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On the Absence of Polarization in Electron Scattering

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In an attempt to explain the discrepancy between Mott's theory and the observational results on the polarization of electrons in the double scattering experiment, we have investigated the depolarizing effect of (1) multiple elastic scattering, (2) inelastic scattering with spin change of the incident electron and (3) exchange scattering in which the exchanged electrons have opposite spins. Although multiple scattering actually takes place in the Au foils used in the experiments, the depolarization caused by such scattering is negligible. For a screened Coulomb field and electrons of 100 kv the depolarization in a 2.5×10^{-5} cm Au foil is only 2.2 percent whereas a 90–100 percent depolarization is needed to reconcile theory and experiment. The depolarization caused by inelastic scattering is about 100 times smaller than that caused by elastic scattering. The small depolarization in these two cases is due to the fact that while there are many elastic and inelastic collisions, most of these collisions take place with small scattering angle ϑ and for small angles the depolarization per collision is small (of order ϑ^2). The main contribution to the depolarization by inelastic collisions comes then from large scattering angles at which inelastic scattering is relatively infrequent. The exchange scattering contributes practically no depolarization ($\sim 10^{-12}$). While the depolarization per collision in each exchange process is relatively complete, the number of exchange processes is extremely small due to interference ($\sim 10^{-14}$ times the number of elastic collisions). Therefore the absence of polarization in electron scattering cannot be explained on the basis of the processes considered here.

A discussion of the validity of the first Born approximation is given (Appendix §2). It is shown that in the case that the Born expansion parameter $e^2Z/\hbar v$ is of order unity, the first Born approximation gives a result for the scattered intensity which is accurate to within 20 percent or better. The depolarization is given with an accuracy of 18 percent or better.

I. Introduction

One of the important consequences of the Dirac theory of the electron is the prediction that an initially unpolarized beam of electrons will become partially polarized as the result of an elastic scattering process. If this partially polarized beam of electrons is made to undergo a second scattering, the intensity of the twice scattered beam, according to the theory, should be asymmetrical in the azimuth about the direction of incidence on the second target. In fact, the calculated scattering consists of two parts: The first, independent of the azimuth, is the product of the relativistic Rutherford intensities corresponding to the scattering angles at the two targets. The second part arises from spin-orbit interaction in the scattering field; it depends on the orientation of the magnetic moment with respect to the second scattering plane, that is, it is azimuth dependent. Obviously this term yields a nonvanishing contribution to the scattering only after the spin axes have been given a preferential direction.

The quantitative theory of the polarization of electrons by double scattering has been given by Mott.[1] According to his exact treatment of the scattering in a Coulomb field the expected azimuthal asymmetry is large enough to be observed experimentally if the following conditions are fulfilled: (1) only single scattering should take place in the two targets,[2] (2) the atomic number of the scattering atoms of both targets should be large ($Z/137$ comparable with unity), (3) both scattering angles should be large (of order $\pi/2$), and (4) the energy of the incident electrons should not be too small.[3]

The intensity of the doubly scattered beam is proportional to J where

$$J=1+\delta(\vartheta_1,\,\vartheta_2)\cos\varphi. \tag{1}$$

Here ϑ_1 and ϑ_2 are the two scattering angles, φ is the azimuthal angle and δ is a function of the two scattering angles which has been calculated by Mott. The asymmetry, that is, the quantity given in the experiments is

$$2\delta=(J_0-J_\pi)/J_{\text{Av}}, \tag{2}$$

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[1] N. F. Mott, Proc. Roy. Soc. **135**, 429 (1932).

[2] As will be seen in the following, this condition is perhaps too stringent.

[3] For Au the theoretical asymmetry is negligible below 30 kv, attains a (rather flat) maximum at 130 kv and then very slowly approaches zero for infinite energy.

where the subscripts refer to the azimuthal angle. For $\vartheta_1=\vartheta_2=\pi/2$, the scattering in Au of electrons of 90 to 160 kv should be characterized by an asymmetry of 12 to 16 percent.

Very careful experiments for the purpose of observing the asymmetry have been carried out by Dymond.[4] With electrons of 90, 125 and 160 kv energy scattered through angles $\pi/2$ at two Au foil targets the asymmetry observed was of the same magnitude as the experimental error of one percent. That is, *if there is any asymmetry at all*, it is small enough to be completely masked by the one percent experimental error due to the small deviations from ideal geometry, etc.

In the attempt to reconcile the theoretical results of Mott and Dymond's experiments we may first inquire whether all the conditions mentioned above were fulfilled. The only condition the fulfillment of which may be open to question is that of single scattering. It can be shown (Appendix §3) that an appreciable amount of multiple scattering (about 50 collisions) takes place even in the thin foils used by Dymond (2.5×10^{-5} to 5×10^{-6} cm thickness). It is evident that multiple scattering will result in some depolarization of the beam but just how much remains to be seen.

In addition to multiple scattering we have investigated the depolarizing effect of other processes. These are: (1) inelastic scattering with change of spin of the incident electron and (2) exchange scattering in which the exchanged electrons are of opposite spin.

The result of these considerations may be stated very briefly. Unfortunately, none of the effects considered produces any appreciable depolarization of the electrons and the discrepancy between theory and experiment remains—perhaps more glaring than before.

II. The Theory of Polarization

Before discussing the depolarizing effect of multiple scattering it will be most convenient to recapitulate some of the general theory of double scattering.[1] We consider an electron with wave number k incident along the z axis and scattered elastically in the direction ϑ, φ by some field of force. The wave function describing the electron is asymptotically

$$\psi_\nu\sim s_\nu e^{ikz}+u_\nu(\vartheta,\varphi)e^{ikr}/r,\quad \nu=1\cdots4, \tag{3}$$

where s_ν are the components of the amplitude of the Dirac plane wave. In particular we have

$$s_3=A,\quad s_4=B, \tag{4}$$

where A and B are constants related to the direction of the spin axis (χ_0, ω_0) in the incident beam.

$$-B/A=\cot(\chi_0/2)e^{i\omega_0}. \tag{5}$$

We may further impose the normalizing condition

$$|A|^2+|B|^2=1. \tag{6}$$

The differential cross section for scattering into the solid angle $d\Omega$ is then given by

$$\Phi_0 d\Omega=(|u_3|^2+|u_4|^2)d\Omega. \tag{7}$$

The general solution of the scattering problem may be obtained by a superposition of the two waves corresponding to $A=1$, $B=0$ and $A=0$, $B=1$. Mott has shown that in these two cases the asymptotic forms of the wave functions ψ_3 and ψ_4 are given by:

$A=1$, $B=0$:

$$\begin{aligned}\psi_3&\sim e^{ikz}+f(\vartheta)e^{ikr}/r\\ \psi_4&\sim g(\vartheta)e^{i\varphi}e^{ikr}/r,\end{aligned} \tag{8a}$$

$A=0$, $B=1$:

$$\begin{aligned}\psi_3&\sim -g(\vartheta)e^{-i\varphi}e^{ikr}/r\\ \psi_4&\sim e^{ikz}+f(\vartheta)e^{ikr}/r,\end{aligned} \tag{8b}$$

where f and g are independent of φ and are the same functions in (8*a*) as in (8*b*). By superposition we find for the scattered beam

$$u_3=Af-Bge^{-i\varphi},\quad u_4=Bf+Age^{i\varphi}. \tag{9}$$

For the scattered electron the direction of the spin axis χ, ω is given by (cf. Eq. (5))

$$-u_4/u_3=\cot(\chi/2)e^{i\omega}. \tag{10}$$

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[4] E. G. Dymond, Proc. Roy. Soc. **136**, 638 (1932), **145**, 657 (1934). For further references and a critical survey of the experimental work see H. Richter, Ann. d. Physik **28**, 533 (1937).

