It will be noted that in addition to the ordinary absorbed radiation of half-width $C/4\pi$, there is an additional absorption of radiation over the entire half-width γ. The intensity of this additional absorption is much less than that of the normal absorption, to which it bears the ratio of the square of $G/4\pi\gamma^3$. When H equals zero, this is the ratio of the normal half-width $C/4\pi$ to the half-width γ of the incident radiation. In general this ratio is small and therefore the usual analysis, which assumes $F(z)$ equal to a constant C, may be legitimately applied. This verifies Weisskopf's[9] conclusion.

The above formulas give the order of magnitude of the effects involved. When γ becomes much less than Γ, defined as the corresponding half-width for transitions from B to A, the consideration of the transitions A to B alone becomes a poor approximation to the actual state of affairs. The double transition A to B to A may be treated by a direct application of the foregoing analysis, provided the ordinary approximations are made for the spontaneous jumps B to A. In addition to the usual substitution of $C/4\pi+\Gamma$ for $C/4\pi$ in (27), the most important result is that $\gamma+\Gamma$ replaces γ in the square root in (22). Since Γ will include a term G/γ^2 because of the induced transitions B to A, the square root will apparently be real whenever $G(z)$ is given by (19). Thus there will be no shift of the absorbed line; this is again in agreement with Weisskopf's result[9] when the frequency of his strictly monochromatic radiation coincides with the line center, $(E_B-E_A)/h$.

[9] V. Weisskopf, Ann. d. Physik 9, 23 (1931).

FEBRUARY 15, 1939 PHYSICAL REVIEW VOLUME 55

Static Solutions of Einstein's Field Equations for Spheres of Fluid

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(Received January 3, 1939)

A method is developed for treating Einstein's field equations, applied to static spheres of fluid, in such a manner as to provide explicit solutions in terms of known analytic functions. A number of new solutions are thus obtained, and the properties of three of the new solutions are examined in detail. It is hoped that the investigation may be of some help in connection with studies of stellar structure. (See the accompanying article by Professor Oppenheimer and Mr. Volkoff.)

§1. INTRODUCTION

IT IS difficult to obtain explicit solutions of Einstein's gravitational field equations, in terms of known analytic functions, on account of their complicated and nonlinear character. Even in the physically simple case of static gravitational equilibrium for a spherical distribution of perfect fluid, there are only two explicit solutions which are at present well known. These are Einstein's original cosmological solution for a uniform distribution of fluid with constant density ρ and constant pressure p throughout the whole of space, and Schwarzschild's so-called interior solution for a sphere of incompressible fluid of constant density ρ and a pressure p which drops from its central value to zero at the boundary.[1] To these, by regarding empty space as the limiting case of a fluid having zero density and pressure, we can also add de Sitter's cosmological solution for a completely empty universe, and Schwarzschild's so-called exterior solution for the field in the empty space surrounding a spherically symmetrical body, thus giving four solutions in all.

The present paper has a twofold purpose. In the first place, a method will be given for treating

[1] In addition to these explicit solutions for a spherical distribution of fluid, we also have Lemaitre's interesting explicit solution for a spherical distribution of solid, each concentric layer of which supports its own weight by purely transverse stresses. See Eq. (5.11), Ann. de la Soc. Scient. de Bruxelles A53, 51 (1933).

the nonlinear differential equations, applying to the gravitational equilibrium of perfect fluids, in such a manner as to make it somewhat easy to obtain a variety of explicit solutions. In the second place, it will then be shown that this method of treatment leads directly not only to the four well-known solutions mentioned above, but also to a number of others which may have a measure of physical interest. In particular, it is hoped that some of these new solutions may be of use in trying to understand the internal constitution of stars.[2]

§2. The General Form of Solution for an Equilibrium Distribution of Fluid

If for simplicity units are chosen which make the velocity of light and the constant of gravitation both equal to unity, Einstein's field equations (connecting the distribution of matter and energy as described by the components of the energy-momentum tensor $T_{\mu\nu}$ with the resulting gravitational field as described by the potentials $g_{\mu\nu}$) can be written in the form

$$-8\pi T_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}, \qquad (2.1)$$

where the cosmological constant is denoted by Λ, and where the contracted Riemann-Christoffel tensor $R_{\mu\nu}$ and its trace R are known functions of the potentials $g_{\mu\nu}$ and their first and second derivatives with respect to the space-like and time-like coordinates $x^1\cdots x^4$. Owing to the complicated and nonlinear dependence of $R_{\mu\nu}$ and R on the $g_{\mu\nu}$ and their derivatives, no general precise, explicit solution for these equations is known, and special solutions are difficult to obtain in explicit analytic form as has already been mentioned. The problem of solution is made particularly difficult by the nonlinear character of the equations which prevents the construction of further solutions by the superposition of those already obtained.

