On the Theory and Observation of Highly Collapsed Stars

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The investigation presented here falls into two parts. In the first part (Sections A to H) the general relativistic solution given by Schwarzschild of the problem of a homogeneous sphere of constant density is discussed. For every given density a characteristic mass M_l exists, for which the pressure in the center of the sphere becomes infinite, and for which the velocity of light in the same point becomes equal to zero. In the case of collapsed neutron stars M_l is of the order of a large stellar mass. The *effective* mass and the gravitational energy of the sphere are determined as functions of its *proper* mass. Equations are developed which express the velocity and the wave-length of light as functions of the distance from the center of the sphere. The characteristic mass M_l of collapsed neutron stars is expressed in terms of the charge and mass of the electron and the proton and the universal gravitational constant. Some possible relations of the results obtained with recent cosmological theories are pointed out.

The second part of this paper deals with the possibility of actually observing the formation of collapsed neutron stars. The hypothesis is examined that *supernova* outbursts are caused by the formation of *neutron stars*. A number of reasons are advanced which make this hypothesis attractive. From the data which were obtained from numerous observations of the bright supernova (1937) in the extragalactic spiral IC 4182, a number of consequences of the neutron star hypothesis are developed. In particular the *redshift* in the spectrum of the supernova IC 4182 is interpreted as a *gravitational redshift*. On this interpretation it follows that some of the physical characteristics of the central star of a supernova one year after maximum brightness are (in order of magnitude) as follows: radius 100 km, average density 10^{12} g/cm^3 and effective surface temperature greater than 5×10^6 degrees. The light curves, the spectra and the total generation of energy in supernovae are discussed in the light of the neutron star hypothesis.

A. The Neutron Star Hypothesis

The study of supernovae during the past few years has brought to light many new facts, the correct interpretation of which promises to be of considerable value for the solution of a number of astronomical and physical problems. The remarkable brightness of supernovae and the total energy generated in these stars in the short period of a few weeks made us question some of the current ideas about the generation of energy in stars. As Baade and I have pointed out repeatedly[1, 2] it appears doubtful whether any of these ideas can satisfactorily account for the energy liberated in a supernova outburst. It seemed to us in particular that the tremendous *rate* of generation of energy in supernovae would require an explanation along entirely new lines. We suggested[1, 2] that such an explanation might be found in the consideration of the transformation of an ordinary star which is composed mainly of electrically charged particles into a *collapsed neutron star* of exceedingly high density (10^{14} g/cm^3) and small stellar radius (10^6 cm). It must be emphasized that we here use the term neutron star simply to designate a highly collapsed star, the average density of which is of the order of the density of matter existing inside of ordinary atomic nuclei. The very properties of space in highly collapsed stars may be radically different from the properties of ordinary space. When we therefore speak of the neutron composition of such a star this does not necessarily mean neutrons in the ordinary sense. It must be rather taken as a short designation for an extended state of matter of nuclear density in which every region whose linear dimensions d are larger than about $\delta=e^2/mc^2$ is essentially electrically neutral, where e and m are the charge and the mass of the electron and c is the velocity of light.

In stellar bodies whose masses are of the same order as the mass of the sun $M(\odot)=2\times10^{33}$ g and whose densities are of the order of 10^{14} g/cm^3 gravitational effects become so large, that most of the other effects usually considered in the theory of stellar constitution become of secondary importance. We therefore disregard in the first approximation the effects of ordinary chemical reactions and even of nuclear reactions and

[1] W. Baade and F. Zwicky, Phys. Rev. **45**, 138 (1934) and **46**, 76 (1934); Proc. Nat. Acad. Sci. **20**, 254 (1934) and **20**, 259 (1934).
[2] F. Zwicky, Sci. Mon. **40**, 461 (1935); Proc. Nat. Acad. Sci., **22**, 457 (1936) and **22**, 557 (1936).

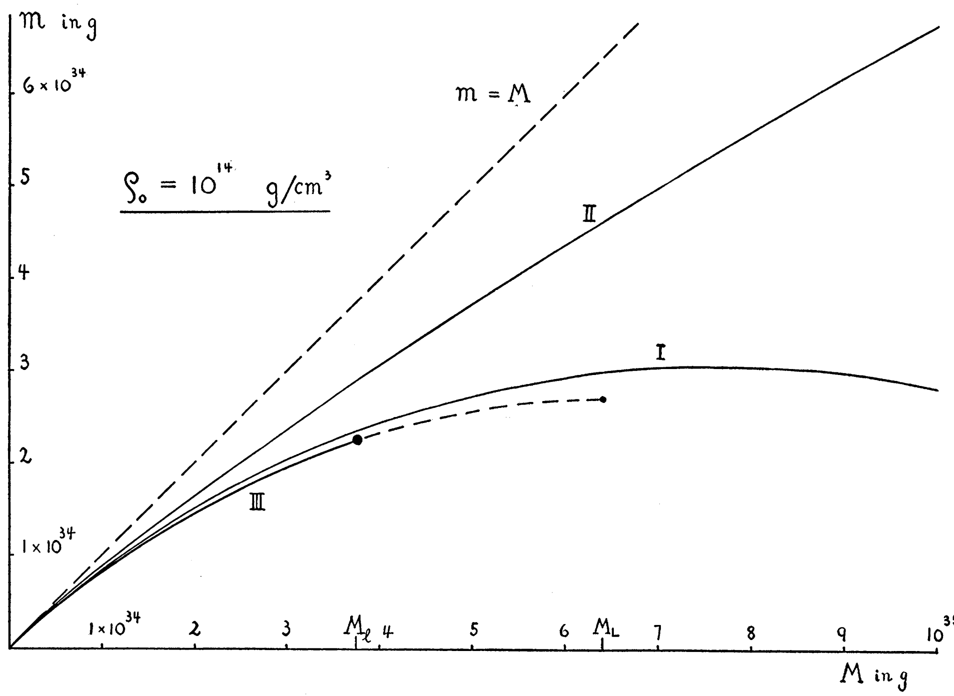

FIG. 1.

limit our discussion essentially to gravitational effects only.

The hypothesis of the formation of neutron stars would run into serious difficulties if one should attempt to retain the classical theory of gravitation. We therefore sketch briefly, why, in the theory of neutron stars it is necessary to introduce general relativistic effects.[3]

B. DIFFICULTIES OF THE CLASSICAL THEORY OF GRAVITATION

Let us consider, for the sake of illustration, a sphere of radius r_1 and of constant *proper* density ρ_0. The total *proper* mass M of the matter contained in this sphere is, by definition, the mass of this matter when dispersed over a very large volume. The gravitational potential energy E_g of our sphere is

$$E_g = -3\Gamma M^2/5r_1 \tag{1}$$

with

$$M = 4\pi\rho_0 r_1^3/3 \tag{2}$$

and $\Gamma = 6.66 \times 10^{-8}$ g^{-1} cm^3 sec.$^{-2}$. We rewrite (1) in terms of ρ_0 and M and we obtain

$$E_g = -\tfrac{3}{5}(4\pi/3)^{\frac{1}{3}}\Gamma M^{5/3}\rho_0^{\frac{1}{3}} = -6.44 \times 10^{-8} M^{5/3}\rho_0^{\frac{1}{3}}. \tag{3}$$

E_g is measured relative to the usual zero configuration in which the mass M is assumed to be dispersed over a very large volume ($\rho = 0$). According to the special theory of relativity the mass equivalent of E_g is

$$\Delta = |E_g|/c^2 = 7.2 \times 10^{-29} M^{5/3}\rho_0^{\frac{1}{3}}. \tag{4}$$

If we indefinitely increase M at constant ρ_0, or ρ_0 at constant M we arrive according to (4) at configurations for which

$$\Delta = M \quad \text{or even} \quad \Delta > M. \tag{5}$$

[3] An outline of the following considerations was first presented in December, 1933, to the Physical Society meeting at Stanford (reference 1). Some details were subsequently published in various papers on supernovae (references 1 and 2). At that time very little was known definitely about supernovae and it seemed premature to discuss in any detail the formation of neutron stars as a possible cause for supernovae. In the meantime, through the discovery with the 18-inch Schmidt telescope on Palomar mountain of eight supernovae, the existence of supernovae as a special class of temporary stars may be regarded as established beyond reasonable doubt (F. Zwicky, Publ. Astr. Soc. Pac. **49**, 204 (1937); **50**, 215 (1938) and **51**, 36 (1939)). Also, the spectral studies of the bright supernovae in IC 4182 (August, 1937) and NGC 1003 (September, 1937) have furnished data (R. Minkowski, Astrophys. J., March (1939); F. Zwicky, Astrophys. J. **88**, 522 (1938)) which fully justify a more detailed examination of the neutron star hypothesis at the present time.

