetic moments of many of the nuclei are undetermined within a certain range of possible values since the relative order of magnitude of the spin-orbit coupling and the separation of rotational levels is unknown. This ambiguity does not appear in the Hartree model, so that the magnetic moments as calculated by the two models are somewhat different. The most notable difference occurs for the spins of C^{13}, N^{13}, which have not yet been measured; the Hartree model predicts a spin of $\frac{1}{2}$ for them whereas the α-model yields a spin of $1\frac{1}{2}$. The α-model appears to give slightly better agreement for those magnetic moments that have been measured.

INTRODUCTION

THE α-particle model[1] has been successful in explaining many qualitative features of light nuclei. It has recently[2] been applied with some success to nuclei that contain α-particles plus or minus a neutron or proton. However, there is no reason to claim that the α-particle model gives an exact, or even an accurate, description of the structure of light nuclei. The description given by the Hartree model, which is more difficult to apply, is certainly more fundamental since it takes the (unknown) forces between individual particles into account explicitly.

In order to determine which model is closer to the truth, it is necessary to calculate with both and to compare the results with experiment. Total angular momenta (spins) and magnetic moments of light nuclei in their ground states have been calculated[3] on the basis of the Hartree model. It is the purpose of this paper to find what differences, if any, arise when the same properties are calculated by means of the α-particle model. This calculation has already been made[4] for Li^7, but will be repeated here for completeness.

It has already been pointed out that the saturated systems which contain an integral number of α-particles have zero angular momentum and magnetic moment in the ground state. The interest here will be in the systems which contain an added or subtracted particle. Since this model is essentially molecular, the notation used will follow that of molecular spectroscopy as closely as possible.

1. GENERAL METHOD

The proper functions, $\psi_{K,m,\Lambda}$ of the orbital-rotational levels of the nuclei considered here have been determined by Hafstad and Teller. Since the systems are spherical or symmetrical tops, they are characterized by at most three quantum numbers:[5] K, the rotational angular momentum; m, its projection on the space-fixed z axis; and Λ, its projection on the symmetry axis of the body, called the ζ axis of the body-fixed coordinate system.

Were there no spin, the characterization of the levels would be complete. However, an added or subtracted particle introduces a spin of $\frac{1}{2}$, which means that K and m are no longer true quantum numbers but that the levels must be described by a linear combination of products of spin and rotational wave functions. These states will always be characterized by a total angular momentum $J=K\pm\frac{1}{2}$, and by M, the projection of J on the space-fixed z axis.

A state corresponding to a given value of M includes those rotational states for which $m=M\pm\frac{1}{2}$; the mixing of these m values is determined by the symmetry of the system and is

[1] J. H. Bartlett, Jr., Nature **130**, 165 (1932); W. Wefelmeier, Zeits. f. Physik **107**, 332 (1937); J. A. Wheeler, Phys. Rev. **52**, 1083 (1937); C. F. von Weizsäcker, Naturwiss. **26**, 209, 225 (1938).
[2] L. R. Hafstad and E. Teller, Phys. Rev. **54**, 681 (1938).
[3] M. E. Rose and H. A. Bethe, Phys. Rev. **51**, 205 (1937).
[4] H. Bethe, Phys. Rev. **53**, 842 (1938).

[5] The notation used here differs from that used in reference 2 in that the J, K there become K, Λ here, respectively.

therefore completely independent of the spin-orbit interaction.

A fixed value of J may arise from two K values, $K=J+\frac{1}{2}$ and $K=J-\frac{1}{2}$. The way in which these two states mix is determined by the spin-orbit interaction. This interaction also determines which of the two $(2J+1)$-fold degenerate levels, $J=K+\frac{1}{2}$, $J=K-\frac{1}{2}$, has the lower energy. The J value, J_0, for the lowest energy is the "spin" of the nucleus.

If φ_σ, $\sigma=\pm\frac{1}{2}$, is the spin proper-function for which σ is the projection of the spin on the space-fixed z axis, the above linear combination may be written as

$$\Psi_{M,\,J=K+\frac{1}{2}}=\sum_{m+\sigma=M}\{C_{K,\,m,\,\sigma}\psi_{K,\,m,\,\Lambda}\varphi_\sigma + C_{K+1,\,m,\,\sigma}\psi_{K+1,\,m,\,\Lambda}\varphi_\sigma\}.$$

According to the above, for a given value of K, the ratios of the $C_{K,\,m,\,\sigma}$ to each other are determined by the symmetry, but the ratios of the $C_{K,\,m,\,\sigma}$ to the $C_{K+1,\,m,\,\sigma}$ depend on the spin-orbit coupling. It happens that, in the cases considered, Λ values are not mixed, as is indicated by the summation indices.