In searching for solutions of the field equations (2.1) that would correspond to an equilibrium distribution of perfect fluid, considerable simplification can be introduced at the start.

In the first place, since the condition of gravitational equilibrium for a fluid will on physical grounds be a static and spherically symmetrical distribution of matter, we can begin by choosing space-like coordinates r, θ and ϕ, and a time-like coordinate t such that the solution will be described by the simple form of line element

$$ds^2=-e^{\lambda}dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+e^{\nu}dt^2, \qquad (2.2)$$

with λ and ν functions of r alone, as is known to be possible in the case of any static and spherically symmetrical distribution of matter. With the simple expressions for the gravitational potentials appearing in (2.2), the application of the field equations (2.1) then leads to the following expressions for the only surviving components of the energy-momentum tensor

$$8\pi T_1^1=-e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)+\frac{1}{r^2}-\Lambda,$$

$$8\pi T_2^2=8\pi T_3^3$$

$$=-e^{-\lambda}\left(\frac{\nu''}{2}-\frac{\lambda'\nu'}{4}+\frac{\nu'^2}{4}+\frac{\nu'-\lambda'}{2r}\right)-\Lambda, \qquad (2.3)$$

$$8\pi T_4^4=e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}-\Lambda,$$

where differentiations with respect to r are indicated by accents.[3]

[2] My own present interest in solutions of Einstein's field equations for static spheres of fluid is specially due to conversations with Professor Zwicky of this Institute, and with Professor Oppenheimer and Mr. Volkoff of the University of California, who have been more directly concerned with the possibility of applying such solutions to problems of stellar structure. Professor Zwicky in a recent note (Astrophys. J. **88**, 522 (1938); see also Phys. Rev. **54**, 242 (1938)) has suggested the use of Schwarzschild's interior solution for a sphere of fluid of constant density as providing a model for a "collapsed neutron star." He is making further calculations on the properties of such a model, and it is hoped that the considerations given in this article may be of assistance in throwing light on the questions that concern him. Professor Oppenheimer and Mr. Volkoff have undertaken the specific problem of obtaining numerical quadratures for Einstein's field equations applied to spheres of fluid obeying the equation of state for a degenerate Fermi gas, with special reference to the particular case of neutron gas. Their results appear elsewhere in this same issue. My own solutions of the field equations, as given in the immediately following, can make only an indirect contribution to the physically important case of a Fermi gas, since it will be seen that they correspond to equations of state which cannot be chosen arbitrarily. My thinking on these matters has, however, been largely influenced by discussions with Professor Oppenheimer and Mr. Volkoff, and it is hoped that the explicit solutions obtained will at least assist in the general problem of developing a sound intuition for the kind of results that are to be expected from the application of Einstein's field equations to static spheres of fluid.

[3] See for example, R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford, 1934), Eq. (95.3).

In the second place, since the matter involved in the distribution is by hypothesis a perfect fluid, we can obtain a direct connection with the properties of the fluid by making use of the general expression

$$T^{\mu\nu}=(\rho+p)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}-g^{\mu\nu}p, \qquad (2.4)$$

which by definition gives the components of the energy-momentum tensor $T^{\mu\nu}$ of a perfect fluid at any point and time of interest in terms of the proper (macroscopic) density ρ, the proper pressure p, and the components of "velocity" dx^μ/ds and dx^ν/ds of the fluid at that point and time. With the simple form of line element (2.2), this then leads to

$$T_1^1=T_2^2=T_3^3=-p, \quad T_4^4=\rho, \qquad (2.5)$$

as expressions for the only surviving components of the energy-momentum tensor in terms of the pressure and density of the fluid.

Substituting (2.5) in (2.3), we now have

$$8\pi p=e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}+\Lambda,$$

$$8\pi p=e^{-\lambda}\left(\frac{\nu''}{2}-\frac{\lambda'\nu'}{4}+\frac{\nu'^2}{4}+\frac{\nu'-\lambda'}{2r}\right)+\Lambda, \qquad (2.6)$$

$$8\pi\rho=e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}-\Lambda$$

as the desired expressions which make direct connection with the properties of the fluid in terms of its pressure and density. In the case of gravitational equilibrium for a distribution of perfect fluid, we are thus provided with the general form of solution for the gravitational potentials $g_{\mu\nu}$ which is described by the line element (2.2), and with three differential equations (2.6) which relate the unknown functions λ and ν appearing in that form of solution, along with the pressure p and density of the fluid ρ, to coordinate position r within the distribution of fluid.

§3. Method of Obtaining Explicit Analytic Solutions

In accordance with the foregoing we now have three differential equations for the four unknown quantities λ, ν, p and ρ as functions of r. The problem will hence become determinate as soon as we introduce one further independent equation corresponding to some additional hypothesis as to the nature of the fluid or of the distribution.