The scattering of an initially unpolarized beam may be described by considering the beam to be composed of two equally intense completely polarized beams with opposing directions of polarization. This direction is arbitrary and it is convenient to choose it so that the *direction* of polarization in each beam is the same after one scattering as before. That is, from (5) and (10)

$$u_4/u_3=B/A. \tag{11}$$

From (9) we find

$$B/A=\pm ie^{i\varphi}. \tag{11a}$$

Choosing the scattering plane as the plane $\varphi=0$, we have

$$B/A=\pm i, \tag{11b}$$

which means that the direction of polarization is normal to the scattering plane. The two polarized beams may now be considered separately.

The amplitude of the once scattered beam is then

$$u_3/A=u_4/B=f\mp ig \tag{12}$$

and the intensities for the two scattered beams are given by

$$\Phi_\pm=|f|^2+|g|^2\mp i(f^*g-fg^*). \tag{12a}$$

Thus after one scattering the beam is polarized with a polarization given by

$$\delta^{\frac{1}{2}}=i(fg^*-f^*g)/(|f|^2+|g|^2). \tag{12b}$$

The existence of a polarization therefore depends not only on the existence of g but also on a difference in phase between f and g.

The scattered intensity averaged over the spin directions is from (12a)

$$\Phi_0(\vartheta)=|f|^2+|g|^2. \tag{12c}$$

Since the two beams are treated independently the intensity for the double scattering is obviously equal to (compare (1))

$$\Phi_0(\vartheta_1)\Phi_0(\vartheta_2)[1+(\delta(\vartheta_1)\delta(\vartheta_2))^{\frac{1}{2}}\cos\varphi]. \tag{12d}$$

III. DEPOLARIZATION BY MULTIPLE ELASTIC SCATTERING

We consider that the scattering in each target consists of many small angle scatterings (multiple scattering) plus one large angle scattering so that the total deflection is $\pi/2$. The small angle scatterings taking place before the first large deflection do not polarize the initial beam and, of course, we are not interested in the depolarizing effect of the small angle scattering after the second large deflection. Therefore our problem is to determine the depolarization of the beam between the two large deflections after it has been polarized by the first of these large deflections.

We define the polarization p in terms of the probability amplitudes for the two spin directions:

$$p=|A|^2-|B|^2. \tag{13}$$

Then the change of polarization per collision is

$$\delta p=(|u_3|^2-|u_4|^2)/(|u_3|^2+|u_4|^2)-p. \tag{14}$$

From (13), (9), (7) and (12c) we find for the change in polarization averaged over the azimuth[5]

$$\delta p=-2p|g|^2/(|f|^2+|g|^2). \tag{15}$$

Multiplying by the number of collisions with scattering angle ϑ which take place in a distance x_1 in the first target and a distance x_2 in the second target, which from (12c) is $2\pi N(x_1+x_2)(|f|^2+|g|^2)\sin\vartheta d\vartheta$, we find for the net polarization

$$P(x_1,x_2)=\exp\left[-4\pi N(x_1+x_2)\int_0^\pi |g|^2\sin\vartheta d\vartheta\right], \tag{16}$$

if the polarization is complete initially. Here N is the number of scattering atoms per unit volume. Averaging over the effective thickness of each target for 45° incidence (and emergence) we have

$$P=\frac{1}{2d^2}\int_0^{\sqrt{2}d}dx_1\int_0^{\sqrt{2}d}dx_2P(x_1,x_2), \tag{17}$$

where d is the actual thickness of the target. Thus we find for the degree of polarization

$$P=\left(\frac{1-e^{-\gamma d}}{\gamma d}\right)^2, \tag{18}$$

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[5] There is no correlation between the direction of the spin axis and the azimuth of the scattering plane.

where

$$\gamma = 4\sqrt{2}\pi N \int_0^\pi |g|^2 \sin \vartheta d\vartheta. \tag{18a}$$

From the above treatment it is apparent that P represents the ratio of the polarization of the beam just before the second large deflection to that just after the first large deflection. Therefore the asymmetry is now given by

$$2\delta = P 2\delta_{\mathrm{Mott}}. \tag{19}$$

In the following we shall refer to $1-P$ as the degree of depolarization.

Calculation of degree of depolarization

The calculation of g may be carried out by formulating the scattering problem in terms of the phases of the scattered (partial) waves at infinity. We denote the radial wave functions of the third and fourth components of the Dirac wave function by $G_\varkappa$. For the two types of solutions of the wave equation, that is, for the total angular momentum quantum number $j=l+\frac{1}{2}$ and $l-\frac{1}{2}$, $\varkappa$ is given by

$$\begin{aligned} \varkappa &= l & &\text{for} & j&=l+\tfrac{1}{2} \\ \varkappa &= -l-1 & &\text{for} & j&=l-\tfrac{1}{2}. \end{aligned} \tag{20}$$

At large distances r from the scattering atom $G_\varkappa$ has the asymptotic form

$$G_\varkappa \sim 1/r \cos\left(kr-(l+1/2)\pi+\eta_\varkappa\right). \tag{21}$$

Defined in this manner, the phase $\eta_\varkappa$ vanishes for the free particle.[6]

Mott[1] has shown that superpositions of the partial waves, $G_\varkappa$ multiplied by a suitable spherical harmonic, may be found which have the asymptotic form (8) with $g(\vartheta)$ given by

$$g(\vartheta) = i/2k \sum_0^\infty P_l^1(\cos\vartheta)\left[e^{2i\eta_l}-e^{2i\eta_{-l-1}}\right], \tag{22}$$

where

$$P_l^1(\cos\vartheta) = \sin\vartheta (d/d\cos\vartheta) P_l(\cos\vartheta) \tag{22a}$$

and P_l is the ordinary Legendre polynomial. With this expression for g the integral occurring in (18a) becomes

$$\int_0^\pi |g|^2 \sin\vartheta d\vartheta$$

$$= 2/k^2 \sum_1^\infty (l(l+1)/(2l+1)) \sin^2 \delta\eta_l, \tag{23}$$

where

$$\delta\eta_l = \eta_l - \eta_{-l-1}. \tag{24}$$

To determine the phase difference $\delta\eta_l$ we consider the differential equation obtained for $G_\varkappa$ by eliminating the radial function of the first and second Dirac components from the well-known pair of simultaneous first-order equations.[7] The equation obtained may be written in the form

$$d^2G'/dr^2 + Q(r)G' = 0, \tag{25}$$

$$G = (2mc^2+E-V)^{-\frac{1}{2}} G'/r. \tag{25a}$$

E is the kinetic energy of the electrons and V is the scattering potential.

$$\begin{aligned} Q(r) = k^2 &- \frac{\varkappa(\varkappa+1)}{r^2} - \frac{2V(E+mc^2)}{\hbar^2c^2} \\ &+ \frac{\varkappa+1}{r}\,\frac{dV/dr}{2mc^2+E-V} \\ &+ \frac{1}{4}\,\frac{(dV/dr)^2+2(2mc^2+E-V)d^2V/dr^2}{(2mc^2+E-V)^2}. \end{aligned} \tag{26}$$

For the scattering potential V we must use a screened Coulomb field since at the small angles in which we are interested, the scattering by the atomic electrons is just as important as the nuclear scattering. We shall take

$$V = -(e^2Z/r)e^{-r/b}, \tag{27}$$

where b is the screening length (see Eq. (39) below).