The following particular cases will occur:

$$\Psi_{\frac{1}{2},\,\frac{1}{2}}=a_0\psi_{0,\,0,\,\Lambda}\varphi_{\frac{1}{2}}+a_1[\sqrt{\tfrac{1}{3}}\psi_{1,\,0,\,\Lambda}\varphi_{\frac{1}{2}} -\sqrt{\tfrac{2}{3}}\psi_{1,\,1,\,\Lambda}\varphi_{-\frac{1}{2}}], \quad (1a)$$

$$\Psi_{1\frac{1}{2},\,1\frac{1}{2}}=b_1\psi_{1,\,1,\,\Lambda}\varphi_{\frac{1}{2}}+b_2[1/\sqrt{5}\,\psi_{2,\,1,\,\Lambda}\varphi_{\frac{1}{2}} -2/\sqrt{5}\,\psi_{2,\,2,\,\Lambda}\varphi_{-\frac{1}{2}}]. \quad (1b)$$

Normalization of the wave functions requires that

$$a_0^2+a_1^2=1;\quad b_1^2+b_2^2=1. \quad (2)$$

Knowledge of the eigenfunctions makes it possible to calculate the average value of the z component of the magnetic moment in that ground state of the system for which $M=J_0$ (J_0 is the angular momentum in the ground state):

$$\bar{\mu}_z=(\Psi_{J_0,\,J_0};\,\mu_z\Psi_{J_0,\,J_0}).$$

μ_z is additively composed of the moment due to the spin of the added or subtracted particle and of that due to the rotation of the system as a whole. Of these contributions, the first is connected to the spin with a fixed proportionality factor g_s, depending on whether the particle is a neutron or proton, and may be calculated immediately as

$$g_s\sum_{K=J_0-\frac{1}{2}}^{J_0+\frac{1}{2}}\sum_{m+\sigma=M}(C_{K,\,m,\,\sigma})^2\sigma. \quad (3a)$$

In order to calculate the contribution due to rotation, the gyromagnetic ratio of magnetic to angular momentum for rotation of the system must be determined in the body-fixed ξ, η, ζ axes. If the extra particle or hole is assumed, in the average, to be distributed uniformly over the system (see Section 4), the values of g_ξ and g_η are completely determined and $g_\xi=g_\eta=$ ratio of total charge to total mass. However, the angular momentum about the ζ axis is sometimes composed additively of motion of the system as a whole and of orbital motion of the extra particle with respect to the system. Since the g factor is different for the two motions, g_ζ will depend on this distribution of angular momentum and will in general be different from g_ξ.

If we choose to write the Landé factor, g_ζ, as $(g_\zeta-g_\xi)+g_\xi$ the last term and the ξ, η components of the magnetic moment give the simple contribution

$$g_\xi\sum_{K=J_0-\frac{1}{2}}^{J_0+\frac{1}{2}}\sum_{m+\sigma=M}(C_{K,\,m,\,\sigma})^2 m, \quad (3b)$$

since the three together transform from one coordinate system to the other simply as a constant (g_ξ) times the angular momentum.

For the calculation of the contribution due to $g_\zeta-g_\xi$, the direction cosine between the body-fixed ζ axis and space-fixed z axis must be used. These matrices have been calculated[6] for the asymmetrical top; they have elements referring to the transition K to $K\pm1$ as well as elements diagonal in K, but they are completely diagonal in m, Λ. If $(K, m, \Lambda|D_{z\zeta}|\Lambda, m, K')$ be an element of these matrix direction cosines, the contribution due to $(g_\zeta-g_\xi)$ is:

$$\sum_{K,\,K',\,m+\sigma=M}(C_{K,\,m,\,\sigma})(C_{K',\,m,\,\sigma}) \Lambda(K, m, \Lambda|D_{z\zeta}|\Lambda, m, K')(g_\zeta-g_\xi). \quad (3c)$$

[6] H. B. G. Casimir, Zeits. f. Physik **59**, 623 (1930). For the table of direction cosines in the notation used here, see G. Placzek and E. Teller, Zeits. f. Physik **81**, 209 (1933).

In particular, the following elements of the direction cosine matrix will be used:[6]

$$
\begin{aligned}
&(1,1,1|D_{z\zeta}|1,1,1)=\tfrac{1}{2},\\
&(2,1,1|D_{z\zeta}|1,1,2)=\tfrac{1}{6},\\
&(2,2,1|D_{z\zeta}|1,2,2)=\tfrac{1}{3},\\
&(1,1,1|D_{z\zeta}|1,1,2)=(2,1,1|D_{z\zeta}|1,1,1)\\
&\qquad\qquad=\tfrac{1}{2}(\tfrac{3}{5})^{\frac{1}{2}}.
\end{aligned}
\tag{4}
$$

Equations (1 a and b), (3 a, b, c) and (4) give for $M=J_0=\frac{1}{2}$, $\Lambda=1$:

$$\mu=\bar{\mu}_z=\tfrac{1}{2}g_s\{a_0^2-\tfrac{1}{3}a_1^2\}+\tfrac{1}{3}a_1^2(g_\xi+g_\zeta) \tag{5a}$$

and for $M=J_0=\frac{3}{2}$, $\Lambda=1$:

$$
\begin{aligned}
\mu=\tfrac{1}{2}g_s\{b_1^2-\tfrac{3}{5}b_2^2\}&\\
+g_\xi\{\tfrac{1}{2}b_1^2+\tfrac{3}{2}b_2^2-\tfrac{1}{5}(3)^{\frac{1}{2}}b_1b_2\}&\\
+g_\zeta\{\tfrac{1}{2}b_1^2+3/10b_2^2+\tfrac{1}{5}(3)^{\frac{1}{2}}b_1b_2\}.&
\end{aligned}
\tag{5b}
$$