From a physical point of view, it might seem most natural to introduce this additional hypothesis in the form of an "equation of state" describing the relation between pressure p and density ρ which could be expected to hold for the fluid under consideration. Or since the properties of the fluid could also depend on position, as would be the case in treating a fluid of varying composition or in making an approximate application to a fluid of varying temperature, it might seem natural to test the consequences of adding some equation connecting p and ρ with r.

From a mathematical point of view, however, the derivatives of λ and ν occur in our Eqs. (2.6) in such a complicated and nonlinear manner that we cannot in general expect to obtain explicit analytic solutions when we complete the set by adding a further equation connecting p with ρ or p and ρ with r, and should usually have to resort to a method of approximate quadrature to obtain solutions in that way. In order to obtain explicit analytic solutions, it proves more advantageous to introduce the additional equation necessary to give a determinate problem in the form of some relation, connecting λ or ν or both with r, so chosen with reference to the occurrence of the derivatives of λ and ν in expressions (2.6) as to make the resulting set of equations readily soluble. By adopting such a mathematically rather than physically motivated procedure, we of course run the risk of obtaining solutions which may not be physically interesting or even physically possible. Nevertheless, having once obtained an explicit solution, it then becomes relatively easy to examine the implied physics and see if this has a character which affords insight into the equilibrium conditions that could be expected for actual fluids.

To carry out the suggested method of attack, it is desirable to re-express Eqs. (2.6) in a somewhat different form which will make it easier to ascertain what conditions on λ and ν can be introduced to secure solubility. In the first place, it is helpful to equate the two different expressions for the pressure p given by (2.6) and thus obtain a single equation for the dependence of λ

and ν on r. In the second place, it is then helpful to rewrite this result in a form which is already nearly integrable so that the desired kind of condition on λ and ν can be more easily seen. Introducing such a re-expression, Eqs. (2.6) are then found to be equivalent to the set

$$\frac{d}{dr}\left(\frac{e^{-\lambda}-1}{r^2}\right)+\frac{d}{dr}\left(\frac{e^{-\lambda}\nu'}{2r}\right)+e^{-\lambda-\nu}\frac{d}{dr}\left(\frac{e^{\nu}\nu'}{2r}\right)=0, \tag{3.1}$$

$$8\pi p=e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}+\Lambda, \tag{3.2}$$

$$8\pi \rho=e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}-\Lambda, \tag{3.3}$$

where the first of these equations (3.1) has been obtained in the manner just described. The one term in (3.1) which is not immediately integrable contains λ and ν in a sufficiently simple manner so that conditions on those quantities can be readily found which will make it easy to obtain a first integral of that equation.

§4. Specific Solutions

This method of attack can now be used to obtain a number of specific solutions. We shall summarize the results thus provided by first stating the additional *assumed equation* expressing the condition on λ or ν that was taken to secure the integrability of (3.1), and then giving the *resulting solution* for e^{λ}, e^{ν}, ρ and p as functions of r which can be obtained by combining the new equation with (3.1), (3.2) and (3.3). New symbols such as A, B, R, c, m, n, etc. will be used in these expressions to denote constants of integration; they are to be regarded as adjustable parameters having real, not necessarily integral, values.

Solution I. (Einstein universe)

Assumed equation

$$e^{\nu}=\text{const.}$$

Resulting solution

$$e^{\lambda}=\frac{1}{1-r^2/R^2}, \qquad e^{\nu}=c^2, \tag{4.1}$$
$$8\pi\rho=3/R^2-\Lambda, \quad 8\pi p=-1/R^2+\Lambda.$$

In this case the assumed equation makes (3.1) immediately integrable since the second two terms drop out. The resulting solution is the well-known one for a static Einstein universe with uniform density and pressure throughout. The solution could correspond to a distribution of actual fluid, with ρ and p non-negative, only with the cosmological constant satisfying the condition $3/R^2>\Lambda>1/R^2$.

Solution II. (Schwarzschild-de Sitter solution)

Assumed equation

$$e^{-\lambda-\nu}=\text{const.}$$

Resulting solution

$$e^{\lambda}=\left(1-\frac{2m}{r}-\frac{r^2}{R^2}\right)^{-1}, \qquad e^{\nu}=c^2\left(1-\frac{2m}{r}-\frac{r^2}{R^2}\right), \tag{4.2}$$
$$8\pi\rho=3/R^2-\Lambda, \quad 8\pi p=-3/R^2+\Lambda.$$

In this case the assumed equation makes it immediately possible to obtain a first integral of (3.1), since it makes the third term at once integrable. The necessary second integration then also proves to be possible and leads to the well-known combined Schwarzschild-de Sitter solution for a de Sitter universe with a spherically symmetrical body at the origin of coordinates. With ρ and p both non-negative the space around this body has to be empty with $3/R^2=\Lambda$ and $\rho=p=0$. With $R=\infty$ the solution goes over into the usual form for the Schwarzschild solution surrounding an attracting particle of mass m, and with $m=0$ it goes over into the usual form for the de Sitter universe.