The effective mass m of our sphere would in these cases become

$$m = M - \Delta \lesseqgtr 0. \tag{6}$$

If we set ρ_0 equal to a nuclear density of the order

$$\rho_n = 10^{14}\ \mathrm{g/cm^3} \tag{7}$$

the condition (6) is realized if M is of the order of a very large stellar mass. Indeed, for $\rho_0 = \rho_n$ we obtain $m = 0$ if

$$M = M_{\mathrm{critical}} = 1.64 \times 10^{35}\ \mathrm{g}. \tag{8}$$

It is approximately $M_{\mathrm{crit}} = 80M(\odot)$. Curve I in Fig. 1 shows a plot of the effective mass m as a function of the proper mass M of our sphere as given by formula (9) for $\rho_0 = \rho_n$.

$$m = M - \tfrac{3}{5}(4\pi/3)^{\frac{1}{3}}\Gamma M^{5/3}\rho_0^{\frac{1}{3}}c^{-2}. \tag{9}$$

From curve I it is obvious that the gravitational "packing fraction" of neutron stars assumes considerable values if M is an appreciable fraction of M_{crit}. Consequently, in such stars we may expect to encounter conditions with which we are not at all familiar in the theory of the idealized models of ordinary stars which are at present customary.

The question naturally arises whether actual configurations of dense matter in bulk, such as neutron stars, can possibly behave as paradoxially as the relation (9) suggests. Although negative masses have been considered in the theory of elementary corpuscles, we have no observational knowledge of matter in bulk, such as stars, of negative effective mass. We must therefore look for a solution of our problem on the basis of further theoretical considerations. In the first place we may proceed a little farther on the basis of the classical theory of gravitation coupled with the result of the special theory of relativity that the mass equivalent of any type of energy E is $\Delta = E/c^2$. This latter fact suggests that the relation (1) for the gravitational energy of our sphere ceases to be strictly correct as soon as the effective mass m differs from the proper mass M. Although, on the special theory of relativity, we do not know how the mass equivalent Δ of the gravitational energy is to be properly distributed over the sphere, we may, for the sake of illustration smear it uniformly over this sphere, with the result that the relations (1) and (9) now assume the following form

$$E_g = -3\Gamma m^2/5r_1 \tag{10}$$

and

$$m = M - 3\Gamma m^2/5r_1c^2 \tag{11}$$

or, since (2) still holds true,

$$m = M - 3\Gamma(4\pi\rho_0/3)^{\frac{1}{3}}m^2/5c^2M^{\frac{1}{3}}. \tag{12}$$

If we put

$$1/\alpha = 3\Gamma(4\pi\rho_0/3)^{\frac{1}{3}}/5c^2, \tag{13}$$

then (12) may be written as

$$m = \tfrac{1}{2}\alpha M^{\frac{1}{3}}[(1+4M^{\frac{2}{3}}/\alpha)^{\frac{1}{2}} - 1]. \tag{14}$$

For values of the proper mass M which are small compared with the critical mass M_{crit} we may expand and we obtain

$$m = M - (3\Gamma/5c^2)(4\pi\rho_0/3)^{\frac{1}{3}}M^{5/3} + \cdots, \tag{15}$$

which in the first approximation is identical with (9). For large values of the proper mass $M \gg M_{\mathrm{crit}}$ it is approximately

$$m = \alpha^{\frac{1}{2}}M^{\frac{2}{3}}. \tag{16}$$

According to (14) the effective mass m of a sphere of constant density therefore never becomes negative, but increases somewhat more slowly than M itself. Curve II in Fig. 1 shows the dependence of m on M according to the relation (14).

From what we have said already, it is clear that the classical theory of gravitation, even if combined with the fundamental results of the special theory of relativity is unsatisfactory, because it does not allow us, to formulate clearly how the mass equivalent of the gravitational potential energy is to be distributed over space. This difficulty has its real root in the fact that gravitation is a *cooperative phenomenon*.[4] Systems which are subject to the actions of such cooperative phenomena can be treated satisfactorily only with the help of theories which make possible the treatment of these systems in their entirety. The general theory of relativity is the only theory which permits a unified treatment of gravitation as a cooperative phenomenon. We therefore must now consider the results of this theory regarding the effective masses of large dense stars.

[4] F. Zwicky, Phys. Rev. **43**, 270 (1933).

C. The Schwarzschild Configuration

We again discuss the case of a sphere of constant density ρ_0. The general relativistic treatment of this case was first given by Schwarzschild.[5] If we neglect the cosmological constant the relativistic line elements for the exterior and the interior of the sphere have the following form:

exterior,

$$ds^2 = -dr^2/(1-2\bar{m}/r) - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + (1-2\bar{m}/r)dt^2; \quad (17)$$

interior,

$$ds^2 = -dr^2/(1-r^2/R^2) - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + [A - B(1-r^2/R^2)^{\frac{1}{2}}]^2 dt^2; \quad (18)$$

where r, θ, ϕ and t are the polar coordinates and the time. The constants involved have the following values

$$A = \tfrac{3}{2}(1-r_1^2/R^2)^{\frac{1}{2}}, \quad B = \tfrac{1}{2}, \quad (19)$$

$$R^2 = 3/8\pi\bar{\rho}_0, \qquad \bar{m} = 4\pi\bar{\rho}_0 r_1^3/3, \quad (20)$$

where r_1 is the radius of our sphere, m is the effective mass of the sphere, which, in distant points ($r \gg r_1$), determines the gravitational field in the ordinary Newtonian way. The bars on $\bar{m}$, $\bar{M}$ and $\bar{\rho}_0$ indicate that the masses involved are expressed in the customary relativistic units rather than in ordinary units. These units for mass are so chosen that the mass m expressed in ordinary units is in the following manner related to the mass $\bar{m}$ which appears in the preceding relations.

$$\bar{m} = m\Gamma/c^2, \quad (21)$$

where

$$\Gamma/c^2 = 7.42 \times 10^{-29} \text{ cm/g}. \quad (22)$$

If we assume that $\bar{\rho}_0$ is a constant throughout the sphere, its proper mass $\bar{M}$ is given by

$$\bar{M} = 4\pi\bar{\rho}_0 \int_0^{r_1} r^2(1-r^2/R^2)^{-\frac{1}{2}} dr. \quad (23)$$

Equations (20) and (23) determine the values of r_1 and $\bar{m}$ in terms of the proper mass $\bar{M}$ and the proper density $\bar{\rho}_0$. We rewrite these equations by introducing a new variable x, where

$$x = r_1/R. \quad (24)$$

Eq. (23) now reads

$$\bar{M} = 4\pi R^3 \bar{\rho}_0 \int_0^x \xi^2(1-\xi^2)^{-\frac{1}{2}} d\xi \quad (25)$$

$$\text{or } \bar{M}/\bar{m} = (3/2x^3)[-x(1-x^2)^{\frac{1}{2}} + \arcsin x]. \quad (26)$$

From (26) it follows that there exists a mathematical limit[6] to the Schwarzschild solution (18) for $x=1$, which is equivalent to the following relations:

$$r_1 = R, \quad r_1^2 = 3/8\pi\bar{\rho}_0, \quad r_1 = 2\bar{m}. \quad (27)$$

The physical significance of such a limit would be that the exterior space of the sphere would entirely disappear, and the interior of the sphere would represent the whole space available.

In addition to the fact that the Schwarzschild solution (18) breaks down already for $x=(8/9)^{\frac{1}{2}}$ it must be remarked that in actual space generally any complete closure of a star cannot take place for the following reason. In the exterior space of the sphere the average density ρ, although it is small, is nevertheless not exactly zero as we have assumed it. The exterior space can therefore not, so to speak, be entirely swallowed up by our sphere. In fact, even if ρ were initially equal to zero, it would necessarily cease to be equal to zero during the process of formation of the limiting Schwarzschild sphere. During this process the gravitational "packing" energy

$$|E_g| = (M-m)c^2 \quad (28)$$

[5] The formulation of the Schwarzschild solution used here may be found in R. C. Tolman, *Relativity, Thermodynamics, Cosmology* (Oxford, 1934).

[6] Actually this limit cannot be realized, because the Schwarzschild solution already breaks down at $x=(8/9)^{\frac{1}{2}}$, for which value of x the pressure becomes infinite at the center of the sphere. In the following section we shall nevertheless proceed to discuss the behavior in the whole range $0 \leqslant x \leqslant 1$ of those properties of our star which do not exhibit any singularity at $x=(8/9)^{\frac{1}{2}}$. This facilitates the mathematical discussion since for $x=1$ the formulae involved take on a simpler form than if we inserted $x=(8/9)^{\frac{1}{2}}$. It must be remembered, however, that $x=(8/9)^{\frac{1}{2}}$ represents the maximum value of x for which any physical significance can be ascribed to the solution for a sphere of constant density which we have used here. In Section *G* we shall later on calculate some of the characteristics which interest us for the actual physical limit $x=(8/9)^{\frac{1}{2}}$. In this connection attention should be called to the fact that the Schwarzschild interior solution (18) does not seem to be the most general solution for a sphere of constant density. A discussion of a more general solution has just been published by Mr. G. M. Volkoff (Phys. Rev. **55**, 413 (1939).

must be transferred from the sphere into its exterior, such that the exterior of the sphere necessarily will contain some energy or mass.