2. Spin-Orbit Interaction

Although the spin-orbit coupling energy arises from both Larmor and Thomas precession, it has been pointed out[7] that, in nuclei, the Larmor term may be neglected as compared to the Thomas term since the Coulomb forces are much smaller than the nuclear forces. The interaction energy is therefore[8] $V=(\mathbf{a}\times\mathbf{v}/2c^2\cdot\mathbf{S}\hbar)$, where $\mathbf{a}$ is the acceleration of a particle with spin $\mathbf{S}$ and velocity $\mathbf{v}$ relative to the α-particles that give rise to $\mathbf{a}$. The acceleration $\mathbf{a}$ is the resultant of the acceleration due to the field of the α-particles plus the Coriolis acceleration,[9] and $\mathbf{v}$ is the sum of the body-fixed velocity of the neutron (proton) and the velocity of the neutron with respect to the α-particles due to rotation of the entire system. If the spin-orbit coupling is expressed in terms of these quantities, it takes the form $V=-\alpha(\mathbf{L}_0\cdot\mathbf{S})-\beta(\mathbf{L}_R\cdot\mathbf{S})$, α, $\beta>0$, where $\mathbf{L}_0$ is the orbital angular momentum of the neutron (proton) relative to the α-particles, and $\mathbf{L}_R$ is the rotational angular momentum of the complete system.

The angular momenta for the rotational states of the nuclei can be taken directly from the results of Hafstad and Teller;[2] however, since they are different for different systems, the form of the spin-orbit interactions will be different. Therefore, for the calculation of the ground states of the nuclei, each system will be considered separately; the order in which they are considered will simply be that of complication.

[7] D. R. Inglis, Phys. Rev. **50**, 783 (1936).

[8] See D. R. Inglis, Phys. Rev. **50**, 784 (1936).

[9] Accelerations of the order of magnitude of the centrifugal terms will be neglected, since they depend on the square of the rotational angular momentum.

3. Ground States

O^{17}, F^{17}

The system consists of four α-particles at the vertices of a tetrahedron plus a particle with a spherical node through the α-particles. The ground state therefore arises from $K=0$ and has an angular momentum equal to the spin of the added particle; thus $J_0=\frac{1}{2}$. The magnetic moment is given by the magnetic moment of the added particle.

C^{13}, N^{13}; $\Lambda=0$, $K=1$[10]

The three α-particles form an equilateral triangle, and the added particle has a node in the plane of this triangle so it can have no orbital angular momentum in the body-fixed system. The angular momentum is then due only to rotation of the system as a whole. This yields the interaction $V=-\beta(K\cdot S)$, $\beta>0$.

The usual angular momentum matrices give the energy matrix:

$$
\begin{aligned}
-(K,m,\sigma|V|\sigma',m',K')&\\
=\beta(K,m,\sigma|\sum_{j=x,y,z}K_jS_j|\sigma',m',K')&\\
=\beta\delta_K{}^{K'}\sum_j(K,m|K_j|m',K)(\sigma|S_j|\sigma')&\\
=\beta\delta_K{}^{K'}\{\tfrac{1}{2}((K+m)(K-m+1))^{\frac{1}{2}}\delta_{m-1}{}^{m'}\delta_{\sigma+1}{}^{\sigma'}&\\
+\tfrac{1}{2}((K-m)(K+m+1))^{\frac{1}{2}}\delta_{m+1}{}^{m'}\delta_{\sigma-1}{}^{\sigma'}&\\
+m\sigma\delta_m{}^{m'}\delta_\sigma{}^{\sigma'}\}.&
\end{aligned}
$$

The characteristic values are: $E=\beta$ for $J=\frac{1}{2}$, $E=-\frac{1}{2}\beta$ for $J=1\frac{1}{2}$ arising from $K=1$; $E=1\frac{1}{2}\beta$ for $J=1\frac{1}{2}$, $E=-\beta$ for $J=2\frac{1}{2}$ arising from $K=2$, etc. The state of lowest energy[11] therefore has $J_0=1\frac{1}{2}$ and, since the energy is diagonal in K, $b_2=0$, $b_1=1$ (Eq. (2)).

[10] The angular momenta given here and in the following correspond to the lowest rotational states as given in reference 2.

[11] This assumes that the separation between rotational levels is of the same order of magnitude of, or greater than, the spin-orbit interaction, so that rotational levels cannot be made to cross by this interaction.

Li^7, Be^7; $\Lambda=0$, $K=1$

In this case, the nuclei lack one neutron or proton of being two complete α-particles. This lack of a neutron or proton may be considered as an added neutron or proton "hole," where the hole is placed in the state from which the subtracted particle has been taken. The hole has a node between the two α-particles so that there is no body-fixed orbital angular momentum. In the complete system, there are then three neutrons (protons) with body-fixed orbital angular momentum equal to zero. Two of these particles are in the same state with antiparallel spin and give no contribution to the spin-orbit coupling energy. The third is coupled to rotation of the complete system since there is no body-fixed orbital motion. The interaction is again $V=-\beta(K\cdot S)$, $\beta>0$ so that the ground state has $J_0=1\frac{1}{2}$, $b_2=0$, $b_1=1$.

