minute. The corresponding figures for scales-of-one are 60 and 1200 counts a minute for the slow and the fast recorders, respectively.

The writers wish to take this opportunity to correct a misprint in reference (2) below. In that reference, Eq. (10) reads $1/n_{1\,\max}$. This should be corrected to read $1/(n_{1\,\max}\times e)$.

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February 2, 1939.

[1] L. Alaoglu and N. M. Smith, Phys. Rev. **53**, 832 (1938).
[2] H. Lifschutz and O. S. Duffendack, Phys. Rev. **54**, 714 (1938).
[3] H. V. Neher, Rev. Sci. Inst. **10**, 29 (1939).

On the Equilibrium of Massive Spheres

It is usually stated[1] that general relativity sets an upper limit to the mass and radius of a sphere of constant proper energy density ρ. This result is obtained by considering only those solutions of Einstein's field equations which give a finite central proper pressure p; the minimum mass and radius for which p first becomes infinite at the center are taken as the limiting values. In his original paper[1] Schwarzschild points out the existence of other solutions with infinite central p, but dismisses them as physically inadmissible because of this singularity without a further discussion of their properties. However, an examination of these solutions, which are described below, shows that they lead to arbitrarily large masses and radii.

On the other hand a cold neutron gas model[2] leads to an upper limit on the size of a static sphere. It is of interest to try to account for the difference in behavior of these two models. If for a material consisting of particles in motion (and which may exert forces on each other) the energy-momentum tensors for the particles, and for the force fields (apart from gravitation) associated with them, are additive and have non-negative traces, then for such a material $T=\rho-3p\geqq 0$ must always hold. The $\rho=$const. model for sufficiently high pressures has $T<0$. This not altogether consistent model corresponds to the case of perfectly incompressible particles packed tightly together and treated essentially nonrelativistically in that the contribution of the forces to ρ is not taken into account. A negative T near the center of the sphere, such as makes possible an arbitrarily large mass for the $\rho=$const. model, may be regarded as analogous to a negative (repulsive) mass which keeps the sphere from collapsing.

Both the original Schwarzschild solutions, and the new singular solutions leading to arbitrarily large spheres, may be obtained by a method considerably simpler than the one used by Schwarzschild. Einstein's equations for constant ρ and for the line-element

$$ds^2=-e^{\lambda(r)}dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+e^{\nu(r)}dt^2$$

reduce to:[2]

$$e^{\nu(r)}=\frac{(1-2u_b/r_b)\rho^2}{[p(r)+\rho]^2} \tag{1}$$

$$dU/dx=3x^2 \tag{2}$$

and

$$\frac{dP}{dx}=-\frac{P+3}{2x(x-U)}[Px^3+U], \tag{3}$$

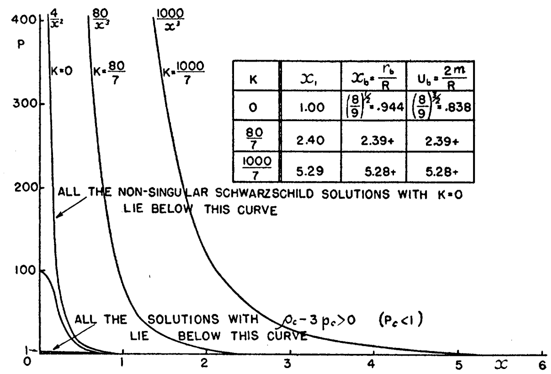

FIG. 1.

where x, U, P, are dimensionless quantities defined by $x=r/R$, $U=2u/R$, $P=8\pi R^2p$; $u\equiv\frac{1}{2}r(1-e^{-\lambda})$, R is a characteristic length determined in terms of the given constant ρ by $R^2=3/8\pi\rho$; and relativistic units, making $c=1$, $G=1$, are used throughout. The radius r_b of the sphere is determined by that value x_b of x where P first vanishes, and the gravitational mass of the sphere is given by $m=u_b$, the value of u at $x=x_b$. The integral of (2) above is $U=x^3-K$, where $K\geqq 0$ in order that $e^{-\lambda}=[1-r^2/R^2+KR/r]$ should nowhere change sign. The usual procedure[1] corresponds to making $K=0$, and then Eq. (3) may be at once explicitly integrated, and gives the well known Schwarzschild interior solution:

$$P=\frac{3(P_c+1)(1-x^2)^{\frac12}-(P_c+3)}{(P_c+3)-(P_c+1)(1-x^2)^{\frac12}},$$

where P_c is the value of P at $x=0$. If $P_c\rightarrow\infty$, which is commonly taken as the limiting solution, then as $x\rightarrow 0$, $P\sim 4/x^2$, and the corresponding radius and mass are fixed by $x_b=(8/9)^{\frac12}$, and $U_b=(8/9)^{\frac32}$. This solution makes $e^{\nu}\rightarrow 0$ and $e^{\lambda}\rightarrow 1$ as $r\rightarrow 0$.

The new singular solutions are obtained by making $K>0$. Eq. (3) now can not be explicitly integrated in terms of known functions, but it may be shown that solutions exist which near the origin behave like $P\sim 7K/x^3$. Numerical integration of (3) for several values of K shows that x_b is very closely given by the largest positive root x_1 of $x-U=x-x^3+K=0$. Thus $x_b\sim x_1$, $U_b=x_b^3-K\sim x_1^3-K=x_1$. For very large K, $x_1\sim K^{\frac13}$, so that $x_b\sim K^{\frac13}$, $U_b\sim K^{\frac13}$, and it is seen that no upper limit on the size of the sphere exists, if the singular solutions for the pressure are not excluded. For these solutions both $e^{\lambda}\rightarrow 0$ and $e^{\nu}\rightarrow 0$ as $r\rightarrow 0$. Fig. 1 gives a general idea of the various solutions, including the Schwarzschild limiting solution and the last one for which $\rho_c-3p_c\geqq 0$ ($P_c\leqq 1$) holds.

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[2] J. R. Oppenheimer and G. M. Volkoff, this issue, Phys. Rev. **55**, 374 (1939). Eqs. (1), (2) and (3) are Eqs. (7), (9) and (10) of that article rewritten for constant ρ.