Limiting cases for which $\rho_0 \neq 0$ inside of the sphere and $\rho_e = 0$ outside, therefore would necessitate a rediscussion on the basis of a somewhat more refined picture regarding the distribution of mass over the various parts of the space considered. For instance as a next approximation a sphere of constant density ρ_0 imbedded in a universe of average density $\rho_e \ll \rho_0$ might be investigated. Nevertheless, as long as x is sensibly smaller than unity the corrections thus introduced will be negligible, and for small values of x the solution (17) and (18) as given by Schwarzschild will remain applicable to real cases. The discussion of the Schwarzschild formulae for $x<1$ leads to the following results.

As we have already stated, the expression $(M-m)c^2$ may be regarded as the energy which is released by the sphere into exterior space during the process in which the mass M dispersed over a large volume is condensed into the sphere of radius r_1. It is perhaps useful to show that for $x \ll 1$ the Schwarzschild expression for E_g is in the first approximation identical with the ordinary Newtonian value (1) of the gravitational packing energy of a sphere of constant density. Indeed, if we expand (26) in terms of x we obtain

$$\bar{M}/\bar{m} = M/m = 1 + 3x^2/10 + 9x^4/56 + \cdots. \quad (29)$$

From (20) and (24) we derive the explicit value of x in terms of ρ_0 and m. It is

$$x = (\pi/3)^{1/6} 2^{5/6} \bar{\rho}_0^{1/6} \bar{m}^{\frac{1}{3}}, \quad (30)$$

or in terms of $\bar{m}$ and r_1

$$x^2 = 2\bar{m}/r_1. \quad (31)$$

If we return again to ordinary units of mass, (31) is equivalent to

$$x^2 = 2\Gamma m / r_1 c^2. \quad (32)$$

Therefore, in the first approximation

$$M/m = 1 + 3\Gamma m / 5 r_1 c^2. \quad (33)$$

Since for $x \ll 1$ we have $M - m = \Delta \ll M$, we finally obtain

$$m = M - 3\Gamma M^2 / 5 r_1 c^2, \quad (34)$$

which is identical with the expression (9) for the effective mass of a sphere of constant density. Agreement, for weak gravitational fields, with the Newtonian theory is a necessary prerequisite of any general theory of gravitation. The general relativistic expressions (28) and (26) for the gravitational energy of spheres of uniform density are therefore in so far satisfactory as they converge towards the Newtonian expression (9) in the case of proper masses M which are small compared with the critical mass M_{crit} (6).

We now discuss somewhat more fully the dependence of the effective mass m in the Schwarzschild solution on the proper mass M. It can easily be shown that the difference between the Newtonian expression and the Schwarzschild expression for the gravitational E_g is inappreciable for all possible values of x. The real difference between the two expressions is not a numerical one. The essential difference rather consists in the fact that in the Newtonian case we may, at constant density ρ_0, increase the proper mass M of our sphere indefinitely without introducing any infinite values for the pressure or making the velocity of light anywhere equal to zero. In the general theory of relativity this is not possible because of the existence of critical masses some characteristics of which we shall discuss below.

For values $x>1$ the Schwarzschild solution breaks down mathematically since the expression $(1-x^2)^{\frac{1}{2}}$ which occurs in the line element becomes imaginary. For the mathematical real limit $x=1$ we obtain from (26)

$$\bar{M}_L/\bar{m}_L = M_L/m_L = 3\pi/4. \quad (35)$$

This means that the gravitational energy E_{gL} which is liberated in the process of contraction of matter of the proper mass M_L from the completely dispersed state into the configuration $x=1$ is equal to

$$-E_{gL} = (M_L - m_L)c^2 = M_L(1 - 4/3\pi)c^2 \quad (36)$$

or

$$-E_{gL} = 0.58 M_L c^2. \quad (37)$$

Consequently somewhat more than half the proper mass would be transformed into energy during the formation of the sphere $x=1$.

The relation between M_L and the density ρ_0 follows from (27) and (35) with the result that

$$M_L^2 \rho_0 = \pi (3c^2/8\Gamma)^3. \quad (38)$$

The Newtonian expression for the gravitational energy E_{gL}' of a sphere of mass M_L and of constant density ρ_0 is, according to (3)

$$E_{gL}' = -\tfrac{3}{5}(4\pi/3)^{\frac{1}{3}}\Gamma M_L^{5/3}\rho_0^{\frac{1}{3}} \tag{39}$$

or with substitution of (38)

$$E_{gL}' = -(9/40)(4/3)^{\frac{1}{3}}\pi^{\frac{2}{3}}M_L c^2 = -0.53 M_L c^2. \tag{40}$$

The difference between the general relativistic and the Newtonian values of the gravitational energy for $M = M_L$, therefore, is

$$E_{gL} - E_{gL}' = -0.05 M_L c^2, \tag{41}$$

which is less than ten percent of the respective values of E. This result is perhaps somewhat unexpected if we consider that for highly collapsed configurations the Schwarzschild solution presents conceptually and actually aspects so fundamentally at variance with the Newtonian theory. The differences between the two theories are especially striking when the properties of light signals from our sphere to a distant observer outside are investigated. We shall come back to this problem in Sections F and G.

We now show that the effective mass m of a sphere of constant density ρ_0 and of proper mass M as given by Schwarzschild's solution is for all possible values of ρ_0 and M slightly smaller than the effective mass m given by the Newtonian expression (9) or the corrected Newtonian expression (14). For this purpose we investigate the respective values of the derivatives $(dm/dM)_{\rho_0=\mathrm{const}}$. From the Schwarzschild solution (26) we derive

$$(dM/dm)_{\rho_0=\mathrm{const}} = \frac{1}{2}\left[\frac{3}{x^3} - \frac{9m}{x^4}\frac{dx}{dm}\right] \times\left[-x(1-x^2)^{\frac{1}{2}} + \arc\sin x\right] + \frac{3m}{2x^3}\left[-(1-x^2)^{\frac{1}{2}} + \frac{x^2+1}{(1-x^2)^{\frac{1}{2}}}\right]\frac{dx}{dm}. \tag{42}$$

From (30) it follows

$$dx/dm = x/3m. \tag{43}$$

Substituting (43) in (42) we obtain

$$(dm/dM)_{\rho_0=\mathrm{const}} = (1-x^2)^{\frac{1}{2}} \tag{44}$$

or with (31) and (44)

$$(dm/dM)_{\rho_0=\mathrm{const}} = \left[1 - 2\Gamma(4\pi\rho_0/3)^{\frac{1}{3}} m^{\frac{2}{3}}/c^2\right]^{\frac{1}{2}}. \tag{45}$$

On the other hand, we derive from the Newtonian expression (9) the following relation

$$(dm/dM)'_{\rho_0=\mathrm{const}} = 1 - \Gamma(4\pi\rho_0/3)^{\frac{1}{3}} M^{\frac{2}{3}}/c^2. \tag{46}$$

Since always $m < M$ it is seen that for all values of M and ρ_0

$$(dm/dM)' > (dm/dM) \tag{47}$$

or, in turn

$$m(\rho_0, M)_{\mathrm{Newton}} > m(\rho_0, M)_{\mathrm{Schwarzschild}}. \tag{48}$$

Again for values of $M \ll M_{\mathrm{crit}}$ it is approximately

$$(dm/dM)'_{\mathrm{Newton}} = (dm/dM)_{\mathrm{Schwarzschild}}, \tag{49}$$

which is identical with our former conclusion (34) that for small proper masses M the general theory of relativity and the Newtonian theory lead to the same gravitational energy of our sphere. The curve III in Fig. 1 represents $m(M)$ of a sphere of constant density $\rho_0 = 10^{14}$ g/cm^3 as given by the Schwarzschild solution.

D. THE SCHWARZSCHILD MASS M_L FOR NEUTRON STARS

According to (35) it would be, for $x = 1$

$$m_L = 4M_L/3\pi \tag{50}$$

and from (30)

$$x = (\pi/3)^{1/6}2^{5/6}\rho_0^{1/6}m_L^{\frac{1}{3}}(\Gamma/c^2)^{\frac{1}{2}} = 1. \tag{51}$$

Substituting (50) in (51) we obtain

$$M_L = 2^{-9/2}3^{\frac{3}{2}}\pi^{\frac{1}{2}}(\Gamma/c^2)^{-\frac{3}{2}}\rho_0^{-\frac{1}{2}} \tag{52}$$

or numerically

$$M_L = 6.39\times 10^{41}\rho_0^{-\frac{1}{2}}. \tag{53}$$

If we adopt $\rho_0 = 10^{14}$ g/cm^3 as a standard nuclear density which is of the order of the density to be expected in collapsed neutron stars, the characteristic proper mass M_L becomes[7]

$$M_L = 6.4\times 10^{34}\ \mathrm{g}. \tag{54}$$

This mass is approximately equal to $32M(\odot)$.

[7] Following the discussion given by R. C. Tolman in *Relativity, Thermodynamics and Cosmology* (Oxford, 1934), p. 247, I have recently stated (F. Zwicky, Astrophys. J. **88**, 522 (1938)) that M_L represents a natural upper limit for the mass of neutron stars inasmuch as for this mass a neutron star would entirely close itself and thus represent the whole space available. In view of the reasons given in Section E such an extreme configuration can, however, probably never be realized.