N^{15}, O^{15}; $K=1$

The system consists of four α-particles at the vertices of a tetrahedron plus a neutron or proton hole. The nucleus is then a spherical top and has no preferred axis; thus the angular momentum may be quantized along any body-fixed axis, but the rotational proper functions are degenerate in this quantum number, Λ.

Just as in the above case, there is one particle in a state of unsaturated spin which is entirely responsible for the interaction energy. However, the unsaturated particle moves relative to the α-particles and therefore gives rise to an orbital angular momentum which combines with rotation of the system as a whole to form the total angular momentum. Without a very detailed treatment, the question of how the total angular momentum is distributed between these two cannot be settled. It will be assumed that the two angular momenta contribute to the total angular momentum in the ratio $\theta/(1-\theta)$ where $0\leqslant\theta\leqslant 1$.

There are now two contributions to the energy; one due to rotation; $V_R=-\beta(K\cdot S)(1-\theta)$, and the other arising from orbital motion, $V_0=\pm\alpha(K\cdot S)\theta$. In order to determine the sign of V_0, it is necessary to consider the wave function of the hole in a little more detail.

Let $\psi_{l,\lambda}(k)$ be one of the possible orbital wave functions of the kth neutron or proton. Since a rotation by $2\pi/3$ about an axis through one α-particle and perpendicular to the plane of the other three permutes the three α-particles, these orbital functions must form a representation of this rotation. Therefore, under this rotation of $\Delta\omega=2\pi/3$, the $\psi_{1,\lambda}(k)$ must transform like $\exp(i\lambda\omega)$; $\lambda=0, 1, -1$. The fourth function, ψ_{00}, will be symmetric under all covering operations of the tetrahedron.

To get the lowest energy for the system, the hole is placed in a state of $|\lambda|=1$. Therefore the proper functions of the system may be taken to be

$$\Psi=\sum_P(-1)PP\psi_{00}(1)\alpha(1)\psi_{00}(2)\beta(2)$$
$$\times\psi_{10}(3)\alpha(3)\psi_{10}(4)\beta(4)\psi_{1,-\lambda}(5)\alpha(5)$$
$$\times\psi_{1,-\lambda}(6)\beta(6)\psi_{1,\lambda}(7)\begin{Bmatrix}\alpha(7)\\ \beta(7)\end{Bmatrix};\quad\lambda=\pm 1,$$

where P is the permutation operating on particles 1 to 7 and α and β are the usual spin wave functions.

If the entire system is rotated by $2\pi/3$ about the threefold axis and the neutrons (protons) by $-2\pi/3$ about the same axis, the proper functions of the complete system should remain unchanged since the α-particles form an Einstein-Bose system. This rotation multiplies the neutron (proton) proper function by $\exp 2\pi i\lambda/3$ and the rotational functions by $\exp 2\pi i\Lambda/3$, where Λ is the projection of K on the axis of rotation. The complete function is multiplied by $\exp 2\pi i(\Lambda+\lambda)/3$ which must be unity, and hence $\Lambda=-\lambda$. The only particle with unsaturated spin, that is, the only one that contributes to the spin-orbit energy, is in the state $\psi_{1,\lambda}$. For this particle, the projection of l, the orbital angular momentum, on the above threefold axis of symmetry is proportional to λ. Since the interaction energy is proportional to $-(l\cdot S)$, the equation $\Lambda=-\lambda$ indicates that the energy V_0 is given by $V_0=\alpha(K\cdot S)\theta$, $\alpha>0$.

The total interaction then becomes $V=V_0+V_R=(K\cdot S)[\alpha\theta-\beta(1-\theta)]=\gamma(K\cdot S)$. The sign of γ is determined by the ratio of rotational to orbital motion with $\gamma\lesseqgtr 0$ for $\theta/(1-\theta)\lesseqgtr\beta/\alpha$, respectively.

The quantity β may be assumed to be small as compared to α, since it gives the contribution to the energy due to rotation of the complete system, which corresponds to a much smaller velocity than that of the orbital motion of the particle. Thus $\theta/(1-\theta)$ is probably greater than β/α and $\gamma>0$. According to the section on C^{13}, N^{13} the characteristic values of the interaction energy would then be $E=-\gamma$ for $J=\frac{1}{2}$ and $E=\frac{1}{2}\gamma$ for $J=1\frac{1}{2}$ arising from $K=1$. Thus the ground state would correspond to $J_0=\frac{1}{2}$ and, since K values are not mixed, $a_0=0$, $a_1=1$ (Eq. (2)).

For $\gamma<0$, the situation is exactly the same as for C^{13} etc., and $J_0=1\frac{1}{2}$, $b_2=0$, $b_1=1$.

Be^9, B^9; $\Lambda=1$, $K=1$

In this system, the proton or neutron has a node through the two α-particles; this corresponds to an angular momentum, $\Lambda=1$, about the body-fixed ζ axis. Therefore the greater part of the interaction energy is (since $\alpha\gg\beta$; see the section on N^{15}, O^{15}) $V=-\alpha(\Lambda\cdot S)$, $\alpha>0$.

