The Calculation of Internal Conversion Coefficients

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We treat the internal conversion of nuclear γ-rays by methods suited especially to the cases of the artificially radioactive elements. The paper is divided into five parts: Section 1 discusses the radiative transitions in nuclei of middle atomic weights, where one may expect large angular momentum changes and low energy of excitation; in Section 2 we consider the selection rules for the various types of nuclear radiation, and the conditions on their strictness; in Section 3 we examine the choice of potentials for the representation of the multipole fields; Section 4 contains calculations leading to a nonrelativistic formula for the conversion of electric multipole radiation of any order, applicable for $Z<40$ and γ-ray energy $\ll mc^2$; and very simple relativistic formulae are derived, good for high γ-ray energy but restricted to low binding, $Z<30$, for both magnetic and electric multipoles of any order. We summarize the qualitative behavior of the formulae in a conclusion.

1. Introduction

The internal conversion of γ-rays was first observed as the superposition of homogeneous groups of emitted electrons on the continuous γ-ray spectrum of certain naturally radioactive elements. The intensities of these lines have been calculated[1] by methods appropriate to these cases: i.e., by using relativistic electron theory necessary for these elements where $Z\sim 80$ and the velocity of the electron in the bound state is large, and for radiation of a few low multipole orders. The integrals involved had to be carried out numerically.

The artificial production of γ-radioactive nuclei of low and middle Z made the extension of the calculations to this range seem desirable. Thus the activity found by Alvarez[2] in Ga^{67} is associated with a line of emitted electrons, presumably conversion electrons of a Zn γ-ray which is emitted after the Ga nucleus captures a K electron to go to Zn^{67}. For the detailed interpretation of such reactions it is helpful to have reliable estimates of the conversion coefficient.

Hebb and Uhlenbeck[3] were led to calculate the internal conversion coefficients for the first five electric multipoles on different grounds. The ideas of von Weizsäcker[4] about nuclear isomers, which ascribe the long lifetime of the metastable nuclear state to a difference of several units of angular momentum between metastable and ground states, and in fact the general ideas of Bohr on nuclear structure, indicated that high multipole orders should frequently be involved in nuclear radiation between states of low excitation. Their calculations were made to ascertain the correction to the estimate of lifetime on the basis of radiative decay alone, when the possibility of decay by electron emission is taken into account.

The formulae of Uhlenbeck and Hebb are completely nonrelativistic, applicable in the range $Z<40$ and for γ-ray energies $\ll mc^2$. We extend the formulae of Uhlenbeck and Hebb to all multipole orders, and obtain simple relativistic results for light nuclei, for magnetic multipoles, and for high multipole order.

2. Selection Rules for Multipole Radiation

The interaction between the electron and the nuclear charge and current we represent as the perturbation on the electron motion by the fields of an oscillating electric or magnetic multipole placed at the nucleus. This treatment is based on a simple argument: the classical multipole fields define normal coordinates for Maxwell's equations by an expansion of the fields in spherical harmonics, and the field of an electric or magnetic 2^L pole when quantized represents a light quantum with angular momentum $\mathbf{L}\hbar$.[5] Since the

[1] For results and extensive references see H. R. Hulme, N. F. Mott, F. Oppenheimer and H. M. Taylor, Proc. Roy. Soc. **A155**, 315 (1936).
[2] L. W. Alvarez, Phys. Rev. **54**, 486 (1938).
[3] M. H. Hebb and G. E. Uhlenbeck, Physica **5**, 605 (1938).
[4] C. F. v. Weizsäcker, Naturwiss. **24**, 813 (1936).
[5] W. Heitler, Proc. Camb. Phil. Soc. **32**, 112 (1936).

total angular momentum of the system *nucleus* $+\gamma$-*ray* must be conserved, we expect a nuclear transition between states of total angular momentum $\mathbf{J}$ and $\mathbf{J}'$ to emit a quantum represented by a 2^L pole such that $\mathbf{L}=\mathbf{J}-\mathbf{J}'$.