In regard to some views recently advanced[8] concerning possible changes of dimensionless ratios among the so-called fundamental physical constants it is of interest to express M_L or M_l (see Section G) in terms of these constants. This may be done in the following way. Instead of adopting for ρ_0 a definite numerical value we express ρ_0 approximately in terms of fundamental physical constants. We start with the expression for a length δ of nuclear dimension

$$\delta = e^2/m_e c^2 = 2.81 \times 10^{-13}\ \text{cm}, \tag{55}$$

where e and m_e are the charge and mass of an electron. As the prototype v of a nuclear volume we may take the sphere of radius δ such that

$$v = 4\pi\delta^3/3 = 9.3 \times 10^{-38}\ \text{cm}^3. \tag{56}$$

If the volume v contains the mass m_P of a proton (or a neutron), the average density ρ' in v is

$$\rho' = m_P/v = 1.78 \times 10^{13}\ \text{g/cm}^3. \tag{57}$$

In an actual collapsed neutron star the volume occupied by each neutron is equal to γv where γ is of the order of unity (probably about equal to $\gamma = 5$). We therefore introduce in (52) the following expression for the density

$$\rho_0 = m_P/\gamma v = \rho'/\gamma \tag{58}$$

and we obtain for the prototype (p) of a highly collapsed star

$$M_L(p) = \gamma^{\frac{1}{2}} 3\pi 2^{-7/2} (e^2/\Gamma m_P m_e)^{\frac{3}{2}} m_N \tag{59}$$

or
$$M_L(p) = 0.83 \gamma^{\frac{1}{2}} R^{\frac{3}{2}} m_N, \tag{60}$$

where $R = 2.29 \times 10^{39}$ is the well-known dimensionless number which represents the ratio between the electric and the gravitational attraction between an electron and a proton.

For an actual neutron star the constant γ may be written as a product

$$\gamma = \gamma_1 \gamma_2, \tag{61}$$

where $\gamma_2 \neq 1$ indicates that the neutron, considered as a hard sphere has a proper density $\rho = \rho'/\gamma_2$, whereas $\gamma_1 \neq 1$ indicates that if we try to pack hard spheres as closely as possible there still remain interstices which amount to 26 percent of the total volume. Consequently for an assembly of closely packed spheres

$$\gamma_1 = 100/(100 - 26) = 1.35.$$

Concerning the numerical value of γ_2 we may make no very precise statement. Since many neutrons are packed into ordinary heavy nuclei we might adopt the average nuclear density of such nuclei as a reasonable value for ρ_0, as we have done in calculating M_L as given by (54). It must be remembered, however, that some uncertainty arises regarding the actual value of γ_2 and therefore of ρ_0 because of the tremendous gravitational pressure in neutron stars. This pressure affects the closest approach or the packing of neutrons in a way which cannot be predicted without any further studies of the fundamental properties of neutrons. Also it must be remembered that a neutron star represents a Fermi degenerate state with resulting kinetic energies of the individual neutrons which may be quite comparable with the proper energy $m_N c^2$ of a neutron. The next step of the theoretical approach to neutron stars therefore must take into account the *equation of state* of a degenerate neutron star. This, however, lies beyond the scope of the preliminary outline of the theory of highly collapsed stars given in this paper.

E. Relation to Some Cosmological Speculations

Finally a few remarks concerning the possible *cosmological* importance of the dimensionless ratio R may be in order. Many years ago H. Weyl[9] first remarked that the tremendous value of R suggests a possible connection of R with other equally large numbers. Astronomical observations during the last two decades have brought to light the existence of several numbers of this kind. These observations refer in particular to the redshift of light from distant nebulae and to the distribution of nebulae in space.

Denote with ν the frequency of a specific spectral line, such as the H or K line of calcium, as measured for the light of a terrestrial source. The frequency ν' of the same spectral line in the

[8] F. Zwicky, Phys. Rev. **43**, 1031 (1933) and **53**, 315 (1938); Philosophy of Science **1**, 353 (1934); Proc. Nat. Acad. Sci. **23**, 106–110 (1937).

[9] H. Weyl, *Raum. Zeits. Materie*, 5th edition (Springer, Berlin, 1923), p. 277. See also Naturwiss. **22**, 145 (1934).

light which reaches the earth from a distant nebula in the first approximation is $\nu'=\nu+\Delta\nu$, where

$$\Delta\nu/\nu=-D/R'. \quad (63)$$

If the distance D of the nebula is measured in units of $\delta=e^2/m_ec^2$, then R' according to Hubble and Humason is a dimensionless number of the order

$$R'=6\times 10^{39}. \quad (64)$$

A second large number N whose order of magnitude may be determined by direct observations is the number of elementary particles, such as protons and electrons, which is contained in the sphere the radius of which is $R'\delta$, where

$$R'\delta=1.68\times 10^{27}\text{ cm}\cong 1.7\times 10^9\text{ light years}. \quad (65)$$

If ρ_s is the average density of matter in this sphere we have

$$N\cong(4\pi/3)(R'\delta)^3\rho_s/m_P. \quad (66)$$

From the results of counts of nebulae, combined with an estimate of the mass of average nebulae, we may assume in order of magnitude

$$\rho_s\cong 10^{-28}\text{ g/cm}^3. \quad (67)$$

Consequently $N\cong 1.2\times 10^{78}$, (68)

which is not far different from R^2.

A third large number R'' might ultimately be derived from observations resulting in a possible curvature of space. Such observations have been attempted by Hubble who counted nebulae to the limit reached by the 100-inch telescope. The apparent too rapid increase in the number of nebulae per unit volume at very great distances might be interpreted as indicating the existence of a curvature of space which is different from zero. Since certain objections can be raised against such an interpretation, we shall for the present not longer dwell on the discussion of R''.

As was already stated, Weyl[9] and more recently Dirac[10] and others have suggested that there exist relations among the dimensionless numbers R, R', R'' and N. Such an hypothesis must naturally be regarded as mere speculation unless it allows us to draw further conclusions which are in agreement with observational facts. Such conclusions can actually be drawn, as I have previously pointed out,[8] if we examine Weyl's hypothesis in the light of the fundamental question, whether or not the average physical conditions in our universe are slowly but systematically changing. For instance, if the redshift of light from distant nebulae is to be interpreted as an average velocity of recession of all nebulae from any observer fixed to a given nebula, we should be confronted with such a systematic change (expansion of the universe). On the other hand, if the redshift should find another explanation leaving the dimensions of the universe stationary in the sense that R' is an absolute constant we might find that matter on the average is transforming itself gradually into radiation. In any of these cases we may be forced to conclude that either R', or N, or both are changing. If invariant relations exist between R, R', R'' and N it follows that all of these numbers change if any one among them changes. Such a conclusion, however, leads to the possibility of further tests of Weyl's hypothesis. One such possibility is indicated by the following line of argument.

In the sense of Sections D and G we may regard the expression $R^{\frac{3}{2}}m_N$ as the prototype of a stellar mass. Since, according to the mass-luminosity relation the brightness of a star is essentially determined by its mass, a change in R should result in a change of the average brightness of stars. In the specific case that R' and R slowly change their values because of a mutual recession of all nebulae this leads to a change in the average brightness of stars and nebulae as a function of the distance D from the observer. Numbers of nebulae to given successively fainter limiting magnitudes in this way would be theoretically related to the magnitude of the redshift observed on the same limiting nebulae. This conclusion from Weyl's hypothesis therefore is subject to direct tests through counts of nebulae to ever decreasing apparent brightness coupled with observations on the redshift. It is important to notice that in such a test the very difficult determination of the distances of faint nebulae is entirely dispensed with.

We briefly mention one other cosmological aspect of the theory of neutron stars. O. Stern[11]

[10] P. A. M. Dirac, Nature **139**, 323 (1937).

[11] O. Stern, Zeits. f. Elektrochemie **31**, 448 (1925).

first pointed out that the application of the principles of statistical mechanics and of thermodynamics to the equilibrium between matter and radiation in the universe leads to the conclusion that in a state of equilibrium practically all matter would be found to have been transformed into radiation. The fact that stars are radiating energy continuously into empty space, combined with our present ignorance of any phenomena which transform this radiation back into matter might be taken as a confirmation of Stern's statement that in the progression of the universe towards a stationary state the systematic transformation of matter into radiation would seem inevitable. The problem raised by Stern was subsequently discussed by many investigators[12] without any essential difference in the final result being achieved. All attempts to save matter from its final annihilation without abandoning entirely the principles of statistical mechanics were blocked by the fact that according to these principles the ratio of the concentration c_m of matter in the ordinary forms and of the concentration c_R of matter transformed into radiation were determined by a Boltzmann factor of undefeatable magnitude, namely

$$c_R/c_m \sim e^{mc^2/KT}. \qquad (69)$$

Here m is the mass of an elementary particle, c the velocity of light, $K=1.37\times10^{-16}$ erg Boltzmann's constant and T the average absolute temperature in the universe. Inserting for m the mass of the electron m_e and assuming for T as high a value as 100° abs. we find that the Boltzmann factor in (62) is still equal to about

$$e^{m_ec^2/100K} \sim e^{6\times10^7}, \qquad (70)$$

which means that in a sphere of the enormous radius R' not even a single electron could exist in statistical competition with radiation. As a possible escape from this unpleasant conclusion I pointed out some time ago[13] that it is not permissible to consider only matter in its ordinary free state in the preceding discussion. Matter may indeed assume states of high statistical weight through gravitational clustering. If this clustering leads to configurations, as it does in limiting neutron stars, where the velocity of light becomes zero, the gravitational packing energy of a particle of mass m may become equal to its original proper energy mc^2. Such configurations of matter consequently are distinguished by Boltzmann factors of precisely the same magnitude as (69) and matter in this form may successfully enter into statistical competition with radiation. It is obvious, however, that a satisfactory solution of the whole problem must also include an estimate not only of the Boltzmann factors but also of the so-called *a priori* weights of matter and of radiation.