In order to calculate this energy matrix, the spin vector, S, with components S_x, S_y, S_z in space must be projected on the body-fixed ζ axis by means of the direction cosines D (see Section 1). This projection, S_ζ, is given by

$$S_\zeta=\sum_{j=x,\,y,\,z} D_{\zeta j}S_j,$$

which yields

$$(\Lambda\cdot S)=\Lambda S_\zeta=\Lambda\sum_j D_{\zeta j}S_j$$

and

$$(K\;\; m,\Lambda,\sigma|(\Lambda\cdot S)|\sigma',\Lambda',m',K')=\delta_\Lambda{}^{\Lambda'}\Lambda\sum_j(K,m,\Lambda|D_{\zeta j}|\Lambda,m',K')(\sigma|S_j|\sigma'),$$

since $D_{\zeta j}$ is diagonal in Λ and σ and S_j is diagonal in K, m, Λ. The $(\sigma|S_j|\sigma')$ are the usual spin matrices and the matrix elements of $D_{\zeta j}$ may be obtained from the table of direction cosines.[6] From these and the fact that the energy must be Hermitian,

$$\begin{aligned}(K,m,\Lambda,\sigma|(\Lambda\cdot S)|\sigma',\Lambda,m',K+1)&=(K+1,m',\Lambda',\sigma'|(\Lambda\cdot S)|\sigma,\Lambda,m,K)\\&=\frac{\Lambda[(K-\Lambda+1)(K+\Lambda+1)]^{\frac{1}{2}}}{(K+1)[(2K+1)(2K+3)]^{\frac{1}{2}}}\\&\quad\times\{\tfrac{1}{2}[(K-m+1)(K-m+2)]^{\frac{1}{2}}\delta_{m-1}{}^{m'}\delta_{\sigma+1}{}^{\sigma'}\\&\quad-\tfrac{1}{2}[(K+m+1)(K+m+2)]^{\frac{1}{2}}\delta_{m+1}{}^{m'}\delta_{\sigma-1}{}^{\sigma'}\\&\quad+\sigma[(K-m+1)(K+m+1)]^{\frac{1}{2}}\delta_m{}^{m'}\delta_\sigma{}^{\sigma'}\},\end{aligned}$$

$$\begin{aligned}(K,m,\Lambda,\sigma|(\Lambda\cdot S)|\sigma',\Lambda,m',K)&=\frac{\Lambda^2}{K(K+1)}\{\tfrac{1}{2}[(K+m)(K-m+1)]^{\frac{1}{2}}\delta_{m-1}{}^{m'}\delta_{\sigma+1}{}^{\sigma'}\\&\quad+\tfrac{1}{2}[(K-m)(K+m+1)]^{\frac{1}{2}}\delta_{m+1}{}^{m'}\delta_{\sigma-1}{}^{\sigma'}\\&\quad+\sigma m\delta_m{}^{m'}\delta_\sigma{}^{\sigma'}\}.\end{aligned}$$

In accordance with Section 1, the energy matrix is to be transformed in such a way that it is labeled by J, M and K values. J and M values and those values of K for which $K\neq J\pm\frac{1}{2}$ are not mixed, therefore the matrix will split up into two-dimensional blocks corresponding to fixed values of J, M and to $K=J\pm\frac{1}{2}$. In particular if $V_{KK'}$ are the matrix elements of V corresponding to $K=J-\frac{1}{2}$, $K'=J+\frac{1}{2}$, then for $J=1\frac{1}{2}$, $M=1\frac{1}{2}$; $V_{11}=-\frac{1}{4}\alpha$, $V_{22}=\frac{1}{4}\alpha$, $V_{12}=V_{21}=-\frac{1}{4}3^{\frac{1}{2}}\alpha$.

In order to transform the energy matrix into diagonal form, it is necessary to calculate the a's and b's of Eqs. (1) and (2). The proper functions of the system may be written as $\Psi=b_K\psi_K+b_{K'}\psi_{K'}$. If H is the complete Hamiltonian of the system, and if $H=H_0-\alpha(\Lambda\cdot S)$, H_0 has the proper values E_K, $E_{K'}$ for $K=J-\frac{1}{2}$, $K'=J+\frac{1}{2}$, and the b's (or the a's) may be calculated by substituting $H_0\psi_K=E_K\psi_K$, $H_0\psi_{K'}=E_{K'}\psi_{K'}$ and $\Psi=b_K\psi_K+b_{K'}\psi_{K'}$ in the equation $H\Psi=E\Psi$. Multiplication by ψ_K, $\psi_{K'}$, respectively, and integration yields:

$$\begin{aligned}(V_{KK}-\delta)b_K+V_{K'K}b_{K'}&=0,\\V_{KK'}b_K+(V_{K'K'}-\delta-\epsilon)b_{K'}&=0,\end{aligned}\qquad(6)$$

with $\qquad\epsilon=E_K-E_{K'};\quad\delta=E-E_K.$

The condition that b_K, $b_{K'}$ be nontrivial determines the energy correction δ. This is

$$\begin{vmatrix}V_{KK}-\delta & V_{K'K}\\ V_{KK'} & V_{K'K'}-\delta-\epsilon\end{vmatrix}=0.\qquad(7)$$

The ground state has $J_0=1\frac{1}{2}$, for which the energy is given by

$$\delta=-\tfrac{1}{2}\epsilon-\tfrac{1}{2}[\epsilon^2-(\epsilon-\alpha)\alpha]^{\frac{1}{2}}.\qquad(8)$$

The coefficients in Eq. (1b) for this state depend on the ratio ϵ/α, and, since this ratio is not known, the calculations will be made for the extreme cases $-\epsilon\gg\alpha$, $-\epsilon\ll\alpha$.