For convenience we summarize the arguments leading to the selection rules for multipole radiation.[6] In the gauge in which

$$\varphi(\mathbf{R})=\int dv\frac{[\rho]_{\text{retarded}}}{|\mathbf{R}-\mathbf{r}|}$$

we can write for the scalar potential at a point distant $\mathbf{R}(R, \Omega)$ from the center of a system of particles with charge e_i and position $\mathbf{r}_i(r_i, \omega_i)$, involved in a transition $\Psi_{J'}\rightarrow\Psi_J$:

$$\varphi(\mathbf{R}, t)=\exp\left[\frac{i(E_{J'}-E_J)t}{\hbar}\right]\cdot\int dv\bar{\Psi}_J(\mathbf{r}_i)\times\Psi_{J'}(\mathbf{r}_i)\sum_i e_i\frac{\exp[i\mathbf{k}\cdot(\mathbf{R}-\mathbf{r}_i)]}{|\mathbf{R}-\mathbf{r}_i|}$$

Suppose that there is no contribution to this integral from any $r_i>\delta$ (δ corresponding to the radius of the nucleus). Then we can make a well-known expansion of the exponential[7] to get

$$\varphi(\mathbf{R}, t)=f(R, t)\sum_{\lambda=0}^{\infty}\sum_{m=-\lambda}^{\lambda}C_\lambda f_\lambda(kR)Y_\lambda{}^m(\Omega)\cdot\sum_{n=0}^{\infty}a_n(k\delta)^{\lambda+2n}\int dv\bar{\Psi}_J\Psi_{J'}\left[\sum_i e_iY_\lambda(\omega_i)\left(\frac{r_i}{\delta}\right)^{2n+\lambda}\right].$$

If $|J-J'|=L$, then in the reduction of $\bar{\Psi}_J\Psi_{J'}$ the lowest spherical harmonic occurring is Y_L, hence the lowest value of λ contributing is $\lambda=L$. Values of λ up to $J+J'$ also contribute, but their contributions are reduced by a factor approximately $(k\delta)^{\lambda-L}$ and this is certainly small for most nuclear γ-rays (for a one-Mev γ-ray, $k\delta<1/15$). This gives a leading term in the scalar potential $\sim Y_L$, just that of a classical electric 2^L pole in this gauge. To find the remaining terms in the scalar potential of such a multipole, we consider all $\lambda>L$ and collect, successively, terms in each power of $k\delta$ that occurs.

[6] Compare, e.g., H. A. Kramers, *Hand- und Jahrbuch der Chemischen Physik*, Vol. I/II, p. 419.

[7] H. Bateman, *Partial Differential Equations*, p. 388.

Going now to the vector potential, the transformation properties of the current vector enter. The current $\rho\mathbf{v}$ transforms like Y_1. For $|J-J'|=L$, one can now obtain three values of λ corresponding to the lowest power of $(k\delta)$, namely L, $L+1$ and $L-1$. In this gauge the magnetic multipoles have zero scalar potential, and correspond to the single term Y_L in the vector potential. The electric multipoles come from Y_{L+1} and Y_{L-1}.

Like the total angular momentum, the parity of the nuclear states is a constant of the motion; from the reflection properties of the vector potential, we can see that the electric 2^L poles have parity $(-)^L$, the magnetic $(-)^{L+1}$. Thus to a given parity change and ΔJ there corresponds a minimum electric and minimum magnetic multipole as shown in Table I. The higher multipoles, up to $L=J+J'$, permitted by the parity could contribute, but their amplitudes contain the small factor $(k\delta)^{L-L_{\min}}$.

Unless there are some further special symmetries of the nucleus, the lowest multipole allowed will be the most important. We know[8] that the dipole moment of all but the lightest nuclei is reduced by the approximate coincidence of the centers of mass and charge, so that electric dipole and electric quadripole radiation can be comparable, but all other electric multipoles smaller. The fair agreement with experiment[1] in the natural radioactive series supports this. For the magnetic moments, which involve not only the charge and motion, but also the spin properties of the nuclear matter, no analogous reduction of the dipole moment is to be expected. On the other hand, we have the same experiments of Ellis *et al.* to show that, while magnetic dipole and electric quadripole are both allowed

TABLE I. *Minimum allowed multipoles for given parity change and* ΔJ $(L=J-J')$.

Parity Change	Minimum Allowed Multipole L Even	L Odd
Even	el. 2^L pole mag. 2^{L+1} pole	el. 2^{L+1} pole mag. 2^L pole
Odd	el. 2^{L+1} pole mag. 2^L pole	el. 2^L pole mag. 2^{L+1} pole

[8] H. A. Bethe, Rev. Mod. Phys. 9, 222 (1937).

for a transition $\Delta J=1$, even parity change, relatively little magnetic radiation is present.