We may solve (25) by the W. K. B. method. If r_1 is the (largest) root of $Q(r)$ we have for $r > r_1$,

$$G_\varkappa = \frac{\text{const.}}{rQ^{\frac{1}{2}}}(2mc^2+E-V)^{-\frac{1}{2}} \times \cos\left[\int_{r_1}^{r} Q^{\frac{1}{2}} dr - \pi/4\right]. \tag{28}$$

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[6] For the free particle $G_\varkappa \sim r^{-\frac{1}{2}} J_{|\varkappa+\frac{1}{2}|}(kr)$, where J is the Bessel function

[7] Cf. e.g., H. A. Bethe, *Handbuch der Physik*, Vol. 24/1, p. 311.

In order to evaluate the integral in (28) we note first of all that the last two terms in (26) are small compared to the third term. As is evident from (26) the relative magnitude of the small terms will be largest for small r. In this case we have

$$dV/dr \sim V/r, \qquad d^2V/dr^2 \sim 2V/r^2. \tag{29}$$

Then in the last term the important part is that involving d^2V/dr^2, *viz.:*

$$(dV/dr)^2/4mc^2 d^2V/dr^2 \sim V/8mc^2 \lesssim (Z/137)(k\hbar/mc)1/8l \ll 1, \tag{29a}$$

since for 100 kv $k\hbar/mc \sim 1$. In (29a) we have evaluated V at the minimum r, i.e., $r_1 \sim l/k$. (See Eqs. (33a) and (34a) below.) The ratio of the fifth to the third term is

$$\tfrac{1}{4}(\hbar/mcr)^2 \lesssim \tfrac{1}{4}(k\hbar/mc)^2 1/l^2 < 1/l^2. \tag{29b}$$

Similarly, the ratio of the fourth term to the third is less than $1/l$ which is also small compared to one.

The third term is therefore the leading term of those involving the potential V. This term will be comparable with the free particle terms $\sim k^2$ only for the smallest values of l, *viz.:*

$$\frac{\text{third term}}{\text{first term}} \lesssim \frac{2Ze^2}{r_1}\frac{m}{\hbar^2k^2}e^{-l/kb} \sim \frac{2}{l}\left(\frac{E_K}{E}\right)^{\frac{1}{2}}e^{-l/kb}, \tag{29c}$$

where E_K is the ionization potential for the K shell; $E_K \approx 40$ kv for Au.

We may now expand Q in powers of the small terms. We set

$$Q = q + q', \tag{30}$$

where

$$q = k^2 - \varkappa(\varkappa+1)/r^2 + 2k_0e^{-r/b}/r, \tag{30a}$$

$$k_0 = e^2Z(E+mc^2)/\hbar^2c^2. \tag{30b}$$

Then

$$\begin{aligned}\delta\eta_l &= \int_{r_1}^{\infty}(Q_l^{\frac{1}{2}} - Q^{\frac{1}{2}}_{-l-1})dr \\ &\cong \tfrac{1}{2}\int_{r_1}^{\infty}q^{-\frac{1}{2}}(q'_l - q'_{-l-1})dr \qquad (31)\\ &= \frac{l+\frac{1}{2}}{E+2mc^2}\int_{r_1}^{\infty}q^{-\frac{1}{2}}(dV/dr)dr/r,\end{aligned}$$

since q is the same for both values of $\varkappa$.

Denoting the integrand in (31) by I we use the identity

$$\int_{r_1}^{\infty}Idr = \int_{r_0}^{\infty}I_0dr + \left(\int_{r_1}^{\infty}Idr - \int_{r_0}^{\infty}I_0dr\right), \tag{31a}$$

where $I_0 = I(k_0=0)$ and $r_0 = r_1(k_0=0)$ is the root of q for $k_0=0$. We denote the contribution of the first integral to the phase difference by $\delta\eta_l^{(0)}$ and that of the last two integrals by $\delta\eta_l^{(1)}$; the latter will be important only for small l (cf. above). When the last two integrals are considered together, the difference between the integrands is appreciable only when the term $2k_0e^{-r/b}/r$ is important compared to the free particle terms in (30a). This is the case for $r \sim r_1 \sim r_0$. Then, since we consider l small, in $I - I_0$ we may set the exponential $e^{-r/b} = 1$ and neglect the term $1/b$ in dV/dr since these two neglections partially compensate each other and the error thus induced is $\sim (r_0/b)^2 \sim (l/kb)^2$ $(l/20)^2$ for energies of the order 100 kv, (cf. (33a) below).

With

$$\delta\eta_l = \delta\eta_l^{(0)} + \delta\eta_l^{(1)}, \tag{32}$$

we have

$$\delta\eta_l^{(0)} = \frac{l+\frac{1}{2}}{E+2mc^2}\frac{e^2Z}{k}\int_{r_0}^{\infty}\frac{dr}{r}\left(\frac{1}{r}+\frac{1}{b}\right)\frac{e^{-r/b}}{(r^2-r_0^2)^{\frac{1}{2}}}. \tag{33}$$

In the following we shall replace $\varkappa(\varkappa+1)$ by $(l+\frac{1}{2})^2$ for both values of $\varkappa$, this procedure being appropriate for the W. K. B. solutions. Then

$$r_0 = (l+\tfrac{1}{2})/k. \tag{33a}$$

The contribution to the phase difference $\delta\eta_l^{(1)}$ is given by

$$\delta\eta_l^{(1)} = \frac{l+\frac{1}{2}}{E+2mc^2}e^2Z\left[\int_{r_1}^{\infty}\frac{dr}{r^3}q^{-\frac{1}{2}}(k_0) - \int_{r_0}^{\infty}\frac{dr}{r^3}q^{-\frac{1}{2}}(k_0=0)\right]_{b=\infty}. \tag{34}$$

It may be noted that r_1 differs very little from r_0:

$$r_1 \approx r_0(1 - k_0/k^2r_0) \approx r_0[1 - (Z/137l)k\hbar/mc]. \tag{34a}$$

The integration in (33) is carried out in the following manner. With the substitution $r = r_0\cosh x$ the integral in (33) becomes

$$\frac{1}{r_0^2}\int_0^{\infty}\operatorname{sech}^2 x(1+\lambda\cosh x)e^{-\lambda\cosh x}dx,$$

where $\lambda = r_0/b = (l+\frac{1}{2})/kb$. Integrating the first term by parts we find

$$\delta\eta_l^{(0)} = \frac{\lambda k}{l+\frac{1}{2}}\frac{e^2Z}{E+2mc^2}\int_0^{\infty}\cosh x\,e^{-\lambda\cosh x}dx. \tag{33b}$$