For $-\epsilon\gg\alpha$ (set $\alpha\approx0$), $\delta\approx-\frac{1}{2}\epsilon-\frac{1}{2}|\epsilon|=0$ since ϵ is negative, and, by Eqs. (6), $b_2=0$. According to Eq. (2), the normalized wave functions are given by $b_1=1$.

Similarly, Eqs. (6) and (8) may be solved for $\alpha\gg-\epsilon$ with the result that $b_1=\frac{1}{2}3^{\frac{1}{2}}$, $b_2=\frac{1}{2}$.

B^{11}, C^{11}; $\Lambda=1$, $K=1$

These nuclei are assumed to have three α-particles at the vertices of an equilateral triangle and a hole with a nodal plane intersecting the axis of figure. One particle in a state of unsaturated spin is again responsible for the spin-orbit interaction energy. Just as in the case of N^{15}, O^{15} the angular momentum is assumed to be divided between orbital motion in the body-fixed system and rotation of the system as a whole in the ratio $\theta/(1-\theta)$, $0\leqslant\theta\leqslant 1$. There are then the two contributions to the energy $V_R=-\beta(K\cdot S)(1-\theta)$ and $V_0=\pm\alpha(\Lambda\cdot S)\theta$. The difference between the V_0 of N^{15}, O^{15} and the V_0 here is due to the fact that there is now a definite axis in the body, and the hole has its nodal plane through this axis, that is, its angular momentum about the axis. The sign of V_0 may be determined just as in the case of N^{15}, O^{15} with the simplification that only rotations about the axis of symmetry need be considered. The result is again $V_0=\alpha(\Lambda\cdot S)\theta$. The total interaction is

$$V=V_0+V_R=\alpha(\Lambda\cdot S)\theta-\beta(K\cdot S)(1-\theta).$$

The second term of the interaction energy has been calculated for C^{13}, N^{13} (β is to be replaced by $\beta(1-\theta)$) and it does not mix K values. Therefore the mixing of $K=J_0+\frac{1}{2}$ with $K=J_0-\frac{1}{2}$ will depend entirely on the first term, which has been treated for Be^9, B^9 (α to be replaced by $-\alpha\theta$).

There are again two possibilities for the lowest state; if the term $\alpha\theta(\Lambda\cdot S)$ is the more important term, $J_0=\frac{1}{2}$, and if $-\beta(K\cdot S)(1-\theta)$ is the predominant term, $J_0=1\frac{1}{2}$ for the ground state. These energies may be calculated as for Be^9, B^9; C^{13}, N^{13}, respectively, and the result is that the angular momentum for the ground state is $J_0=\frac{1}{2}$ or $1\frac{1}{2}$, according as $\theta>2\beta/(2\beta+\alpha)$ or $\theta<2\beta/(2\beta+\alpha)$, respectively.

If $\theta<2\beta/(2\beta+\alpha)$, $J_0=1\frac{1}{2}$ and the b's in Eq. (1b) must be calculated as in Eqs. (6) and (7). When $(-\alpha\theta)$ is substituted for α in Eq. (8):

$$\delta=-\tfrac{1}{2}\epsilon-\tfrac{1}{2}[\epsilon^2+(\epsilon+\alpha\theta)\alpha\theta]^{\frac{1}{2}},$$

where ϵ includes not only the separation of rotational levels but also the correction due to $\beta(1-\theta)(K\cdot S)$ or

$$\epsilon=-2\hbar^2/I-2\beta(1-\theta).$$

Eqs. (6) give:

$$b_1{}^2=\tfrac{3}{4}\frac{(\alpha\theta)^2}{(\alpha\theta)^2-2\alpha\theta\delta+4\delta^2};\quad b_2=-\frac{\alpha\theta-4\delta}{3^{\frac{1}{2}}\alpha\theta}b_1.$$

It seems reasonable to assume that the value of θ appearing here is approximately equal to the θ of N^{15}, O^{15}. It is experimentally known[15] that $J_0=\frac{1}{2}$ for N^{15}; this is possible only if $\theta>\beta/\alpha$. It is still possible, however, for the total angular momentum of B^{11}, C^{11} to be $1\frac{1}{2}$ if θ satisfies the condition $\beta/\alpha<\theta<2\beta/(2\beta+\alpha)$. Since β/α is small, the limits are narrow, and it may be assumed that $\theta=(4/3)\beta/\alpha$ (for convenience) with very little error. A value of θ of this order of magnitude is consistent with the fact that the exchange forces for a hole are small,[2] and therefore correspond to a small orbital velocity.

It is again necessary to know the ratio ϵ/α. For $-\epsilon\gg\alpha$, $b_1\approx1$, $b_2\approx0$; and for $-\epsilon\approx\beta\ll\alpha$, $b_1\approx2/7^{\frac{1}{2}}$, $b_2\approx-(3/7)^{\frac{1}{2}}$. The actual values should lie between these two estimates which are valid for β/α not much greater than 0.1.

For $\theta>2\beta/(2\beta+\alpha)$, $J_0=\frac{1}{2}$ and the coefficients in Eq. (1a) become $a_0=0$, $a_1=1$.