F. Velocity and Redshift of Light Emitted from the Surface of the Star

A light signal which travels radially from the star is described by the conditions

$$ds=0, \qquad d\theta=d\phi=0. \qquad (71)$$

In combination with the expression (17) for the line element in the exterior of the sphere this leads to the following value c' for the velocity of light

$$\overline{c'}=dr/dt=1-2\bar{m}/r \qquad (72)$$

or if time t and mass m are measured in ordinary units

$$c'=dr/dt=c(1-2\Gamma m/rc^2), \qquad (73)$$

where c' and c are also in ordinary units. For $x=1$ it would be $2\bar{m}/r_1=1$ on the surface of the star. Consequently $c'=0$ on the surface. For values of x close to $x=1$ the velocity of light will be almost equal to zero for $r=r_1$ and will gradually grow towards its ordinary value c for $r\gg r_1$. A light signal, for its journey from the surface of the star to an external point $r>r_1$ therefore consumes a time which becomes longer and longer as x grows and which approaches an infinite value as x approaches unity. For the reasons discussed in Sections C and G these extreme conditions are, however, unlikely to be ever actually realized.

The redshift of light from the surface of the star is given by the following relation

$$\lambda'/\lambda=dt/ds=(1-2\bar{m}/r)^{-\frac{1}{2}}. \qquad (74)$$

Here λ is the wave-length of a spectral line observed in a point P far away from the star which an atom would emit if placed itself in or near

[12] R. C. Tolman, Proc. Nat. Acad. Sci. **12**, 670 (1926).
[13] F. Zwicky, Proc. Nat. Acad. Sci. **14**, 592 (1928).

the point P. On the other hand, λ' is the wave-length of the same spectral line again observed in P but emitted by an identical atom placed at the distance r from the center of the star.

If we approach $x=1$ we see that for r approaching r_1 the wave-length λ' grows ever larger and for $x=1$ would become infinite.

The results expressed in (63) and (64) signify that the physical communication between a point P_s on the surface of our collapsed star and a point outside become increasingly difficult as the mass and the density of a star increase, both because the light traveling from P_s to P consumes an ever increasing time and because quanta of light originating in familiar emission processes reach the point P with energies which appear more and more depleted as compared with the energy of quanta produced by the identical processes of emission in the point P itself.

From (73) and (74) the velocity of light c' on the surface of a sphere of constant density and the redshift of light from this surface, for the characteristic value $x=(8/9)^{\frac{1}{2}}$, are

$$c'=c/9, \tag{73a}$$

and

$$(\lambda'/\lambda)_l=3. \tag{74a}$$

What the extreme physically limiting values of c' and λ'/λ for spheres of arbitrary density distribution will turn out to be depends on the solution of the variation problem discussed in Section H, the solution of which would acquaint us with the properties of ultimately collapsed configurations of matter as such.

G. Characteristics of the Schwarzschild Solution for $x=(8/9)^{\frac{1}{2}}$

In Section C (reference 6) we stated that Schwarzschild's solution breaks down already for certain values of $x<1$. This is due to the fact that in the center of our sphere the pressure becomes infinite and the velocity of light equal to zero when $x=(8/9)^{\frac{1}{2}}$ as Eddington[14] has pointed out. The expression for the pressure p in the interior of the sphere is

$$8\pi p=R^{-2}[3B(1-r^2/R^2)^{\frac{1}{2}}-A]/[A-B(1-r^2/R^2)^{\frac{1}{2}}], \tag{75}$$

where A and B are given by (19). The pressure therefore becomes infinite, if

$$A-B(1-r^2/R^2)^{\frac{1}{2}}=0 \tag{76}$$

or

$$r^2=9r_1^2-8R^2. \tag{77}$$

As long as $8R^2>9r_1^2$ Schwarzschild's solution is satisfactory, inasmuch as there exists no real value of r for which p becomes infinite. For $8R^2\leq 9r_1^2$ the radius r of the sphere on whose surface the pressure becomes infinite assumes real values. The range of validity of Schwarzschild's solution (18) therefore is described by

$$0\leq x=r_1/R\leq(8/9)^{\frac{1}{2}}. \tag{78}$$

From (19) it is

$$A=B=\tfrac{1}{2} \quad \text{for} \quad x=(8/9)^{\frac{1}{2}}. \tag{79}$$

The velocity c'' of light in the interior of our sphere, from Eq. (18), in ordinary units of time, is

$$c''=c[A-B(1-r^2/R^2)^{\frac{1}{2}}](1-r^2/R^2)^{\frac{1}{2}} \tag{72a}$$

or for $x=(8/9)^{\frac{1}{2}}$, with substitution of (79)

$$c''=0 \quad \text{for} \quad r=0. \tag{73b}$$

With $x=(8/9)^{\frac{1}{2}}$, we obtain the following characteristic values for the proper mass M_l and the effective mass m_l of a sphere of constant density ρ_0

$$m_l=0.61M_l \tag{80}$$

and

$$M_l=3.75\times 10^{41}\rho_0^{-\frac{1}{2}}. \tag{81}$$

The gravitational packing energy E_{gl} for the sphere of constant density ρ_0 and of proper mass M_l consequently is

$$-E_{gl}=(M_l-m_l)c^2=0.39M_lc^2. \tag{82}$$

For a neutron star with a density of the order $\rho_0=10^{14}$ g/cm³ it would be

$$M_l=3.75\times 10^{34}\text{ g}\leqq 19M(\odot) \tag{83}$$

$$m_l=2.26\times 10^{34}\text{ g} \tag{84}$$

and from (44)

$$(dm/dM)_l=\tfrac{1}{3}. \tag{85}$$

[14] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge, 1923), p. 169.

H. Further Problems Concerning Collapsed Neutron Stars

From the preceding considerations the following mathematical and physical problems suggest themselves.

We are obviously interested in finding that configuration of matter of the proper mass M which possesses the lowest possible effective mass m. A body whose m/M is an *absolute minimum* represents the configuration of lowest energy which matter may assume without being completely transformed into radiation. If we assume for the moment that the general theory of relativity will supply the correct answer, the problem to determine this configuration of lowest energy is equivalent, mathematically, to a problem of the calculus of variations. The fact that the condition $p=\infty$ determines the proper mass M_l for which the velocity of light becomes zero at the center of our sphere suggests that we must, for $M=\text{const.}$ and $p=\infty$ at the center of the sphere, find that distribution $\rho_0(r)$ of the proper density as a function of r which will make the effective mass m an absolute minimum. The limiting condition $p=\infty$ might possibly have to be replaced by the velocity of light $c=0$, if the two conditions are not equivalent.

Physically it might, however, happen that a configuration of minimum m cannot be formed in a finite time. For instance in the case of the characteristic Schwarzschild sphere ($M=M_l$) the velocity of light becomes equal to zero at the center and with it the velocities of all physical reactions tend towards zero. Since the corresponding velocities in points $r\neq 0$ do not become zero simultaneously, the theory of the process of formation of the characteristic sphere M_l must necessarily consider nonstatic configurations which are caused by the queer nonisotropic behavior of the time scale.

Finally we must not forget that actual matter is subject to restrictions expressed commonly by *equations of state*. For instance, matter in the highly collapsed state is subject to some type of quantum degeneracy as has been pointed out by a number of investigators.[15] The ultimate problem before us therefore is to find the configuration of absolutely lowest energy which at the same time satisfies the equilibrium conditions demanded by the actual equations of state of matter in highly collapsed stars. No attempt is here made to determine this configuration.

I. Neutron Stars and Actual Stars

The following problems suggest themselves. (a) Do stars exist which are configurations of lowest energy or at least configurations approaching this limit? (b) Do any of the ordinary stars possess cores of highly collapsed matter of the type described in the preceding? (c) How is the transformation from stars composed of ordinary matter into highly collapsed stars accomplished?

We shall not here give any discussion of the first two questions. The third question, however, may find its answer through observations of supernovae. Some of the reasons which make the formation of neutron stars as a cause for supernovae an attractive possibility are roughly as follows.

(1) On the neutron star hypothesis it is easy to account for the tremendous liberation of energy which we have observed in the supernova outbursts.

(2) A second characteristic of supernovae is that the transformation from a given stellar configuration into an apparently entirely different configuration takes place at a fantastic speed as compared with the slow changes in the characteristics of ordinary stars. The explanation of the explosive character of supernovae may be related to the fact, unique in neutron stars, that such stars possess a nearly vanishingly small opacity.