From Heine's integral representation of the Hankel function,[8] the integral in (33b) is equal to $-\pi/2H_1^{(1)}(i\lambda)$. Then we obtain

$$\delta\eta_l^{(0)}=-\frac{\pi}{2}\frac{e^2Z}{b}\frac{H_1^{(1)}(i\lambda)}{E+2mc^2}. \tag{35}$$

The integral (34) may be evaluated in terms of elementary functions. We find

$$\delta\eta_l^{(1)}=\frac{1}{(l+\frac{1}{2})^2}\left(\frac{e^2Z}{\hbar c}\right)^2\frac{E+mc^2}{E+2mc^2}\Theta \tag{36}$$

$$\tan\Theta=-(l+\tfrac{1}{2})k/k_0,\quad \pi/2<\Theta<\pi. \tag{36a}$$

From (35) and (36) we note that the phase difference $\delta\eta_l$ is small for all l. Since both $\delta\eta_l^{(0)}$ and $\delta\eta_l^{(1)}$ decrease monotonically with increasing l, the largest $\delta\eta_l$ will be $\delta\eta_1$. For energies of the order 100 kv and for any reasonable screening length b the argument of the Hankel function in (35) is small for small l.

$$\delta\eta_l^{(0)}=\frac{e^2Z}{\hbar c}\left(\frac{E}{E+2mc^2}\right)^{\frac{1}{2}}\frac{1}{l+\frac{1}{2}}.\quad \text{(small } l) \tag{37}$$

For $l=1$ and $E=100$ kv (37) and (36) give $\delta\eta_1^{(0)}=0.116$ and $\delta\eta_1^{(1)}=0.176$ so that $\delta\eta_1=0.29$. Therefore we may replace $\sin\delta\eta_l$ in (23) by $\delta\eta_l$ for all l. For very large l, $l\gg kb$ the phase difference behaves like $e^{-l/kb}$. For small l, $l\ll kb$ the summand in (23) is comparatively small because of the factor $l(l+1)/(2l+1)$. For the important values of l, $l\sim kb$ the summand varies very slowly and we may replace summation by integration.

For the quantity γ occurring in the polarization factor P (compare (18)) we find from (18a), (23), (35) and (36):

$$\gamma=4\sqrt{2}\pi N\left(\frac{e^2Z}{E+2mc^2}\right)^2 \times(\log 2kb-C-\tfrac{1}{2}+2\zeta S_1+\zeta^2S_2), \tag{38}$$

where C is Euler's constant $=0.577$ and

$$\zeta=\frac{e^2Z}{\hbar c}\frac{E+mc^2}{E^{\frac{1}{2}}(E+2mc^2)^{\frac{1}{2}}}, \tag{38a}$$

$$S_1=-\frac{\pi}{2}\sum_1^\infty\frac{\lambda H_1^{(1)}(i\lambda)}{(l+\frac{1}{2})^2}\Theta, \tag{38b}$$

$$S_2=\sum_1^\infty\Theta^2(l+\tfrac{1}{2})^{-3}. \tag{38c}$$

The sums S_1 and S_2 may be evaluated numerically. For b we use the Thomas-Fermi atomic radius

$$b=\hbar^2/me^2Z^{\frac{1}{3}}=1.23\times10^{-9}\text{ cm for Au.} \tag{39}$$

Then for $E=100$ kv we have $kb=21$, $\zeta=1.05$, $S_1=1.98$ and $S_2=1.76$. Then

$$\gamma=890\text{ cm}^{-1}. \tag{40}$$

For the thickest foil used by Dymond, $d=2.5\times10^{-5}$ cm and

$$\gamma d=2.2\times10^{-2}. \tag{41}$$

Since the main contribution to the sum S_1 (and to S_2 as well) arises from small[9] l where $H_1^{(1)}(i\lambda)\sim2/\pi\lambda$, it follows that S_1 and therefore γ is practically independent of the screening length b. In addition, for the energies of interest, γ is insensitive with energy and *in general γd will be of the order* 10^{-2}. For such small values of γd the degree of depolarization $1-P$ (Eq. (18)) is just γd. Hence multiple elastic scattering produces a negligible depolarization of the electron beam. To resolve the discrepancy between theory and experiment a degree of depolarization of about 95 percent or more is necessary, i.e., γd should be $\gtrsim 4$. It is very unlikely that the approximations made in the above derivation can introduce an error of more than a factor two or three, certainly not a factor 100.

The explanation of the small depolarization we have found lies in the fact that while several small angle collisions occur the depolarization in each collision is too small to give an appreciable total depolarization. (Compare further Appendix §1).

IV. INELASTIC SCATTERING

In this section we shall investigate the effect of inelastic scattering on the azimuthal asym-

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[8] Jahnke-Emde, *Tables of Functions*, p. 218.

[9] In the sum S_1 the first term contributes about half and the first five terms about 90 percent of the total value of the sum. In S_2 the contribution of the first term is 75 percent and of the first five terms about 99 percent of the total value of the sum.

metry of the doubly scattered electron beam. Inelastic scattering may reduce the asymmetry in two ways: (1) a simple reduction of the energy before the large deflections take place and (2) depolarization caused by change of spin of the incident electron in the inelastic collision.

The first effect would reduce the asymmetry since the large deflections would then occur at lower energies at which the polarization is indeed smaller (cf. reference 3). However, the rate of energy loss[10] in Au is two Mev per g/cm^2 and with Dymond's foils the total energy loss is only one kv. Therefore this reduction in the asymmetry is entirely negligible.

We consider now the inelastic scattering in which the incident electron changes its spin. If we denote the differential cross section for this process by $\Phi_n' \sin \vartheta d\vartheta$, the depolarization per collision is $2\Phi_n'/\Phi_n$ where Φ_n is the total scattering cross section for excitation of the nth atomic state. In a manner similar to that used in the derivation of (18) we have for the degree of depolarization

$$1-P=1-\left(\frac{1-e^{-\gamma d}}{\gamma d}\right)^2, \tag{42}$$

where now

$$\gamma=4\sqrt{2}\pi N\sum_n\int_0^\pi \Phi_n'(\vartheta)\sin\vartheta d\vartheta. \tag{42a}$$

The summation in (42a) is to be taken over all excitations.

To calculate the differential cross section we may use the Born approximation. That this procedure will yield a sufficiently good result may be concluded from the fact that the values of the angular momentum of the incident electron for which the inelastic scattering is important are larger than the corresponding momenta for elastic scattering. Since larger angular momentum implies smaller phases of the scattered partial waves, it follows that the Born approximation is better for the inelastic scattering than for the elastic scattering—and for the latter the Born approximation is quite accurate even for the values of atomic number and energy with which we are concerned (see Appendix §2).