4. Magnetic Moments

For the calculation of the magnetic moments, all that is needed now are the g factors. When these are given, they may be substituted in Eqs. (5a) or (5b) after applying the results of Section 3.

In order to calculate the g factors due to rotation, it will be assumed for simplicity that the added particle or hole is uniformly distributed over the system of α-particles with the result that the g factor for pure rotation of the system is given by the ratio of total charge to total mass. The g factor for orbital motion is one for a proton and zero for a neutron, and that for spin is given by twice the magnetic moment μ_π or μ_ν as the case may be. The magnetic moments are given in the adjoining table in units of nuclear magnetons, $\mu_0=e\hbar/2M_0c$ (M_0=mass of proton), but several require special attention.

The identification of the θ of N^{15}, O^{15} with that of B^{11}, C^{11} gave $\theta \approx (4/3)(\beta/\alpha)$ if $J_0=\frac{1}{2}$ and $1\frac{1}{2}$ for N^{15}, O^{15} and B^{11}, C^{11}, respectively. Since β/α is small, θ has been assumed to be zero in giving the magnetic moments for this case. The g factors of these four nuclei depend on the distribution of angular momentum between orbital and rotational motion and are given by:

$$\begin{aligned}
&N^{15}: g_\zeta = g_\xi = \theta + (7/15)(1-\theta),\\
&O^{15}: g_\zeta = g_\xi = (8/15)(1-\theta),\\
&B^{11}: g_\zeta = \theta + (5/11)(1-\theta),\ g_\xi = 5/11,\\
&C^{11}: g_\zeta = (6/11)(1-\theta),\ g_\xi = 6/11.
\end{aligned}$$

For B^{11}, C^{11} with $J_0=1\frac{1}{2}$, the magnetic moments have been listed for all ϵ between $-\epsilon=\beta$ and $-\epsilon \gg \alpha$.

The magnetic moments of Be^9, B^9 are given in Table I for $\alpha \gg |\epsilon|$, $\alpha \ll |\epsilon|$. The α-particle model then predicts that the actual magnetic moments should lie between these two extremes, and the experimental values may be used to determine the ratio ϵ/α.

5. F^{19}, Ne^{19}

The considerations of Hafstad and Teller did not include the nuclei F^{19}, Ne^{19}; since the magnetic moment of F^{19} has been measured,[12] it is of interest to extend the calculations to these nuclei, which consist of five α-particles minus a proton and neutron, respectively. The arrangement of the five α-particles is such that three are at the vertices of an equilateral triangle and the other two are placed one above and the other below the plane of the triangle. In order to set up the invariant wave functions of the hole, it is necessary to consider rotations of $2\pi/3$ about the axis connecting the two α-particles and reflection in the plane of the three α-particles.

If the three α-particles in the plane are labeled 1, 2, 3 and the other two, 4, 5, and if the proper function of the hole on the ith α-particle is φ_i, the orbitals for a given neutron (proton) may be written as

$$\begin{aligned}
\psi_1 &= \varphi_1+\varphi_2+\varphi_3,\\
\psi_2 &= \varphi_1+e^{2\pi i/3}\varphi_2+e^{-2\pi i/3}\varphi_3,\\
\psi_3 &= e^{2\pi i/3}\varphi_1+e^{-2\pi i/3}\varphi_2+\varphi_3,\\
\psi_4 &= \varphi_4+\varphi_5,\\
\psi_5 &= \varphi_4-\varphi_5.
\end{aligned}$$

Since ψ_1 and ψ_4 are invariant under rotation by $2\pi/3$ about the axis of symmetry and under reflection in the plane of symmetry, any linear combination of them is invariant under these operations. They will thus be mixed by the true Hamiltonian to form two states, one with no nodes and the other with two nodal planes, one above and one below the plane of symmetry. Since the other orbitals have one nodal plane and the hole is to be put into the state of highest

[12] J. M. B. Kellogg, I. I. Rabi, N. F. Ramsey, Jr., J. R. Zacharias, Bull. Am. Phys. Soc. **13**, 7, 10 (1938).

TABLE I. *Spins and magnetic moments of the light nuclei by the α-particle model. The last column contains the results of the Hartree model for the magnetic moments and for those values of J_0 (in parentheses) which are different in the two models. All magnetic moments are given in units of nuclear magnetons. The results of the Hartree model are taken from Rose and Bethe, reference 3, with the numerical values corrected to $\mu_\pi=2.78$, $\mu_\nu=-1.75$.*