(3) The spectra of supernovae are entirely different from those of any known stellar spectra. It will become clear from the following discussion, that on the neutron star hypothesis the physical conditions which determine the character of the emission and absorption of light on the effective surface of supernovae are indeed so unusual that the resulting spectra must be entirely different from anything observed so far.

(4) The progressive redshift of all of the *permanent* features (blue bands) of the spectra of supernovae found first by R. Minkowski[16]

[15] G. Gamow, *Atomic Nuclei* (Oxford, 1937), p. 234; L. Landau, Nature **141**, 334 (1938).

[16] R. Minkowski, Astrophys. J. **89** 156 (1939); F. Zwicky *ibid.* **88**, 522 (1938).

must at the present be regarded as one of the most important features. On the neutron star hypothesis this redshift is readily explainable as a general relativistic gravitational redshift.

(5) For a common nova the difference $m_f - m_{\max}$ between the photographic magnitudes of its final stage and its maximum brightness apparently never exceeds fifteen magnitudes. Although we do not, as yet, possess any data regarding the final stages (remnants) of supernovae good evidence is at hand that the *stellar remnants* of supernovae in some cases must be fainter than the supernovae at maximum by at least twenty magnitudes. On the basis of the neutron star hypothesis this may find its explanation in the fact mentioned in (3) that neutron stars in themselves are incapable of retaining any excessive radiation density.

Before we discuss some of the preceding suggestions in more detail a rough estimate may be made regarding the critical value of stellar masses for which the formation of neutron stars is energetically more favorable than the transformation from one stellar configuration into another which is caused by any known type of nuclear reaction.

We may assume in order of magnitude that the most efficient nuclear reaction which can take place throughout the whole mass M of a given star liberates an energy

$$E \cong 10^{-3} Mc^2. \tag{86}$$

According to (4) the energy E_g liberated through the formation of a neutron star of density $\rho_0 = 10^{14}$ g/cm^3 becomes greater than E for masses M which satisfy the inequality

$$M > M_0 = 5.2 \times 10^{30} \text{ g} \cong M(\odot)/400. \tag{87}$$

This condition is therefore fulfilled by most of the ordinary stars.

K. Redshift in the Spectrum of Supernovae

The spectra of about half a dozen supernovae have been observed thus far. Photographic records over periods of many months, however, have been secured only for the recent supernovae in IC 4182 and NGC 1003. A detailed account of these observations was published by R. Minkowski.[16] He finds that the spectra of all supernovae observed so far are surprisingly similar and that they exhibit a number of wide emission bands in the red and in the blue, respectively. Whereas the red bands in course of time show considerable changes in relative intensity and finally, about nine months after maximum brightness fade away entirely, the blue bands seem to be of a more *permanent* character, except for the fact that their respective intensity maxima are subject to a gradual shift towards longer wavelengths. The total shift of the blue bands towards the red end of the spectrum amounts to about 100A in the period of eight months. The average rate of change of the red shift during a few months following the date of maximum brightness is essentially the *same* for the supernovae which appeared in the spirals NGC 4273, 1003 and IC 4182 regardless of the fact that the absolute brightness of these supernovae at maximum was different by as much as a factor forty.

To fix our ideas, the processes which lead to a progressive redshift of the blue emission bands may schematically be pictured in the following manner. There exists a characteristic surface of separation Σ_s in supernovae[2] such that the material outside this surface is expelled into interstellar space, while matter inside of Σ_s gradually contracts. It is this matter inside of Σ_s which constitutes the stellar remnant of the supernova on whose surface the blue emission bands originate. While this stellar remnant contracts, the gravitational field on its surface becomes increasingly stronger and causes the frequency of characteristic spectral lines to diminish continuously.

For the following considerations we shall make use of the data[16, 17] obtained for the absolutely brightest of all supernovae which are known at the present time. This supernova flared up in the faint spiral IC 4182. It presumably reached its maximum brightness on August 22, 1937, at which time its absolute photographic magnitude was $\mathfrak{M}_p = -16.6$ corresponding to six hundred million times the radiation of the sun in the photographic spectral range. Although we have not yet accurately determined the radiation of this supernova in the red parts of the spectrum it follows from our data that at maximum its

[17] W. Baade and F. Zwicky, Astrophys. J. **88**, 411 (1938).

total radiation (per second) $L_{\max}$ in the spectral range from $\lambda = 7000\text{A}$ to $\lambda = 3300\text{A}$ is approximately equal to a billion times that of the sun ($L(\odot) = 3.8 \times 10^{33}$ ergs/sec.). We therefore have

$$L_{\max} \cong 3.8 \times 10^{42} \text{ ergs/sec.} \qquad (87)$$

By the end of May, 1938 the absolute photographic brightness had become about $\mathfrak{M}_p = -9.5$. The total radiation in the visible range at this time therefore had decreased to approximately $L = L_{\max}/1000$ or

$$L \cong 3.8 \times 10^{39} \text{ ergs/sec.} \qquad (88)$$

In the same period, from August 22, 1937 to May 30, 1938, the blue bands had shifted towards the red by an amount approximately equal to $\Delta\lambda = 100\text{A}$. For purposes of calculation we shall set

$$\kappa = \Delta\lambda/\lambda = 0.02. \qquad (90)$$

In comparison we notice that the corresponding values of κ for the sun and the companion of Sirius are 2.1×10^{-6} and 6×10^{-5}, respectively.

On the basis of the neutron star hypothesis it is suggestive to interpret this redshift in supernovae as a gravitational redshift and to assume that the emission bands which are subject to this redshift originate on the surface of the central stellar remnant of the supernova. This is in keeping with the fact that these bands are much more permanent in character than the red bands whose pronounced changes in intensity combined with the absence of a redshift suggest that these red bands originate in more outlying layers of gas which have been expelled from the central star.

We now proceed to develop some further consequences of this hypothesis.

L. Resulting Radius of the Stellar Remnant of the Supernova in IC 4182

The change of the wave-length of light on its journey from the surface of a star of effective mass m and radius r_1 to a distant point $r \gg r_1$ is given by (74). We write for the wave-length λ' in the external point, in terms of the wave-length λ on the surface of the star

$$\lambda' = \lambda + \Delta\lambda, \qquad (91)$$

where

$$\kappa = \Delta\lambda/\lambda = (1 - 2\Gamma m/r_1c^2)^{-\frac{1}{2}} - 1. \qquad (92)$$

Since we shall have to deal with small values of κ only, we may expand this expression into the one commonly used in discussions of stellar gravitational redshifts. It is

$$\kappa = \Gamma M/r_1c^2. \qquad (93)$$

In conformity with the approximation used, the effective mass m has been replaced by the proper mass M. According to (93) and (90) we therefore obtain for the stellar remnant of the supernova IC 4182 (in the stage of June, 1938)

$$r_1 = 7.4 \times 10^{-29} M/\kappa = 3.7 \times 10^{-27} M. \qquad (94)$$

Offhand we do not know what the mass M of the stellar remnant of our supernova is and we therefore cannot immediately determine its radius, its density and some of its other physical characteristics. On the neutron star hypothesis it is, however, possible to estimate the maximum and minimum values of M.

As will be shown in the next section the minimum value $M_{\min}$ satisfies the inequality

$$M_{\min} > 2.16 \times 10^{32} \text{ g} = M(\odot)/9. \qquad (95)$$

On the other hand, the neutron star hypothesis suggests that according to the considerations given in Section G we have a characteristic mass M_l. The conclusion that in a neutron star of this mass the velocity of light and with it the velocity of all reactions becomes zero at the center of the star might be taken as an indication that the characteristic mass M_l (83) represents a natural limit for the upper value M_u of the mass of the stellar remnant, such that

$$M_u < M_l = 3.8 \times 10^{34} \text{ g} = 19M(\odot). \qquad (96)$$

In our special case we also possess some direct observational evidence which corroborates the correctness of (96). From the fact that the original star of the supernova in IC 4182 before its outburst was fainter than the apparent photographic magnitude 21, combined with our knowledge of the distance of IC 4182 (900,000 parsecs), we conclude that the absolute magnitude of the said original star must have been fainter than -4. It therefore was not at all one of the brightest stars in IC 4182, and its mass M_u, according to the mass luminosity relation, presumably was

$$M_u < 15M(\odot) = 3 \times 10^{34} \text{ g}. \qquad (97)$$

In the case of supernovae in elliptical nebulae, such as the most recent supernova, which appeared in NGC 4636, we may even say that the original star did not have an absolute magnitude brighter than about -2. If, therefore, the redshift in supernovae is a gravitational redshift, Table I describes the range of possible radii of the stellar remnant of the supernova in IC 4182 and presumably of supernovae in general.