The differential cross section (per solid angle) in Born approximation is given by[11]

$$\Phi_n'=\frac{4e^4}{[(\mathbf{k}-\mathbf{k}')^2+(E-E'/\hbar c)^2]^2}\times\left(\frac{E+mc^2}{\hbar^2c^2}\right)^2|M_n|^2, \tag{43}$$

$$M_n=\int d\tau u_n^*\sum_j \exp(i\mathbf{q}\cdot\mathbf{r}_j)(a_0+\mathbf{a}\cdot\boldsymbol{\alpha}_j)u_m, \tag{43a}$$

$$a_0=s(\mathbf{k})s^*(\mathbf{k}'),\quad \mathbf{a}=-s^*(\mathbf{k}')\boldsymbol{\alpha}s(\mathbf{k}) \tag{43b}$$

and $$\mathbf{q}=\mathbf{k}-\mathbf{k}',\quad q=2k\sin\vartheta/2. \tag{43c}$$

Here E, $\mathbf{k}$ and E', $\mathbf{k}'$ are the energies and wave vectors before and after the scattering; $s(\mathbf{k})$ and $s(\mathbf{k}')$ are the amplitudes of the Dirac plane waves for positive and negative spin, respectively; u_m and u_n are atomic wave functions for the initial and final state, respectively, the components being Schrödinger-like wave functions since very little energy is given to the atom; the Dirac operator $\boldsymbol{\alpha}_j$ acts on the spin coordinate of the jth electron. In (43) and (43c) we have used the fact that the energy loss is small: $k\approx k'$, $E\approx E'$.

In §4 of the appendix it will be shown that the second term of (44a) is much smaller than the first term over practically the entire range of integration over ϑ. Then the retardation term in the denominator of (43) may be neglected and the matrix element (43b) becomes

$$M_n=a_0F_n, \tag{44}$$

where F_n is the "generalized form factor"

$$F_n=\int d\tau u_n^*\sum_j \exp(i\mathbf{q}\cdot\mathbf{r}_j)u_m. \tag{44a}$$

With $$a_0=\frac{1}{2}\frac{E}{E+mc^2}\sin\vartheta e^{i\varphi} \tag{45}$$

the differential cross section (43) becomes

$$\Phi_n'=\frac{1}{4}\left(\frac{e^2}{E+2mc^2}\right)^2\cot^2(\vartheta/2)|F_n|^2. \tag{46}$$

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[10] See e.g., H. A. Bethe, reference 7, p. 522.

[11] H. A. Bethe, reference 7, p. 495.

Summing over all excitations we have, using the Hartree model[12]

$$\sum_{n\neq m} |F_n|^2 = \sum_m (1-|F_m|^2), \tag{47}$$

where F_m is the atomic form factor for the mth electronic state

$$F_m = \int d\tau \exp(i\mathbf{q}\cdot\mathbf{r})|\psi_m(\mathbf{r})|^2, \tag{47a}$$

wherein ψ_m is the wave function of a single atomic electron in the mth state. The integration in (47a) is now to be taken over the coordinates of a single electron. The summation on the right-hand side of (47) is to be taken over all occupied states.

For large q the atomic form factor F_m is negligible due to interference. We may set

$$F_m = 0 \qquad \text{for} \qquad q > 1/b_m, \tag{48}$$

where b_m is the radius of the mth electron shell. For smaller q we may expand the exponential in (47a) and obtain

$$F_m = 1 - \tfrac{1}{2}q^2 \int d\tau\, r_q^2 |\psi_m|^2, \quad q < 1/b_m \tag{48a}$$

where r_q is the component of $\mathbf{r}$ in the direction of $\mathbf{q}$. The term linear in r_q vanishes from symmetry. The matrix element of r_q^2 may be set equal to b_m^2. Thus we have

$$1-|F_m|^2 = \begin{cases} q^2 b_m^2 & q < 1/b_m \\ 1 & q > 1/b_m \end{cases}. \tag{49}$$

Actually $1-|F_m|^2$ is less than the smaller of the two values on the right-hand side of (49).

For the evaluation of the total cross section for excitation of the atomic electron in initial state m we have

$$\int_0^\pi \Phi_n' \sin\vartheta d\vartheta = \tfrac{1}{4}\left(\frac{e^2}{E+2mc^2}\right)^2 \sum_m (I_0+I_1), \tag{50}$$

where

$$I_0 = b_m^2 \int_0^{\vartheta_m} \cot^2(\vartheta/2) q^2 \sin\vartheta d\vartheta \tag{50a}$$

is the contribution to the scattering from small angles and

$$\sin\vartheta_m/2 = 1/(2kb_m). \tag{50b}$$

The contribution from the large angle scattering is

$$I_1 = \int_{\vartheta_m}^{\pi} \cot^2\vartheta/2 \sin\vartheta d\vartheta. \tag{50c}$$

Carrying out the integrations in (50a) and (50c) we find

$$I_0 + I_1 = 4 \log 2kb_m. \tag{50d}$$

In carrying out the summation over m occurring in (50) we obtain the geometric mean of the shell radii b_m which we may replace for order of magnitude purposes by b, the screening length introduced in Section III (cf. Eq. (39)). That is,

$$\sum_m (I_0+I_1) = 4Z \log 2kb. \tag{50e}$$

For the quantity γ we obtain from (42a)

$$\gamma = 4\sqrt{2}\pi NZ(e^2/E+2mc^2)^2 \log 2kb. \tag{51}$$

For $E = 100$ kv and the foil thickness $d = 2.5 \times 10^{-5}$ cm we find

$$\gamma = 5.0 \text{ cm}^{-1}, \quad \gamma d = 1.3\times 10^{-4}. \tag{52}$$

We see then that the depolarization by inelastic collisions is negligible. We observe that the depolarization caused by inelastic scattering is smaller than that caused by elastic scattering by about a factor $1/Z$ (cf. Eqs. (38) and (51)). This can be understood from the fact that the small angle scattering gives a small contribution (cf. Eqs. (50a), (50c)) to the depolarization. At large scattering angles, it is well known that the total elastic scattering is about Z times larger than the inelastic, and the depolarization in *one* scattering process is the same for elastic and inelastic collisions at a given scattering angle, (see Appendix, Eqs. (107), (107a)).

V. Exchange Scattering

A change in the polarization of an electron beam can take place as a result of exchange scattering only if the exchanged electrons have opposite spins. Since the spin forces are weak for

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[12] The transitions which are forbidden by the exclusion principle (state n initially occupied) have not been subtracted. Therefore the cross section is *overestimated.*

the atomic electrons the spin of each electron will not change in the capture and emission processes. This also has the important consequence, that such exchange processes as we are considering can take place only with atomic electrons in incomplete shells. Exchange between the incident electron and an atomic electron of opposite spin in a closed shell will result in two atomic electrons having the same set of quantum numbers and this is forbidden by the Pauli exclusion principle. Thus in Au only the 6*s* valence electron can participate in the type of exchange scattering which leads to depolarization.

In contrast to the case of depolarization by elastic and inelastic scattering, the depolarization *per collision* due to exchange scattering will not be small but the *net depolarization* will be extremely small because very few exchange collisions of the type considered will take place. This is due essentially to the strong interference between the wave functions for the valence state and the continuum.

To obtain a quantitative estimate of the importance of exchange scattering we make the following (necessarily) rough calculation of the exchange cross section. The differential cross section is given by[13]

$$\Phi_{\mathrm{ex}}(\vartheta)\sin\vartheta d\vartheta=(8\pi^2 a_0^2)^{-1}|M_{\mathrm{ex}}|^2\sin\vartheta d\vartheta, \qquad (53)$$

where $a_0=\hbar^2/me^2$ is the Bohr radius. A factor $\frac{1}{2}$ has been inserted in (53) because the spin of the incident electron must be opposed to that of the valence electron and this is true in half the cases.