Nucleus	J_0	Magnetic Moment	α-model	Hartree model
Li^7	$1\frac{1}{2}$	$\mu_\pi+3/7$	3.21	3.08
Be^7,	$1\frac{1}{2}$	$\mu_\nu+4/7$	-1.18	-1.05
Be^9	$1\frac{1}{2}$	$\mu_\nu+2/9<\mu<(3/5)\mu_\nu+1/3$	$-1.53<\mu<-0.7$	-1.40
B^9	$1\frac{1}{2}$	$(3/5)\mu_\pi+0.87<\mu<\mu_\pi+7/9$	$2.55<\mu<3.56$	3.43
B^{11}	$\frac{1}{2}$	$-(1/3)\mu_\pi+(2/33)(3\theta+5)$	-0.44 $(\theta=1)$	
B^{11}	$1\frac{1}{2}$	$(1/3)\mu_\pi+0.57<\mu<\mu_\pi+5/11$	$1.5<\mu<3.2$ $(\theta=0)$	3.43
C^{11}	$\frac{1}{2}$	$-(1/3)\mu_\nu+(2/11)(2-\theta)$	0.77 $(\theta=1)$	
C^{11}	$1\frac{1}{2}$	$\mu_\nu+6/11<\mu<(1/3)\mu_\nu+0.68$	$-1.2<\mu<0.1$ $(\theta=0)$	-1.40
C^{13}	$1\frac{1}{2}$	$\mu_\nu+6/13$	-1.29	1.05 $(J_0=\frac{1}{2})$
N^{13}	$1\frac{1}{2}$	$\mu_\pi+7/13$	3.32	-0.73 $(J_0=\frac{1}{2})$
N^{15}	$1\frac{1}{2}$	$\mu_\pi+7/15$	3.25	
N^{15}	$\frac{1}{2}$	$-(1/3)\mu_\pi+(2/45)(8\theta+7)$	-0.62 $(\theta=0)$	-0.26
O^{15}	$1\frac{1}{2}$	$\mu_\nu+8/15$	-1.22	
O^{15}	$\frac{1}{2}$	$-(1/3)\mu_\nu+(16/45)(1-\theta)$	0.94 $(\theta=0)$	0.59
O^{17}	$\frac{1}{2}$	μ_ν	-1.75	
F^{17}	$\frac{1}{2}$	μ_π	2.78	
F^{19}	$\frac{1}{2}$	μ_π	2.78	
Ne^{19}	$\frac{1}{2}$	μ_ν	-1.75	

energy, which is the one with the greatest number of nodes, it will be in the second of the above mixed states. This state is nondegenerate and the orbital remains unchanged under a rotation around the figure axis. Since the total eigenfunction must be symmetric under all those rotations which correspond to an exchange of α-particles, it is seen that the state $\Lambda=0$, and therefore $K=0$ is allowed. Therefore $J_0=\frac{1}{2}$ and:

$$F^{19}: \mu=\mu_\pi,$$
$$Ne^{19}: \mu=\mu_\nu.$$

The experiments that have been made[12] on F^{19} give $J_0=\frac{1}{2}$, but the magnetic moment is somewhat smaller than predicted. In order to explain this by the pure α-particle model, a more detailed account must be taken of the spin-orbit interaction. The interaction that has been used here does not mix the state with angular momentum $J=\frac{1}{2}$ arising from $K=0$ with the state $J=\frac{1}{2}$ which arises from the higher rotational level $K=1$. However, if the centrifugal terms which have been neglected[9] are taken into account, these two states of the same symmetry may be mixed. Since the state $J=\frac{1}{2}$ arising from $K=1$ has a negative magnetic moment (just as for N^{15}, B^{11}), any slight mixing will decrease the magnetic moment in the ground state.

Conclusion

The numerical results in the table of magnetic moments are given for $\mu_\pi=2.78$ [12] and $\mu_\nu=-1.75$.[13] The results of the Hartree model as given by Rose and Bethe[14] are given in the last column of the table for comparison. The Hartree model gives the same angular momenta and approximately the same magnetic moments for all nuclei except C^{13}, N^{13}. Rose and Bethe give different angular momenta in the ground state for these nuclei with the result that the magnetic moments are much different. However, in the α-model, the state $J=\frac{1}{2}$ is very close to the ground state, $J_0=1\frac{1}{2}$, since the separation is of the order of magnitude of β.

It has already been pointed out[4] that the α-particle value of the magnetic moment of Li^7 is somewhat better than that given by the Hartree model. The two models agree with each other and with experiment in the case of F^{19}. The value of the spin of N^{15} also is in agreement with experiment,[15, 16] but the magnetic moment has not yet been measured.

The results of the α-model for B^{11}, Be^9 may be made to fit a large range of possible values by a suitable choice of θ and α/ϵ, whereas the results given for the Hartree model are unambiguous. However, it must be pointed out that the calculations[3] by the Hartree model do not take into account the mixing of P and S states due to spin-orbit coupling. This has been considered here and is completely responsible for the arbitrariness of the result.

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Note added in proof.—The recent measurement of the magnetic moment of Be^9 by Kusch, Millman and Rabi[17] comes well within the range of possible values given by the α-model if the spin is assumed to be $1\frac{1}{2}$.

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[13] This value is based on the mixing of S and D states in the deuteron as calculated by J. Schwinger, and makes use of the magnetic moments of the proton and deuteron as given in reference 12. The author wishes to express his appreciation to J. Schwinger for giving him this result before publication.

[14] See reference 3. Rose and Bethe assumed $\mu_\pi=2.85$, $\mu_\nu=-2.0$; however, the values given in Table II have been corrected to $\mu_\pi=2.78$, $\mu_\nu=-1.75$ for comparison.

[15] R. W. Wood and G. H. Dieke, J. Chem. Phys. **6**, 908 (1938).

[16] On the assumption that $\theta>\beta/\alpha$; see the section on N^{15}.

[17] P. Kusch, S. Millman and I. I. Rabi, Phys. Rev. **55**, 666 (1939).