M. Density of the Stellar Remnant of Supernovae

According to the gravitational interpretation of the redshift in the spectrum of supernovae the average density ρ_a of their stellar remnants is given by

$$\rho_a = 3M/4\pi r_1^3 = 3(c^2/\Gamma)^3\kappa^3/4\pi M^2 \qquad (98)$$

or numerically, with the observed value $\kappa = 0.02$

$$\rho_a = 4.7\times 10^{78} M^{-2}. \qquad (99)$$

Table II gives the range of possible values of the average density ρ_a. From this table it follows that on our interpretation of the redshift as a general relativistic effect, combined with the neutron star hypothesis, masses smaller than $M_{\min}$ must be ruled out, since otherwise the introduction of densities surpassing those of neutron stars would be necessary. If it should be found that densities greater than 10^{14} g/cm^3 are possible, the value of $M_{\min}$ must be correspondingly decreased.

The actual range of possible masses is further narrowed down by the fact that the density cannot be assumed to be uniform throughout the stellar remnant of a supernova. A star whose density throughout the star is about 10^{14} g/cm^3 would be a *pure* neutron star. Such a star, because of its lack of opacity, could not retain any radiation longer than a few seconds, a conclusion which obviously contradicts the fact that the stellar remnant of the supernova in the stage which we are investigating (end of May, 1938) still was a million times as bright as the sun and at the time of writing (February, 1939) continues to radiate at a rate surpassing that of the sun by at least a factor 10,000. Here again attention must be called to the fact that our whole discussion is based on the assumption that the observed redshift is a gravitational redshift. If this assumption should prove to be incorrect, we could of course not conclude that the blue emission bands, which are subject to the redshift, have their origin on the surface of the stellar remnant of the supernova. These bands might then for instance originate in an extended gaseous envelope which presumably surrounds the stellar remnant of a supernova.

Table I. *Possible radii of the stellar remnant of the supernova in IC 4182. M is the proper mass of the central stellar remnant.*

M in g	r_1
2.2×10^{32}	8 km
$2\times 10^{33} = M(\odot)$	74 km $= 10^{-4} r(\odot)$
3×10^{34}	1100 km

Table II. *Possible values of the average density in supernovae.*

M in g	ρ_a in g/cm^3
$2.16\times 10^{32} (= M_{\min})$	10^{14}
$2\ \ \times 10^{33} (= M\odot)$	1.18×10^{12}
$3\ \ \times 10^{34} (= M_u)$	5.2×10^{9}

From these considerations we conclude that the possible mass M of the stellar remnant of the supernova IC 4182 must fall into the range

$$2.16\times 10^{32}\ \text{g} < M < 3\times 10^{34}\ \text{g}, \qquad (100)$$

a range which is practically identical with that covering all of the ordinary stars.

N. On the Total Generation of Energy in Supernovae

It was shown in Section C (41) that the relativistic and the Newtonian values for the gravitational energy of contraction E_g of a sphere of constant density differ at most by a few percent. Especially for small values of the energy, that is for $|E_g| \ll Mc^2$ we may always replace the relativistic value by the Newtonian value and write

$$E_g = -3\Gamma M^2/5r_1 = -6.44\times 10^{-8} M^{5/3}\rho_0^{\frac{1}{3}}. \qquad (101)$$

Substituting (93) in (101) we obtain

$$E_g = -3\kappa Mc^2/5, \qquad (102)$$

and for the difference between proper mass M and effective mass m

$$\Delta M = M - m = -E_g/c^2 = 3\kappa M/5. \qquad (103)$$

TABLE III. *Significant energy values for the supernova in IC 4182 as functions of possible masses M of its stellar remnant.*

M IN G	E_t IN ERGS	E_{MAX}	Mc^2	E_{VIS}
$2.16\times10^{32}(=M_{\text{min}})$	$=2.33\times10^{51}$	2.33×10^{51}	1.9×10^{53}	10^{49}
$2\quad\times10^{33}(=M(\odot))$	$\geqq2.16\times10^{52}$	9.52×10^{52}	1.8×10^{54}	10^{49}
$3\quad\times10^{34}(=M_u)$	$\geqq3.24\times10^{53}$	8.60×10^{54}	2.7×10^{55}	10^{49}
$3.8\times10^{34}(=M_l)$	$\geqq4.08\times10^{53}$	1.31×10^{55}	3.4×10^{55}	10^{49}

For the stellar remnant of the supernova IC 4182, with $\kappa=0.02$ it is

$$\Delta M=0.012M. \tag{104}$$

For $M=M_{\text{min}}=2.16\times10^{32}$ g the energy of contraction released by the supernova IC 4182 from August, 1937 until June, 1938 would therefore be

$$-(E_g)_{\text{min}}=\Delta Mc^2=2.33\times10^{51}\text{ ergs}. \tag{105}$$

This energy, for $M=M_{\text{min}}$ is identical with the total energy which is released through the contraction of the initial star into a pure neutron star. As we have previously stated it is actually $M>M_{\text{min}}$ and the total energy E_t released by the supernova in the stated period must therefore be greater than $-(E_g)_{\text{min}}$. Consequently we have

$$E_t>2.33\times10^{51}\text{ ergs}. \tag{106}$$

For comparison we add

$$E_{\text{vis}}=10^{49}\text{ ergs} \tag{107}$$

and

$$Mc^2>1.9\times10^{53}\text{ ergs} \tag{108}$$

where E_{vis} is the energy actually liberated in the form of visible light and Mc^2 is the total energy of annihilation of the original star.

The fact that the total energy liberated in a supernova outburst according to our theory is at least 230 times the observed energy in the visual range of the spectrum confirms some of our conclusions published on previous occasions.[1, 2]

In Table III we list some significant energy values for the supernova in IC 4182 as functions of possible masses M of its stellar remnant.

The various energies listed have the following meaning. The total energy liberated in our supernova according to the neutron star hypothesis in the period from August, 1937 until June, 1938 is designated with E_t. Since the density of the stellar remnant presumably is not constant but increases towards its center, it is obviously

$$E_t>|E_g|, \tag{109}$$

where E_g is the gravitational energy of contraction for a sphere of mass M and constant density ρ_0 as given by (102). In the limiting case $M=M_{\text{min}}$, in which the stellar remnant has reached its final stage of contraction $\rho_0=\text{const.}\leqq10^{14}$ g/cm³ it is $E_t=|E_g|$. Further, E_{max} is the energy of contraction in the case that the stellar remnant has reached its final stage $\rho_0=10^{14}$ g/cm³. According to (3) it is

$$E_{\text{max}}=3\times10^{-3}M^{5/3}\text{ ergs}. \tag{110}$$

In the case $M=M_{\text{min}}$ it is $E_{\text{max}}=E_t$, whereas for the upper limit $M=M_l$ given by the Schwarzschild solution we have $E_{\text{max}}=0.39M_lc^2$, as given by (80). The values for Mc^2 indicate the energies which would be liberated if the mass of the star were transformed into radiation in its entirety. Finally E_{vis} is the total energy liberated in the form of light in the wave-length range $6700\text{A}>\lambda>3300\text{A}$ as we have observed it in the period from August, 1937 until June, 1938.

If, for the date of June 1, 1938, we take the mass of the stellar remnant to be $M=M(\odot)$, we may draw the following conclusions from the neutron star hypothesis. Of the maximum energy $E_{\text{max}}=9.52\times10^{52}$ ergs to be liberated ultimately through the contraction into the limiting configuration $\rho_0\leqq10^{14}$ g/cm³ more than twenty percent, that is more than 2.16×10^{52} ergs had been released by June 1, 1938. Of this energy only $0.5\ {}^0/_{00}$ or about 10^{49} ergs had been radiated in form of visible light. The remaining energy $E_t'=E_t(1-5\times10^{-4})\cong E_t$ must therefore have been liberated in the form of other energies such as invisible light, kinetic energy, heating of the stellar remnant, ionization, nuclear transformations, cosmic rays, etc. It will be of great interest to determine, by elaboration of the theory and by further observations, how E_t is to be distributed over the various forms of energy just mentioned.

The difference $E_{\text{max}}-E_t$ remains available for further evolutionary processes of the stellar remnant. Since all of the supernovae found in extragalactic systems pass out of reach of present day telescopes after the short period of a few years at most, direct information about the final stages of supernovae can obviously be obtained only through the discovery of the remnants of supernovae in our own stellar system. In this connection the suggestion made recently by Baade[18] and Morgenroth[19] that the crab nebula had its origin in a supernova several hundred years ago, may prove to be of great importance.