The matrix element for exchange is essentially given by

$$M_{\mathrm{ex}}=\int d\tau_r d\tau_Z \psi_Z^*(\mathbf{r})\chi_{\mathbf{k}'}^*(\mathbf{r}_Z) \times|\mathbf{r}-\mathbf{r}_Z|^{-1}\psi_Z(\mathbf{r}_Z)\chi_{\mathbf{k}}(\mathbf{r}), \qquad (54)$$

where $\mathbf{r}$, $\mathbf{r}_Z$ denote the coordinates of the electrons initially in the continuum and valence state, respectively. ψ_Z is the wave function for a single electron in the valence state and moving in the field of the rest of the atom. $\chi_{\mathbf{k}}$ and $\chi_{\mathbf{k}'}$ are the wave functions of the incident and outgoing electrons, respectively. We take

$$\chi_{\mathbf{k}}(\mathbf{r})=\exp(i\mathbf{k}\cdot\mathbf{r}), \quad \chi_{\mathbf{k}'}(\mathbf{r}_Z)=\exp(i\mathbf{k}'\cdot\mathbf{r}_Z), \qquad (55)$$

$\mathbf{k}$ and $\mathbf{k}'$ representing the initial and final wave vectors.

Because of the rapid oscillation of χ the contribution to the matrix element will be small in the region where ψ_Z is slowly varying, but will be appreciable near the nucleus where ψ_Z also varies rapidly. We therefore represent the valence state wave function by

$$\psi_Z(r)=\psi_Z(0)e^{-\beta r}, \quad \beta=Z/a_0. \qquad (56)$$

With the wave functions (55) and (56) the matrix element (54) may be evaluated to give approximately

$$M_{\mathrm{ex}}\approx\frac{32\pi^2}{k^2}|\psi_Z(0)|^2\frac{\beta}{(\beta^2+q^2)^2}, \qquad (57)$$

where $\mathbf{q}=\mathbf{k}-\mathbf{k}'$ (cf. Eq. (43c)).

The value of the 6*s* wave function ψ_Z at the nucleus may be obtained from considerations of the type used in the theory of hyperfine structure.[14] For *s* electrons we find

$$\psi_Z^2(0)=\frac{Z_i(1+z)^2\kappa(\frac{1}{2},Z_i)}{\pi a_0^3 n^{*3}}=\frac{\beta^3\kappa}{\pi Z^2 n^{*3}}, \qquad (58)$$

where z is the degree of ionization (here $=0$), Z_i the effective nuclear charge (here $=Z$), n^* is the effective quantum number and $\kappa(\frac{1}{2},Z)$ is a relativistic correction factor given in Eq. (246) of reference 14. For the 6*s* state of Au, $n^*=1.21$ and $\kappa=2.0$.

From (53), (57) and (58) we find for the differential cross section

$$\Phi_{\mathrm{ex}}(\vartheta)=128\frac{\kappa^2}{Z^4 n^{*6}}\left(\frac{R_y}{E}\right)^2\frac{a_0^2}{(1+\xi^2)^4}, \qquad (59)$$

where $\xi=(2ka_0/Z)\sin\vartheta/2$ and $R_y=e^4m/2\hbar^2$ is the Rydberg energy.

From (59) we obtain for the total cross section

$$\Phi_{\mathrm{ex}}^{\mathrm{tot}}=\frac{32\pi}{3}\frac{\kappa^2}{n^{*6}}\left(\frac{R_y}{E}\right)^3\left(\frac{a_0}{Z}\right)^2 \times[1-(1+4k^2/\beta^2)^{-3}]. \qquad (60)$$

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[13] See e.g., H. A. Bethe, Ann. d. Physik 5, 325 (1929).

[14] See e.g., H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. 8, 82 (1936), Chapter VIII.

For energies of order 100 kv, $k/\beta \approx 1$ and the square bracket in (60) is practically equal to one.

For the degree of depolarization we have (compare (42a))

$$\gamma d = 2\sqrt{2} N d \Phi_{\mathrm{ex}}{}^{\mathrm{tot}}$$

$$= \frac{64\sqrt{2}\pi}{3} \frac{\kappa^2}{n^{*6}} \left(\frac{R_y}{E}\right)^3 \left(\frac{a_0}{Z}\right)^2 Nd. \quad (61)$$

For $E = 100$ kv, $d = 2.5 \times 10^{-5}$ cm we find $\gamma d = 2 \times 10^{-12}$, an entirely negligible depolarization. This result is due to the rarity of exchange collisions. Since most of the collisions are elastic we see by comparison of (60) and the total cross section for elastic collisions (Appendix, Eq. (112)) that the fraction of collisions which are of the exchange type is

$$\frac{\Phi_{\mathrm{ex}}{}^{\mathrm{tot}}}{\Phi_{\mathrm{el}}{}^{\mathrm{tot}}} = \frac{8}{3} \frac{\kappa^2}{n^{*6}} \left(\frac{R_y}{E}\right)^3 \left(\frac{\hbar v}{e^2 Z}\right)^2 Z^{-4/3}. \quad (62)$$

Thus for $E = 100$ kv only one in 4×10^{13} collisions is of the exchange type.

APPENDIX

§1. Depolarization per collision at small angles

In order to see that in each elastic collision only a small depolarization occurs, it is advantageous to consider the angular dependence of the depolarization. The depolarization in a deflection at angle ϑ was given in Eq. (15) as

$$-\delta p/p = 2|g|^2/(|f|^2 + |g|^2). \quad (101)$$

The function g is given by (22). The Faxen-Holtzmark expression for f is

$$f(\vartheta) = i/2k \sum_0^\infty P_l(\cos\vartheta)$$

$$\times [(l+1)(e^{2i\eta_l} - 1) + l(e^{2i\eta_{-l-1}} - 1)]. \quad (102)$$

From (21) and (28) the phase $\eta_\varkappa$ is given by

$$\eta_\varkappa = \lim_{r\to\infty} \left(\int_{r_1}^{r} Q^{\frac{1}{2}} dr - \int_{r_0}^{r} Q_0{}^{\frac{1}{2}} dr \right), \quad (103)$$

where the subscript 0 refers to $Z = 0$. We can expand the integrand in terms of the most important potential term and find for both values of $\varkappa$ (compare (30))[15]

$$\eta_\varkappa = k_0 \int_{r_0}^{\infty} \frac{dr}{r} e^{-r/b} Q_0{}^{\frac{1}{2}} = \frac{e^2 Z}{\hbar v} \frac{\pi i}{2} H_0{}^{(1)}(i\lambda). \quad (103a)$$

For the sake of simplicity we first eliminate η_{-l-1}. We can see very easily that the phase difference $\eta_l - \eta_{-l-1} = \delta\eta_l$ is small compared to η_l for all l. The ratio $\delta\eta_l/\eta_l$ is largest for $l = 1$. From (103a) and ((37) *et seq*) we find for the usual values of the constants, $\delta\eta_1/\eta_1 = 0.10$. Therefore in (102) and (22) we expand $e^{2i\delta\eta_l}$ and f and g become

$$f = (i/2k) \sum_0^\infty (2l+1) P_l(\cos\vartheta)(e^{2i\eta_l} - 1), \quad (104)$$

$$g = -(1/k) \sum_0^\infty P_l{}^1(\cos\vartheta)\delta\eta_l e^{2i\eta_l}. \quad (104a)$$