We shall see that for low energies internal conversion increases very sharply with multipole order. Nevertheless we are still justified in keeping only the lowest multipole, for the reason that while the ratio of the number of electrons ejected to the number of quanta ejected increases with multipole order for long wave-length, the actual *number of electrons ejected falls off*. We have seen that the intensity of γ-rays falls off with multipole order l by a factor $(k\delta)^{2l}$; the number of emitted electrons may be expected to fall off (and we shall see that this is roughly so) by a factor $(k_e\delta)^{2l}$, where k_e is the reciprocal *electron* wave-length. For energies >1 Mev, $k_e\sim k$; for low energies k decreases linearly with energy and k_e as $k^{\frac{1}{2}}$, and both electron and γ-ray wave-lengths increase. Only for energies >20 Mev are $k\delta$, $k_e\delta\sim 1$. For energies lower than this we need consider, for either γ-ray or electron emission, only the lowest multipole order neither forbidden by selection rules nor reduced by any special arguments of symmetry, because here even the electron wave-length is much longer than the nuclear dimensions.

3. Representation of Multipole Fields

In previous work on internal conversion, where only dipole and quadripole fields were required, the "Maxwell" representation was employed; in this representation the electric dipole vector potential is given by

$$\mathbf{A}_1=[0,\,0,\,(e^{ikr}/kr)]e^{-i\omega t}+\text{complex conjugate.} \quad (1)$$

The quadripole field is

$$\mathbf{A}_2=[0,\,0,\,(\partial/\partial z)(e^{ikr}/kr)]e^{-i\omega t}+\text{complex conjugate.} \quad (2)$$

Higher electric multipole fields are obtained by a corresponding number of differentiations with respect to z. (The scalar potential is obtained in each case from the Lorentz condition.)

These z axis multipoles are, if spherical symmetry otherwise prevails, entirely equivalent to those directed along the x or y axes where successive differentiations are correspondingly along the x or y axes. However, one might erroneously think them also equivalent to multipoles of a "mixed" type, e.g.,

$$\mathbf{A}_2'=[0,\,0,\,(\partial/\partial x)(e^{ikr}/kr)]e^{-i\omega t}+\text{complex conjugate,}$$

reasoning from analogy to the case of static multipoles. That this is not so can best be seen from the fact that $\mathbf{A}_2'$ cannot be obtained from $\mathbf{A}_2$ by rotation. In fact, $\mathbf{A}_2'$ is not pure electric quadripole at all, but contains both electric quadripole and magnetic dipole.

Consequently the process of successive differentiation does not yield in any direct way all of the $2l+1$ independent multipoles of the 2^l type which must exist in order that the multipole fields afford a complete set. Moreover, the fields which can be obtained in this way are unsuitable in calculations for general multipole order.[9]

The problem of finding explicit expressions for the general multipole fields has been solved by Heitler,[10] a modification of whose results we use, and by Hansen. Choosing a gauge in which $\operatorname{div}\mathbf{A}'=\varphi'=0$, we obtain expressions for the vector potential of a 2^l electric pole in the following form:[11]

$$\begin{aligned}
A_z'&=e^{-i\omega t}[q_{l-1}{}^m Y_{l-1}{}^m f_{l-1}(kr)+q_{l+1}{}^m Y_{l+1}{}^m f_{l+1}(kr)],\\
A_+'&=A_x'+iA_y'=e^{-i\omega t}[Q_{l-1}{}^{m+1}Y_{l-1}{}^{m+1}f_{l-1}+Q_{l+1}{}^{m+1}Y_{l+1}{}^{m+1}f_{l+1}], \quad (3)\\
A_-'&=A_x'-iA_y'=e^{-i\omega t}[P_{l-1}{}^{m-1}Y_{l-1}{}^{m-1}f_{l-1}+P_{l+1}{}^{m-1}Y_{l+1}{}^{m-1}f_{l+1}],
\end{aligned}$$

where $\omega=ck$, $Y_l{}^m$ is the spherical harmonic, and

[9] One might hope to calculate matrix elements involving $\mathbf{A}_l$ by repeated partial integrations to reduce out the $(l-1)$th order derivative. The integrands are, however, singular at $r\rightarrow 0$, so that surface terms occur in each partial integration, rendering the method impracticable.

[10] See reference 5. A similar solution has been given independently by W. W. Hansen, Phys. Rev. **47**, 139 (1935).