Solution III. (Schwarzschild interior solution)

Assumed equation

$$e^{-\lambda}=1-r^2/R^2.$$

Resulting solution

$$e^{\lambda}=\frac{1}{1-r^2/R^2}, \quad e^{\nu}=[A-B(1-r^2/R^2)^{\frac{1}{2}}]^2, \tag{4.3}$$
$$8\pi\rho=3/R^2-\Lambda,$$
$$8\pi p=\frac{1}{R^2}\left(\frac{3B(1-r^2/R^2)^{\frac{1}{2}}-A}{A-B(1-r^2/R^2)^{\frac{1}{2}}}\right)+\Lambda.$$

In this case the assumed equation immediately simplifies (3.1) by making the first term drop out, and a first integral can at once be obtained, after multiplying through by $e^{\nu}\nu'/2r$. The second integration then also proves to be easily possible and leads at once to the well-known Schwarzschild interior solution for a sphere of fluid of constant density ρ and a pressure p which decreases with r. With the constant $B=0$ the solution degenerates into the Einstein universe and with $A=0$ into the de Sitter universe.

It should be noted that Schwarzschild's interior solution as given by (4.3) is not the most general solution for a sphere of constant density, since a more general assumed equation, $e^{-\lambda}=1-r^2/R^2+C/r$ with C an arbitrary constant, would also correspond to $\rho=\text{const.}$[4] The consequences of not setting $C=0$ in this expression have been studied by Mr. Volkoff and will be communicated elsewhere.

Solution IV

Assumed equation

$$e^{\nu}\nu'/2r=\text{const.}$$

Resulting solution

$$e^{\lambda}=\frac{1+2r^2/A^2}{(1-r^2/R^2)(1+r^2/A^2)},$$

$$e^{\nu}=B^2(1+r^2/A^2), \tag{4.4}$$

$$8\pi\rho=\frac{1}{A^2}\frac{1+3A^2/R^2+3r^2/R^2}{1+2r^2/A^2}+\frac{2}{A^2}\frac{1-r^2/R^2}{(1+2r^2/A^2)^2}-\Lambda,$$

$$8\pi p=\frac{1}{A^2}\frac{1-A^2/R^2-3r^2/R^2}{1+2r^2/A^2}+\Lambda.$$

In this case, we obtain the first of the new solutions to be considered. The assumed equation makes (3.1) immediately integrable by eliminating the third term. The solution for λ and ν then becomes easy. With suitable values for A and R the solution represents a sphere of compressible fluid with the pressure dropping to zero at the boundary as will be discussed more fully later.

Solution V

Assumed equation

$$e^{\nu}=\text{const. } r^{2n}.$$

Resulting solution

$$e^{\lambda}=\frac{1+2n-n^2}{1-(1+2n-n^2)(r/R)^N},\quad e^{\nu}=B^2r^{2n}, \tag{4.5}$$

where $N=\dfrac{2(1+2n-n^2)}{1+n}$,

$$8\pi\rho=\frac{2n-n^2}{1+2n-n^2}\frac{1}{r^2}+\frac{3+5n-2n^2}{(1+n)R^2}\left(\frac{r}{R}\right)^M-\Lambda,$$

$$8\pi p=\frac{n^2}{1+2n-n^2}\frac{1}{r^2}-\frac{1+2n}{R^2}\left(\frac{r}{R}\right)^M+\Lambda.$$

where $M=\dfrac{2n(1-n)}{1+n}$.

In this case the substitution of the values of e^{ν} and ν' which are given by the assumed equation makes (3.1) immediately integrable if we multiply through by $r^{-M}/(n+1)$. The solution is a natural one to use in investigating spheres of fluid with infinite density and pressure at the center, as will be discussed in more detail later.