Now since most of the collisions take place for small scattering angles we are interested in the values of f and g for ϑ small. Making the appropriate expansions we have from (22a)

$$f = (i/2k) \sum_0^\infty (2l+1)(e^{2i\eta_l} - 1), \quad (105)$$

$$g = (\vartheta/2k) \sum_0^\infty l(l+1)\delta\eta_l e^{2i\eta_l}. \quad (105a)$$

Since η_l and $\delta\eta_l$ behave like $e^{-l/kb}$ for large l, the series (105a) converge and we find from (101) that at small scattering angles the depolarization per collision is small of order ϑ^2. This result is, therefore, true for any scattering field for which the scattered intensity is finite for small angles. For scattering in a pure Coulomb field, the series (105a) will, of course, diverge. In this case, for small ϑ,[16]

$$|f| = (2e^2 Z/\hbar v)(1/\vartheta^2 k), \quad (106)$$

$$|g| = (e^2 Z/\hbar v)(E/E + mc^2)(1/\vartheta k), \quad (106a)$$

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[15] The error incurred by this expansion is small for the important values of $l(\sim kb)$. The error is greatest for $l = 1$; with $E = 100$ kv the phase η_1 from (103*a*) is 2.62 whereas a numerical integration of (103) gives the exact value $\eta_1 = 2.87$. The difference between the exact value of $\eta_\varkappa$ and the value given by (103*a*) decreases rapidly with increasing l.

[16] Reference 1, Eq. (6.8).

so that the depolarization per collision for small deflections is of order ϑ^2 for all scattering fields.

For the numerical value of the depolarization in one elastic collision we find for 100 kv (see Eqs. (109) and (109a) below)

$$2|g/f|^2=0.013\vartheta^2. \tag{107}$$

For inelastic collisions we have, of course, the same result. The depolarization per inelastic collision is from Section IV (cf. (45) and remarks preceding Eq. (42)),

$$2\Phi_n'/\Phi_n=2|a_0|^2=\tfrac{1}{2}(E/E+mc^2)^2\sin^2\vartheta \approx 0.013\vartheta^2. \tag{107a}$$

The small depolarization found in the foregoing may now be understood as follows. The depolarization depends on the product of the scattering cross section and the depolarization per scattering (101). The former is large only for small ϑ and just in this region the latter is small.

§2. Validity of the Born approximation

In the foregoing the statement has been made that the Born approximation is accurate for the elastic scattering even though the Born expansion parameter $e^2Z/\hbar v\approx 1$ for the energies and atomic number in which we are interested. We shall now show that this is so not only for the scattered intensity $|f|^2+|g|^2$ but also for the depolarization $\sim|g|^2$. For large scattering angles the Born approximation is known to give a result for the scattered intensity which is in agreement with the exact formula of Mott (nuclear scattering).

In the Born approximation the phases η_l and $\delta\eta_l$ are assumed small and for f and g we have from (105) and (105a)

$$f_B=-(2/k)\sum_0^\infty(l+\tfrac{1}{2})\eta_l, \tag{108}$$

$$g_B\cong(\vartheta/2k)\sum_0^\infty(l+\tfrac{1}{2})^2\delta\eta_l. \tag{108a}$$

Replacing sums by integrals and using (103a) and (35) we find[17]

$$\begin{aligned} f_B&=-\pi ikb^2(e^2Z/\hbar v)\int_{1/kb}^\infty \lambda H_0^{(1)}(i\lambda)d\lambda \\ &=-\pi(e^2Z/\hbar v)bH_1^{(1)}(i/kb) \\ &\approx 2(e^2Z/\hbar v)kb^2, \end{aligned} \tag{109}$$

since $kb\gg 1$;

$$\begin{aligned} g_B&=-\frac{\pi}{4}kb^2\frac{e^2Z}{\hbar c}\left(\frac{E}{E+2mc^2}\right)^{\frac{1}{2}}\vartheta\int_{1/kb}^\infty \lambda^2H_1^{(1)}(i\lambda)d\lambda \\ &=-\frac{\pi i}{4}\frac{e^2Z}{\hbar c}\left(\frac{E}{E+2mc^2}\right)^{\frac{1}{2}}\frac{\vartheta}{k}H_2^{(1)}(i/kb) \\ &\approx(e^2Z/\hbar c)(E/E+2mc^2)^{\frac{1}{2}}kb^2\vartheta. \end{aligned} \tag{109a}$$

Of course g_B and f_B depend on the screening length b, but their ratio is essentially independent of the screening and in agreement with (106) and (106a) as it should be, cf. Sauter.[18]

In order to test the Born approximation for our problem, the exact value of the sums occurring in (104a) were found numerically.[19] For $kb=100$ and $e^2Z/\hbar v=1$ corresponding to E about 100 kv we find

$$|f|=1.81\times 10^4/k,\quad |g|=0.167\times 10^4\vartheta/k. \tag{110}$$

The Born approximation yields

$$f_B=2.0\times 10^4/k,\quad g_B=0.182\times 10^4\vartheta/k. \tag{110a}$$

The use of the Born approximation therefore overestimates f by only ten percent and g by 9 percent. For the scattering the error is 21 percent and for the depolarization 18 percent. The screening radius b has been taken rather large in the above example. For smaller screening lengths the phases for a given l are smaller and the Born approximation better.

This result is perhaps not surprising insofar as it is well known that the condition for the validity of the Born approximation $e^2Z/\hbar v\ll 1$ is

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[17] It is obviously consequent in the first Born approximation to use for the phase and phase difference the limiting forms for small scattering potential, hence $\delta\eta_l=\delta\eta_l^{(0)}$.

[18] F. Sauter, Ann. d. Physik **18**, 61 (1933).

[19] For the calculation of these sums the part $\delta\eta_l^{(1)}$ of the phase difference may be neglected since this part of $\delta\eta_l$ contributes to g the amount

$$g_1\sim\frac{1}{2}\left(\frac{e^2Z}{\hbar c}\right)^2\frac{E+mc^2}{E+2mc^2}\frac{\vartheta}{k}\sum_1^{kb}\Theta e^{2i\eta_l}.$$

The absolute value of the sum is of order or less than πkb and for the numerical case considered in (110) $g_1\lesssim 30\vartheta/k$ as compared to $g\sim 1700\vartheta/k$ from $\delta\eta_l^{(0)}$.

a sufficient but not a necessary condition. In the problem considered here it is possible to see qualitatively the reason for the fair agreement between (110) and (110a). In (104) and (104a) the main contribution to each of the sums arises from those values of l for which the phase η_l is $\pi/2$ or smaller and in this region the assumption of small phases is sufficiently well fulfilled.

It may be pointed out that the Born approximation is not accurate for the calculation of the polarization of the electrons, but only for the depolarization (or scattered intensity). The polarization as was mentioned in the foregoing, depends on the relative phase of f and g whereas the depolarization depends on the value of g itself. The relative phase of f and g is in fact zero in the first Born approximation[20] (in contrast to the value of g itself) and at least the second approximation is necessary for the calculation of the asymmetry.