O. Possible Surface Temperatures of the Stellar Remnants of Supernovae

The neutron star hypothesis of supernovae discussed in the preceding sections results in estimates of the possible radii of the stellar remnants of supernovae. The combination of these estimates with the observed visible radiation of the stellar remnants allows us to estimate lower limits of their surface temperatures. For instance the supernova in IC 4182 had an absolute magnitude $\mathfrak{M}_p \cong -9.5$ in June, 1938. Its total radiation L at that time therefore was

$$L > 3.78 \times 10^{39} \text{ ergs/sec.} \quad (111)$$

From Table I its radius for the date mentioned must have been of the order 8 km $< r_1 <$ 1100 km. From this it follows that the effective surface temperature T_e of the stellar remnant of the supernova must have satisfied the inequality

$$4\pi r_1^2 \sigma T_e^4 \geqslant L. \quad (112)$$

Substituting (93), we obtain

$$4\pi\sigma T_e^4 (\Gamma M/\kappa c^2)^2 \geqslant L \quad (113)$$

or

$$T_e \geqslant (L/4\pi\sigma)^{\frac{1}{4}} (\Gamma M/\kappa c^2)^{-\frac{1}{2}}, \quad (114)$$

where $\sigma = 5.72 \times 10^{-5}$ ergs/cm^2 sec. is the Stefan-Boltzmann radiation constant. Substitution of (90) and (111) in (114) leads to

$$T_e > 7.86 \times 10^{23} / M^{\frac{1}{2}}. \quad (115)$$

Table IV. *Lowest possible values of the effective temperature T_e of the stellar remnant of the supernova in IC 4182 in June, 1938.*

M in g	T_e in degrees abs.
2.16×10^{32}	$> 5.55 \times 10^7$
$2 \quad \times 10^{33}$	$> 1.76 \times 10^7$
$3 \quad \times 10^{34}$	$> 4.54 \times 10^6$
3.8×10^{34}	$> 4.03 \times 10^6$

[18] W. Baade, Astrophys. J. **88**, 285 (1938).
[19] O. Morgenroth, Die Sterne **17**, 255 (1937).

The lowest possible values of the effective temperature T_e of the stellar remnant of the supernova in IC 4182 for the stage of June, 1938 are listed in Table IV. The values for T_e thus obtained check in order of magnitude the values estimated previously from entirely different considerations,[2] an agreement which strengthens our confidence in the neutron star hypothesis.

It is obvious that with temperatures as high as given by Table IV, the emission spectra from supernovae must be of an entirely different character from those characterizing ordinary stars, whose effective surface temperatures lie in the approximate range from 1000° to 100,000°. An exhaustive study of the effects of these exceedingly high temperatures on the character of the emission of light from supernovae has been made by R. Minkowski[16] in his paper on the spectra of the two recent supernovae in IC 4182 and NGC 1003.

It should be added that in addition to the high temperatures the conditions of pressure and density on the surface of the stellar remnants of supernovae according to the neutron star hypothesis are very unusual. For instance, the acceleration of gravity on the surface of the stellar core of the supernova is, in Newton's approximation,

$$g = \Gamma M / r_1^2 = \kappa c^2 / r_1 \quad (115)$$

or

$$g = \kappa^2 c^4 / \Gamma M. \quad (116)$$

For $\kappa = 2 \times 10^{-2}$ and $M = M(\odot)$ we therefore have

$$g = 2.44 \times 10^{12} \text{ cm sec.}^{-2} \quad (117)$$

as compared with

$$g(\odot) = 2.7 \times 10^4 \text{ cm sec.}^{-2} \quad (118)$$

on the surface of the sun.

Finally attention should be called to the enormous intensity of the radiation near the surface of the stellar remnants of supernovae. For the supernova IC 4182, in the stage June, 1938, the intensity of radiation at $r = r_1$ according to (111) was

$$S = L/4\pi r_1^2 > 6 \times 10^{24} \text{ ergs/cm}^2 \text{ sec.} \quad (119)$$

if the value $r_1 = 7.4 \times 10^6$ cm corresponding to $M = M(\odot)$ is introduced. For comparison we mention that the intensity of the radiation on the surface of the sun is

$$S(\odot) \leqq 6.3\times10^{10} \text{ ergs/cm}^2 \text{ sec.} \quad (120)$$

We again refer to the recent study by Minkowski[16] where the effects of radiation intensities of the order of S on the character of the emission spectra of supernovae are discussed. The effects on the width of spectral lines as well as the fundamental importance of induced emission or "negative" absorption in the spectra of supernovae have been particularly emphasized by Minkowski.

P. Uniformity Among Supernovae

Although we possess extended series of observations for a few supernovae only, already a remarkable uniformity in the behavior of these stars begins to delineate itself. This uniformity becomes apparent through the study of the absolute magnitudes, the light curves and the spectral characteristics of supernovae.

From the available material Baade[18] has determined the mean photographic magnitude $\overline{\mathfrak{M}}_p$ of supernovae as well as the dispersion σ in $\mathfrak{M}_p$. His figures are

$$\overline{\mathfrak{M}}_p = -14.2 \quad (121)$$

and

$$\sigma \leq 1.1. \quad (122)$$

Baade also suggests that σ may be smaller yet, because the data on which the value (122) is based are partially fragmentary. Statistical studies of about 2000 films obtained with the Schmidt telescope at Palomar concerning the easily detectable occurrence of absolutely bright supernovae even at very great distances indicate in fact that the frequency of unusually bright supernovae is smaller than one should expect from a dispersion of the order of (122). The small value of the dispersion indicates a considerable uniformity in the physical behavior of supernovae. Incidentally this fact will eventually make possible the use of the apparent maximum brightness and of the shape of the light curves of supernovae as a reliable gauge of distances.

The light curves of supernovae of the same absolute brightness are remarkably similar. All of the light curves are characterized by an extended maximum of about two weeks followed by an initial decline in brightness of about 0.1 magnitude per day for the first month after the maximum. This decline coupled with a subsequent leveling off of the light curves according to Baade is definite enough to make possible the determination of the brightness at maximum, even if only relatively fragmentary light curves are available.

The evolution of the spectra of supernovae with respect to the progressive redshift of the blue bands also seems to be uniform enough to make possible the determination of the date of maximum brightness, as Minkowski's analysis[16] has revealed.

Furthermore, as Minkowski points out, the redshift about one year after maximum brightness of all supernovae seems to reach asymptotically a value of about 150 angstroms.

From the neutron star hypothesis the uniformity just described in the behavior of supernovae becomes comprehensible. If in particular, as has recently been suggested,[20] the possible masses of the neutron cores of supernovae can be limited *a priori* to a range considerably narrower than that given by (100), the uniformity of supernovae may find a very satisfactory explanation. In view of these possibilities it appears at any rate desirable to spare no efforts in the continuation of the search program for supernovae. It will be particularly important to discover supernovae at very early stages of their outbursts in the hope of securing spectra which can be interpreted in all of their essential details.

We recapitulate that the study of supernovae, in addition to the enrichment of our knowledge of the observable characteristics of this interesting class of stars, may eventually prove to be of considerable interest for the following theoretical reasons.

(a) The elucidation of the causes for supernova outbursts will mean the understanding of the generation of stellar energy for at least one class of stars. So far the problem of the generation of energy cannot be considered as having been solved satisfactorily for any class of stars.

(b) If the neutron star hypothesis of the origin of supernovae can be proved, it will be possible to subject the general theory of relativity to tests which according to the considerations presented in this paper deal with effects which in order of magnitude are large compared with the

[20] G. M. Volkoff, Phys. Rev. **55**, 421A (1939).

tests so far available. The general theory of relativity, although profound and exceedingly satisfactory in its epistemological aspects, has so far practically not lent itself to any very obvious and generally impressive applications. This unfortunate discrepancy between the formal beauty of the general theory of relativity and the meagerness of its practical applications makes it particularly desirable to search for phenomena which cannot be understood without the help of the general theory of relativity.

(c) The possibility that cosmic rays originate in supernovae constitutes an added incentive for the continued pursuit of investigations of these extraordinary stars.

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Resistance, Emissivities and Melting Point of Tantalum

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The relation between true and brightness temperatures of tantalum was determined by means of pyrometric observations on the inside and outside of a long thin-walled tube heated electrically. The brightness temperature as a function of heating current was determined by sighting on an electrically heated wire of known diameter. Potential leads of fine tungsten wire welded to tantalum filaments permitted a determination of the electrical properties as a function of temperature. The electrical properties at the melting point yielded a value for this quantity from the extrapolated temperature *vs.* electrical property relations. Value for the melting point determined is 3269°K.

INTRODUCTION

MEASUREMENTS of the resistance, power radiation and brightness temperature of tantalum between room temperature and the melting point are described here. The work was undertaken when certain inconsistencies with previously published data for tantalum were discovered during experiments on the rate of evaporation of this metal which are described in an accompanying paper.[1] The relationship between resistivity and power radiation which was observed during this work is not the same as that published by Worthing.[2, 3] For a given value of power radiation the specific resistance reported here is about four percent lower than Worthing's value. It seems probable that the constitution of tantalum as made today may differ from that of the product available at the time of the earlier work. Forsythe and Watson[4] have reported an apparent change in the properties of tungsten which took place in an unexplained manner between 1922 and 1934. The earlier values of resistivity were the higher. Tantalum apparently has undergone a similar type of change.

Brightness temperature observations were also made, and are in good agreement with the earlier work both of Worthing and of Utterback and Sanderman.[5] An apparent disagreement between the latter two experimenters is due to a discrepancy between the data as shown in the table and in one of the graphs of Worthing's paper.

Except where noted the tantalum used in these experiments was regular stock from the Fansteel Metallurgical Corporation.

[1] D. B. Langmuir and L. Malter, Phys. Rev. **55**, 748 (1939).
[2] A. G. Worthing, Phys. Rev. **28**, 190 (1926).
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[4] W. E. Forsythe and E. M. Watson, J. Opt. Soc. Am. **24**, 114 (1934).
[5] C. L. Utterback and L. A. Sanderman, Phys. Rev. **39**, 1008 (1932).