Solution VI

Assumed equation

$$e^{-\lambda}=\text{const.}$$

Resulting solution

$$e^{\lambda}=2-n^2,\quad e^{\nu}=(Ar^{1-n}-Br^{1+n})^2, \tag{4.6}$$

$$8\pi\rho=\frac{1-n^2}{2-n^2}\frac{1}{r^2}-\Lambda,$$

$$8\pi p=\frac{1}{2-n^2}\frac{1}{r^2}\frac{(1-n)^2A-(1+n)^2Br^{2n}}{A-Br^{2n}}+\Lambda.$$

In this case, it is convenient to substitute the assumed equation in the form $e^{-\lambda}=(2-n^2)^{-1}$, and

[4] See R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford, 1934), § 96. In connection with the treatment of Schwarzschild's interior solution given in that place, it should be mentioned that the precise upper limits for r_1^2 and $2m$ should really be taken only eight-ninths as large as given in (96.14), since the solution ceases to have physical significance when the pressure p becomes infinite at $r=0$, i.e., when A becomes equal to B rather than when A becomes imaginary.

perform the indicated differentiations in (3.1). A first integral of the equation can then be obtained after multiplying through by $r^{n+1}e^{\nu/2}$, and the second integral after multiplying the result by r^{-1-2n}. The solution gives a very simple expression for the dependence of ρ on r, and is again a natural one to use in investigating spheres of fluid with infinite density and pressure at center, as will be discussed in more detail later.

Solution VII

Assumed equation

$$e^{-\lambda}=1-\frac{r^2}{R^2}+\frac{4r^4}{A^4}.$$

Resulting solution

$$e^{\lambda}=\frac{1}{1-r^2/R^2+4r^4/A^2}, \tag{4.7}$$

$$e^{\nu}=B^2\left[\sin\log\left(\frac{e^{-\lambda/2}+2r^2/A^2-A^2/4R^2}{C}\right)^{\frac{1}{2}}\right]^2,$$

$$8\pi\rho=\frac{3}{R^2}-\frac{20r^2}{A^4}-\Lambda,$$

$$8\pi p=-\frac{1}{R^2}+\frac{4r^2}{A^4}+\frac{4e^{-\lambda/2}}{A^2}(B^2e^{-\nu}-1)^{\frac{1}{2}}+\Lambda.$$

In this case, the substitution of the assumed equation into the first term of Eq. (3.1) makes it reduce to a constant times r, and the equation then becomes integrable after multiplication by $e^{\nu}\nu'/2r$. The dependence of p on r, with $e^{-\lambda/2}$ and $e^{-\nu}$ explicitly expressed in terms of r, is so complicated that the solution is not a convenient one for physical considerations.

Solution VIII

Assumed equation

$$e^{-\lambda}=\text{const. } r^{-2b}e^{\nu}.$$

Resulting solution

$$e^{-\lambda}=\frac{2}{(a-b)(a+2b-1)}-\left(\frac{2m}{r}\right)^{a+2b-1}-\left(\frac{r}{R}\right)^{a-b},$$

$$e^{\nu}=B^2r^{2b}e^{-\lambda},$$

$$8\pi\rho=\left\{1-\frac{2}{(a-b)(a+2b-1)}-(a+2b-2)\left(\frac{2m}{r}\right)^{a+2b-1}+(a-b+1)\left(\frac{r}{R}\right)^{a-b}\right\}\frac{1}{r^2}-\Lambda, \tag{4.8}$$

$$8\pi p=2be^{-\lambda}/r-8\pi\rho,$$

with $(a+b)(a-1)-2b-2=0$.

In this case, it is convenient to substitute the assumed expression for $e^{-\lambda}$ in (3.1) and perform the indicated differentiations. A first integral of Eq. (3.1) can then be obtained after multiplying through by r^a where a is connected with b by the last of Eqs. (4.8); and the second integral can then be obtained after multiplying through by r^{-2a-b}. With $a=2$, $b=0$, the solution degenerates into the Schwarzschild-de Sitter solution as already treated under (4.2).

This is the last of the new solutions which we shall present.

§5. Connection of Interior and Exterior Solutions

With an appropriate choice of parameters, some of the foregoing solutions would correspond to a distribution of fluid in which the pressure p drops from its central value at $r=0$ to the value zero at some particular radius $r=r_b$ where the density ρ still remains finite. Such a solution could then be taken as describing the condition inside a limited sphere of fluid with a definite boundary at $r=r_b$, and in the empty space outside this boundary would be taken as replaced by the Schwarzschild-de Sitter solution, in accordance with Birkhoff's theorem as to the most general spherically symmetrical solution in empty space. It will now be of interest to consider the interconnection of the two forms of solution at the surface of discontinuity at $r=r_b$.

Inside the sphere of fluid, we may take the solution as described by a line element of the general form (2.2)—on which we have based our considerations

$$ds^2=-e^{\lambda}dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+e^{\nu}dt^2, \tag{5.1}$$

where λ and ν are functions of r. Outside the sphere, we may take the solution as described by a line element of the simple Schwarzschild form

$$ds^2=-\frac{dr^2}{1-2m/r}-r^2d\theta^2 -r^2\sin^2\theta d\phi^2+\left(1-\frac{2m}{r}\right)dt^2, \quad (5.2)$$

which arises from the full expression for the Schwarzschild-de Sitter solution (4.2), when we set $\Lambda=3/R^2=0$ in agreement with the known fact that the cosmological constant is too small to produce appreciable effects within a moderate spatial range, and where for convenience we set $c^2=1$ in order that m shall be the mass of the sphere—as measured by its external gravitational field—expressed in the usual units which make the velocity of light and the gravitational constant equal to unity. At the boundary of the sphere at $r=r_b$ both forms of the solution must then give the same results for physical measurements made at that radius.