§3. Total number of collisions

a. Elastic collisions.—The average number of elastic collisions taking place in a foil of thickness d is

$$n_e = 2\sqrt{2}\pi N d \int_0^\pi \Phi_{\mathrm{el}}(\vartheta) \sin\vartheta d\vartheta, \qquad (111)$$

where a factor $\sqrt{2}$ has been inserted because of the obliquity of the path. With $\Phi \cong |f|^2$ and f given by (104) we have in the Born approximation

$$\Phi_{\mathrm{el}}{}^{\mathrm{tot}} = 2\pi \int_0^\pi \Phi_{\mathrm{el}}(\vartheta) \sin\vartheta d\vartheta$$
$$= 4\pi(e^2Z/\hbar v)^2 b^2. \qquad (112)$$

With $d = 2.5 \times 10^{-5}$ cm and $b = \hbar^2/me^2Z^{\frac{1}{3}}$ we find that on the average 40 elastic collisions take place in each foil.

b. Inelastic collisions.—The total number of inelastic collisions in each foil is[13]

$$n_i = 2\sqrt{2}\pi N d \frac{e^4}{mv^2} \sum_n \frac{\nu_n}{A_n} \log \frac{2mv^2}{A_n}, \qquad (113)$$

where A_n is an energy of the order of magnitude of the ionization potential and ν_n the number of electrons in the nth shell. For Au we may take

$A_n =$ 6 volts for the 6s shell
$=$ 20 " " " 5d "
$=$100 " " " 5s, 5p and 4f shells.

The contributions to the sum in (113) from the 6s, 5d and the 5s, 5p and 4f shells are 1.85, 4.95 and 1.7 reciprocal volts, respectively. The contribution from the remaining shells will be about 0.5. The total number of inelastic collisions is then $n_i \sim 16$. This number may be in error by a factor 3 or so in either direction but in any case it is indicative of the fact that several collisions can take place in the foils under consideration.

§4. Relative magnitude of transition "current" and "density" terms

In Section IV it was stated that the current term in (43a), *viz:*

$$\mathbf{a} \cdot \mathbf{J} = \mathbf{a} \cdot \int d\tau u_n{}^* \sum_j \exp(i\mathbf{q} \cdot \mathbf{r}_j) \boldsymbol{\alpha}_j u_m \qquad (114)$$

makes a contribution to the inelastic scattering which is much smaller than that due to the density term a_0F_n (cf. (44), (44a)). Since the relative magnitude of the current term is largest for small scattering angles we may expand the exponential and obtain

$$F_n = i\mathbf{q} \cdot \mathbf{r}_n, \qquad (115)$$

where

$$\mathbf{r}_n = \int \sum_j \mathbf{r}_j u_n{}^* u_m d\tau \qquad (115\mathrm{a})$$

is the dipole moment corresponding to the transition $u_m \rightarrow u_n$. For the current term we have, with $\boldsymbol{\alpha}_j = -\dot{\mathbf{r}}_j/c$

$$\mathbf{J} = i(\Delta E/\hbar c)\mathbf{r}_n, \qquad (116)$$

where ΔE is the energy loss. Therefore the ratio of the current to density term is

$$\frac{\mathbf{a} \cdot \mathbf{J}}{a_0 F_n} = \frac{\Delta E}{\hbar c a_0} \frac{\mathbf{a} \cdot \mathbf{r}_n}{\mathbf{q} \cdot \mathbf{r}_n} = \frac{\Delta E}{\hbar c a_0} \frac{|\mathbf{a}|}{q}, \qquad (117)$$

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[20] N. F. Mott, Proc. Roy. Soc. **124**, 425 (1929). Also F. Sauter, reference 18.

since the direction of $\mathbf{r}_n$ is not significant. From (43b) we find

$$|\mathbf{a}| = \hbar c[q^2-(\Delta E/\hbar c)^2]^{\frac{1}{2}}/2(E+mc^2). \quad (118)$$

With a_0 given by (45) we find

$$\frac{\mathbf{a}\cdot\mathbf{J}}{a_0F_n} < \rho = \frac{\Delta E}{E\sin\vartheta}. \quad (117a)$$

The right-hand side of (117a) is appreciable only for

$$\vartheta \lesssim \Delta E/E \sim \vartheta_m{}^2, \quad (118)$$

where ϑ_m is the angle introduced in (50b). In the average $\vartheta_m \sim 1/kb$ so that $\rho \sim 1$ for

$$\vartheta \sim 1/400 = 0.14°.$$

For scattering angles $\vartheta > \vartheta_m{}^2$, $\mathbf{a}\cdot\mathbf{J} \ll a_0F_n$. Therefore over practically the entire range of integration over ϑ the current term need not be considered at all.

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The Interaction Between Rotation and Oscillation in Deutero-Formaldehyde

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The unsymmetrical broadening of the Q branch toward higher frequencies in the parallel type bands near 1100 cm^{-1} in the absorption spectrum of deutero-formaldehyde is taken as a manifestation of convergence toward lower frequencies rather than, as is usual, in the case of infra-red bands, toward higher frequencies. This phenomenon is accounted for by taking into account an interaction between rotation and oscillation of the Coriolis type arising from the accidental degeneracy of the three vibration frequencies, belonging to three different symmetry classes, falling near 900 cm^{-1}, 1000 cm^{-1} and 1100 cm^{-1}.

I. INTRODUCTION

THE absorption spectrum near 10μ in the deutero-formaldehyde spectrum consists of three bands which badly overlap. Two of these are characteristic of oscillations of the electric moment normal to the axis of symmetry of the molecule, while the third is characteristic of a vibration parallel to the axis of symmetry. The first two of these consist essentially of a group of prominent lines protruding above a background which at present cannot be resolved spectroscopically, and the third contains P, Q and R branches. It has already been pointed out elsewhere[1] that the spacings between the prominent lines in the perpendicular bands in this region are noticeably greater than the spacings between the lines in a similar band near 4.6μ and it has been suggested that the explanation for this is undoubtedly the same as that for the similar effect observed in two overlapping perpendicular bands in the spectrum of ordinary formaldehyde near 8.0μ.[2] The effect is here quite satisfactorily explained by taking account in the energy of a Coriolis interaction between the two frequencies. In the third of the above bands referred to in the spectrum of deutero-formaldehyde the convergence is in the direction of lower frequencies

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[1] E. S. Ebers and H. H. Nielsen, J. Chem. Phys. **6**, 311 (1938).

[2] H. H. Nielsen, J. Chem. Phys. **5**, 818 (1937). The type of interaction between rotation and oscillation discussed recently by W. H. J. Childs and H. A. Jahn (Nature **141**, 916 (1938)) in connection with the methane spectrum is entirely similar to the one discussed in this work relative to H_2CO. The formaldehyde case is especially interesting in that both perturbing frequencies are optically active so that the convergence in opposite directions in the two bands, described by C. and J., can here actually be observed. In the case of the methane molecule the perturbing frequencies are two- and three-fold degenerate, respectively, while in this case they are nondegenerate. The importance of such terms in the interpretation of spectra was already emphasized by E. B. Wilson, Jr., in one of his early papers on this subject.