Since the pressure p of the fluid falls to zero at the boundary, we may calculate the radius of the boundary r_b by setting the general expression for p given by (3.2) equal to zero. With $\Lambda=0$, this then gives us

$$e^{-\lambda(r=r_b)}\left[\frac{\nu'(r=r_b)}{r_b}+\frac{1}{r_b^2}\right]-\frac{1}{r_b^2}=0 \quad (5.3)$$

as the equation which determines the radius of the boundary r_b. At this radius (employing for simplicity a unified system of coordinates r, θ, ϕ, t applicable both inside and outside the sphere) we must then demand the equalities

$$e^{\nu(r=r_b)}=e^{-\lambda(r=r_b)} \quad (5.4)$$

and

$$e^{-\lambda(r=r_b)}=1-2m/r_b \quad (5.5)$$

in order that the two forms of line element shall lead to the same results for measurements made at the boundary with stationary meter sticks and clocks.

We shall make use of the three equations (5.3), (5.4) and (5.5) in the following three sections where we consider specific examples of fluid spheres surrounded by empty space. In using these equations, it is convenient to regard Eq. (5.3) as determining the radius r_b of the sphere of fluid in terms of the parameters appearing in the expressions for e^λ and e^ν in the line element (5.1). Eq. (5.4) may then be regarded as imposing a condition which connects the parameters appearing in e^ν with those in e^λ and with r_b, a condition which can always be readily satisfied since it will be noted that e^ν is originally always arbitrary as to a multiplicative factor. Finally, Eq. (5.5) may then be used to calculate a value of the gravitational mass of the sphere m in terms of r_b and the parameters mentioned. It will be noted from the form of (5.5) that the gravitational mass of the sphere m can in any case not be larger than $r_b/2$, and hence can go to infinity only as the size of the sphere goes to infinity. This, however, does not necessarily imply that the mass M of the fluid before it has been condensed into the sphere could not go to infinity with r_b finite.

§6. Detailed Consideration of Solution IV

We shall take Solution IV as providing the first of the examples of a sphere of fluid surrounded by empty space, which we wish to consider in more detail.

The *line element* describing this solution has the form

$$ds^2=-\frac{1+2r^2/A^2}{(1-r^2/R^2)(1+r^2/A^2)}dr^2-r^2d\theta^2 -r^2\sin^2\theta d\phi^2+B^2(1+r^2/A^2)dt^2. \quad (6.1)$$

Inside the sphere of fluid, the *density* ρ and *pressure* p (with $\Lambda=0$) are given by expressions of the form

$$8\pi\rho=\frac{1}{A^2}\frac{1+3A^2/R^2+3r^2/R^2}{1+2r^2/A^2} +\frac{2}{A^2}\frac{1-r^2/R^2}{(1+2r^2/A^2)^2}, \quad (6.2)$$

and

$$8\pi p=\frac{1}{A^2}\frac{1-A^2/R^2-3r^2/R^2}{1+2r^2/A^2}. \quad (6.3)$$

The *central density* ρ_c and *central pressure* p_c

have the values

$$8\pi\rho_c = \frac{3}{A^2} + \frac{3}{R^2}, \quad \text{and} \quad 8\pi p_c = \frac{1}{A^2} - \frac{1}{R^2}. \qquad (6.4)$$

In terms of these central values, the *equation of state*, connecting the density and pressure of the fluid inside the sphere, can then be written in the convenient simple form

$$\rho = \rho_c - 5(p_c - p) + 8\frac{(p_c - p)^2}{\rho_c + p_c}. \qquad (6.5)$$

The *boundary of the sphere* occurs at

$$r_b = \frac{R}{3^{\frac{1}{2}}}(1 - A^2/R^2)^{\frac{1}{2}}, \qquad (6.6)$$

where the pressure has dropped to the value zero, and the *boundary density* has the value

$$\rho_b = \rho_c - 5p_c + 8p_c^2/(\rho_c + p_c). \qquad (6.7)$$

From the connection with *Schwarzschild's exterior solution* (5.2), holding outside the boundary, we have

$$B^2 = (1 - r_b^2/R^2)/(1 + 2r_b^2/A^2), \qquad (6.8)$$

as a *condition* on the *parameter* B^2 appearing in e^ν. And we have

$$m = \frac{r_b}{2}\left\{1 - \frac{(1 - r_b^2/R^2)(1 + r_b^2/A^2)}{1 + 2r_b^2/A^2}\right\}, \qquad (6.9)$$

as an expression for the *mass of the sphere* as measured by its external gravitational field.

It will be seen from the foregoing that all the properties of the sphere and its surrounding field can be regarded as determined by the choice of the two independent parameters A and R, although it is not necessarily most convenient to express those properties solely in terms of these parameters. With $R^2 > A^2$, it will be found that the pressure and density of the fluid would both fall continuously from their central to their boundary values where the density would still be positive. And with $R^2 < 11.5A^2$, it will be found that the ratio of pressure to density would nowhere exceed one-third. As R^2 and A^2 go to infinity the central density and pressure approach zero, and the sphere becomes larger without limit.

The solution may be a useful one in studying the properties of spheres of compressible fluid since the equation of state (6.5) which connects the density of the fluid with its pressure is relatively simple. This equation of state, however, is of course a very special one, since the coefficients in the terms which give the linear and quadratic dependence of density on pressure are not arbitrary but are of the form which has arisen from the particular assumption that was introduced to secure integrability. Nevertheless, it has been shown by Professor Oppenheimer and Mr. Volkoff, in the article mentioned, that this equation of state leads to results in some respects similar to those which would be obtained from the equation of state for a Fermi gas in cases of intermediate central densities.

§7. Detailed Consideration of a Special Case of Solution V

We shall take Solution V as providing the second of the examples of a sphere of fluid surrounded by empty space which we wish to consider in more detail. In carrying this out we shall make the specific choice $n = \frac{1}{2}$ for the parameter n which appears in the description of the solution (4.5), since this will give a ratio of central density to pressure of special physical interest.

The *line element* describing the solution (with $n = \frac{1}{2}$) has the form

$$ds^2 = -\frac{7}{4 - 7(r/R)^{7/3}}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + B^2 r dt^2. \qquad (7.1)$$

Inside the sphere of fluid the *density* ρ and *pressure* p (with $\Lambda = 0$) are given by expressions of the form

$$8\pi\rho = \frac{3}{7r^2} + \frac{10}{3R^2}\left(\frac{r}{R}\right)^{\frac{1}{3}}, \qquad (7.2)$$

and

$$8\pi p = \frac{1}{7r^2} - \frac{2}{R^2}\left(\frac{r}{R}\right)^{\frac{1}{3}}. \qquad (7.3)$$

The *central density* ρ_c and *central pressure* p_c then become infinite with the ratio

$$p_c/\rho_c = \tfrac{1}{3}. \qquad (7.4)$$

The *equation of state* connecting the density and pressure of the fluid inside the sphere can be written in the form

$$\rho = 3p + \frac{7}{6\pi R^{7/3}}\left\{\frac{1}{4\pi(3\rho+5p)}\right\}^{1/6} \tag{7.5}$$

The *boundary of the sphere* occurs at

$$r_b = R/14^{3/7}, \tag{7.6}$$

where the pressure has dropped to zero and the *boundary density* has the value

$$8\pi\rho_b = 28/(3\times 14^{1/7}R^2). \tag{7.7}$$

From the connection with *Schwarzschild's exterior solution* (5.2), holding outside the boundary, we have

$$B^2 = 14^{3/7}/2R, \tag{7.8}$$

as a *condition* on the *parameter* B^2 appearing in e^ν. And we have

$$m = r_b/4 = R/(4\times 14^{3/7}) \tag{7.9}$$

as an expression for the *mass of the sphere*, as measured by its external gravitational field.

It will be noted in this case that the solution corresponds to a sphere of fluid of infinite density and pressure at the center, having at that point the ratio which would hold for radiation, or for particles of such high kinetic energy that their rest mass may be neglected in comparison with their total mass. Other ratios could, of course, be obtained with a different choice for the parameter n in (4.5).

With the adjustable parameter R lying anywhere in the range $0<R<\infty$, it will be seen that the solution satisfies the necessary physical conditions for a sphere of fluid, with a pressure which drops continuously from an infinite value at the center to zero at the boundary, with a density which drops continuously from an infinite value at the center to a value which is still positive at the boundary, and with a ratio of pressure to density which never exceeds one-third. As R approaches infinity, the ratio of pressure to density approaches one-third throughout and the sphere becomes larger without limit.

The solution may in some cases be a useful one in studying the properties of *finite* spheres of fluid which have *infinite* central pressures and densities. In accordance with the equation of state (7.5), however, the ratio of pressure to density drops too fast with decreasing density to correspond very closely with what would be expected in the case of a Fermi gas having infinite central pressure and density.

§8. Detailed Consideration of a Special Case of Solution VI

We shall now take Solution VI as providing the last of the examples of a sphere of fluid surrounded by empty space which we wish to consider in more detail. In carrying this out we shall make the specific choice $n=\frac{1}{2}$ for the parameter n which appears in the description of the solution (4.6), which will again give us the value one-third for the ratio of central pressure to density.

The *line element* describing the solution (with $n=\frac{1}{2}$) has the form

$$ds^2 = -(7/4)dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 + (Ar^{\frac{1}{2}} - Br^{\frac{3}{2}})^2dt^2. \tag{8.1}$$

Inside the sphere of fluid the *density* ρ and *pressure* p (with $\Lambda=0$) are given by expressions of the form

$$8\pi\rho = 3/7r^2, \tag{8.2}$$

and

$$8\pi p = \frac{1}{7r^2}\,\frac{1-9(B/A)r}{1-(B/A)r}. \tag{8.3}$$

The *central density* ρ_c and *central pressure* p_c then become infinite with the ratio

$$p_c/\rho_c = \tfrac{1}{3}. \tag{8.4}$$

The *equation of state* connecting the density and pressure of the fluid inside the sphere can be written in the form

$$p = \frac{\rho}{3}\,\frac{1-9(3/56\pi)^{\frac{1}{2}}(B/A)\rho^{-\frac{1}{2}}}{1-(3/56\pi)^{\frac{1}{2}}(B/A)\rho^{-\frac{1}{2}}}. \tag{8.5}$$

The *boundary of the sphere* occurs at

$$r_b = A/9B, \tag{8.6}$$

where the pressure has dropped to zero and the *boundary density* has the value

$$8\pi\rho_b = 3^5B^2/7A^2. \tag{8.7}$$

From the connection with *Schwarzschild's exterior solution* (5.2), holding outside the boundary, we have

$$A^2=(3^6/2^4\times 7)(B/A) \qquad (8.8)$$

as a *condition* on the parameter A^2 appearing in e^ν, where B/A may still be taken as arbitrary. And we have

$$m=3r_b/14=A/42B, \qquad (8.9)$$

as an expression for the *mass of the sphere* as measured by its external gravitational field.

Again it will be noted as in the previous case discussed in §7, that the solution corresponds to a sphere of fluid of infinite density and pressure at the center, having at that point the ratio which would hold for radiation or for particles of such high kinetic energy that their rest mass may be neglected in comparison with their total mass. Other ratios could be obtained with a different choice for n in (4.6).

With the adjustable parameter B/A lying anywhere in the range $0<B/A<\infty$, it will be seen that the solution satisfies the necessary physical conditions for a sphere of fluid, with a pressure which drops continuously from an infinite value at the center to zero at the boundary, with a density which drops continuously from an infinite value at the center to a value which is still positive at the boundary, and with a ratio of pressure to density which never exceeds one-third. As B/A approaches zero, the ratio of pressure to density approaches one-third throughout, the sphere becomes larger without limit, and the solution as given by (8.1) approaches the same form as is approached by the solution given by (7.1) as R goes to infinity. This limiting form, which agrees with that given in the article of Professor Oppenheimer and Mr. Volkoff by Eq. (22), might be called the blackbody radiation solution.

The solution in its more general form proves to be a somewhat helpful one for use in connection with the results of Professor Oppenheimer and Mr. Volkoff, since with ρ large the equation of state (8.5) goes over into the approximate form $\rho-3p=\text{const.}\ \rho^{\frac{1}{3}}$ which is that for a highly compressed Fermi gas.

§9. Concluding Remarks

In conclusion we must call attention to two points which are more completely considered in the article of Professor Oppenheimer and Mr. Volkoff. It is necessary to mention these points also here in order to guard against misconceptions as to the nature of the static solutions of Einstein's field equations for spheres of fluid which have been presented in this paper.

In the first place, it should be remarked that the static character of any such solution is in itself only sufficient to assure us that the solution describes a possible state of equilibrium for a fluid, but is not sufficient to tell us whether or not that state of equilibrium would be stable towards disturbances. Further investigation is necessary to settle the question of stability under any given set of circumstances. The question is an important one, since we cannot regard a static solution as representing a physically permanent state of a fluid if the equilibrium turns out to be unstable towards small disturbances, as for example in the well-known case of the Einstein static universe.

In the second place, it should be emphasized that the imposition of static character on the solutions to be considered is from a physical point of view a severe restriction. It is, of course, immediately evident that solutions having a strictly static character could in any case be applicable only in first approximation to spheres of fluid where slow changes are actually taking place. In addition it is to be noted that there might be a possibility for an important class of quasi-static solutions with $e^\nu=g_{44}$ going to zero at the center of the sphere, as in certain of static solutions considered above. Such solutions could be said to have quasi-static character, since changes taking place at the center would exhibit a very slow rate when measured by an external observer. Further discussion of the possible existence and importance of such solutions will be found in the article by Professor Oppenheimer and Mr. Volkoff.