[11] In order to make the potentials real, we add to A_z' its complex conjugate and to each of A_+', A_-' the complex conjugate of the other. However probabilities of transitions involving *absorption* of light will depend only on the terms containing $e^{-i\omega t}$ and we will use only these terms throughout, except in the calculation of the rate of radiation.

the coefficients q, Q, P are given by

$$Q_{l+1}{}^{m+1}(l-m+1)^{\frac{1}{2}}=q_{l+1}{}^{m}(l+m+2)^{\frac{1}{2}},$$
$$P_{l+1}{}^{m-1}(l+m+1)^{\frac{1}{2}}=-q_{l+1}{}^{m}(l-m+2)^{\frac{1}{2}},$$
$$Q_{l-1}{}^{m+1}(l+m)^{\frac{1}{2}}=-q_{l-1}{}^{m}(l-m-1)^{\frac{1}{2}},$$
$$P_{l-1}{}^{m-1}(l-m)^{\frac{1}{2}}=q_{l-1}{}^{m}(l+m-1)^{\frac{1}{2}},$$
$$\left(\frac{l+1}{l}\right)^{\frac{1}{2}}\left(\frac{(2l+3)(2l+1)}{(l+1)^2-m^2}\right)^{\frac{1}{2}}q_{l+1}{}^{m}=\left(\frac{l}{l+1}\right)^{\frac{1}{2}}\left(\frac{(2l+1)(2l-1)}{l^2-m^2}\right)^{\frac{1}{2}}q_{l-1}{}^{m}=\sigma_l{}^m.$$

The constant $\sigma_l{}^m$ measures the multipole moment. The radial function $f_l(kr)$ is:

$$f_l(kr)=H_{l+\frac{1}{2}}(kr)/(kr)^{\frac{1}{2}}$$

where $H_{l+\frac{1}{2}}(kr)$ is the Hankel function of the first kind. This gives rise to outgoing waves singular at the origin rather than to standing waves such as Heitler obtained.

The corresponding expressions for a 2^l magnetic multipole are:

$$A_z'=e^{-i\omega t}b_l{}^mY_l{}^mf_l\cdot m/[l(l+1)]^{\frac{1}{2}},$$
$$A_+'=e^{-i\omega t}B_l{}^{m+1}Y_l{}^{m+1}f_l\cdot m/[l(l+1)]^{\frac{1}{2}},$$
$$A_-'=e^{-i\omega t}C_l{}^{m-1}Y_l{}^{m-1}f_l\cdot m/[l(l+1)]^{\frac{1}{2}}, \quad (4)$$

$$B_l{}^{m+1}=-1/m[(l-m)(l+m+1)]^{\frac{1}{2}}b_l{}^m,$$
$$C_l{}^{m-1}=-1/m[(l+m)(l-m+1)]^{\frac{1}{2}}b_l{}^m.$$

The rate of radiation is calculated in the usual way and is found to be, for electric multipoles

$$\frac{(\sigma_l{}^m)^2}{\pi^2\hbar k}\text{ quanta/sec.} \quad (5)$$

and for magnetic multipoles:

$$\frac{(b_l{}^m)^2}{\pi^2\hbar k}\text{ quanta/sec.} \quad (6)$$

The choice of gauge in which div $\mathbf{A}=0$, besides rendering the potentials compact, is particularly suited for use in problems involving interaction of light with matter, where the Hamiltonian (nonrelativistic) is given, neglecting terms in $\mathbf{A}^2$, by

$$H=\frac{1}{2m}\left[\frac{\hbar}{i}\,\mathbf{grad}-\frac{e}{c}\mathbf{A}\right]^2+e\varphi$$
$$=-\frac{\hbar^2}{2m}\Delta-\frac{e\hbar}{2mic}[\mathbf{grad}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{grad}]+e\varphi.$$

Matrix elements of perturbations by the radiation field have the form

$$\int\bar{\Psi}_f\left\{-\frac{e\hbar}{mic}\mathbf{A}\cdot\mathbf{grad}-\frac{e\hbar}{2mic}(\text{div }\mathbf{A})+e\varphi\right\}\Psi_0dv. \quad (7)$$

If now div $\mathbf{A}$ and φ vanish, the calculation is extremely simplified; the only term remaining is the one in $\mathbf{A}\cdot\mathbf{grad}$.

From this one might conclude that all interaction vanished as the initial velocity approached zero, which disagrees with the results for the conventional gauge. The discrepancy must be looked for in the contribution to the matrix element of the necessarily singular generating function λ of the gauge transformation:

$$\mathbf{A}=\mathbf{A}'+\mathbf{grad}\,\lambda,\quad \varphi=\varphi'-(1/c)(\partial\lambda/\partial t), \quad (8)$$

where $\mathbf{A}$, φ are in the conventional gauge, e.g. (1), (2) and

$$\text{div }\mathbf{A}'=\varphi'=0.$$

Substituting (8) into (7), using charge-current conservation and applying suitable partial integrations, we find that the terms involving λ may be expressed as follows:

$$-\frac{1}{c}\int_V dv(d/dt)(\lambda\bar{\Psi}_f\Psi_0)$$
$$-\frac{e\hbar}{2mic}\int_\Sigma d\boldsymbol{\sigma}\cdot\{\bar{\Psi}_f\Psi_0\,\mathbf{grad}\,\lambda+\lambda\bar{\Psi}_f\,\mathbf{grad}\,\Psi_0-\lambda\Psi_0\,\mathbf{grad}\,\bar{\Psi}_f\},$$

where the surface integral is over the bounding surface Σ of the volume V under consideration. The volume integral, integrated from time 0 to t is an oscillating term which does not build up with t and hence does not contribute to the transition probability. If, as is appropriate for the problem of internal conversion, we let our bounding surface Σ consist of two spheres whose radii approach infinity and zero, respectively, then the integral over the outer surface will vanish because of the regularity of the wave functions and the potentials. We are consequently left with a surface integral over a small sphere surrounding the origin

$$\Lambda=\frac{e\hbar}{mic}\int_{\Sigma(r\to 0)} r^2 d\Omega\left\{\bar{\Psi}_f\psi_0\frac{\partial\lambda}{\partial r}+\lambda\bar{\Psi}_f\frac{\partial\Psi_0}{\partial r}-\lambda\Psi_0\frac{\partial\bar{\Psi}_f}{\partial r}\right\}. \quad (9)$$

To evaluate this, and thus to determine the difference in matrix elements between primed and unprimed gauges, one must know merely the limiting forms for small r of the wave functions and of λ.

Consider now the case of electric multipoles; by comparing (1), (2), etc. with (3), it is easily seen that $\mathbf{A}'$ is more singular than $\mathbf{A}$; hence it follows from (8) that for small r,

$$\mathbf{grad}\,\lambda\sim-\mathbf{A}'.$$

This equation is satisfied by

$$\begin{aligned}\lambda&\sim\frac{\sigma_l^m}{k}\left(\frac{l}{l+1}\right)^{\frac{1}{2}}Y_l^m f_l e^{-i\omega t}\\ &\sim\frac{\sigma_l^m}{k}\left(\frac{l}{l+1}\right)^{\frac{1}{2}}Y_l^m\left[i(-)^{l+1}\frac{(kr/2)^{-l-\frac{1}{2}}}{(kr)^{\frac{1}{2}}\Gamma(-l+\frac{1}{2})}\right]e^{-i\omega t}.\end{aligned} \quad (10)$$

The additive constant is omitted since it does not contribute to (9), which is now completely fixed.

We must still account for the fact that such a difference between gauges exists and must answer the question of which gauge, if either, gives the "right" answer—i.e., that corresponding to the physical problem. We know that any expressions for the potentials of an electrical system in terms of multipoles are approximate, and are valid only outside the region containing the charges and currents. These expressions can therefore be appropriately used only when the effect we wish to calculate (or the predominant part of it) takes place outside this region. If, because of the artificial singularities at the origin introduced in the definition of the multipole potentials, our matrix elements are given a finite contribution from an arbitrarily small region, then we must consider these potentials wrong for our particular problem. If, by a gauge transformation, we are led to another set of potentials for which the integrand of the matrix element is small at the origin, this latter set can be called correct.

In the problem of internal conversion, this condition is satisfied by the unprimed gauge. It is not satisfied by the electric multipoles in the primed gauge; div $\mathbf{A}'$ and φ', while zero throughout space, are integrably infinite at the origin, as can be seen by partially integrating

$$\int_V dv\bar{\Psi}_f\Psi_0\,\mathrm{div}\,\mathbf{A}',$$

where V is a sphere including the origin whose radius is permitted to approach zero.

Nevertheless, the primed gauge still can be advantageously used since the correction necessary to bring it into agreement with the correct gauge is known, namely, (9).

The magnetic multipole potentials in the primed gauge may be used without correction.

4. Calculations

We will calculate the number of electronic transitions per second from the K level to the continuum of an atom of atomic number Z induced by a nuclear 2^l pole with unit rate of radiation. This number, which we call $\alpha_K{}^l$ for electric, and $\beta_K{}^l$ for magnetic multipoles, is defined as the coefficient of internal conversion. Taylor and Mott have shown[12] that it is to be interpreted, in all cases of interest, as the ratio of the number of electrons to the number of quanta leaving the atom, N_e/N_q, and not to $N_e/(N_e+N_q)$ as was originally supposed. In its main effect, the perturbation by the electron serves to induce additional nuclear transitions, not to reduce the number of emitted quanta.

In the first part we limit ourselves to nonrelativistic energies throughout. This gives a range of applicability up to $Z=40$ and γ-ray energies up to 200,000 or 300,000 volts. From these calculations we find an asymptotic expression for high l.

[12] H. M. Taylor and N. F. Mott, Proc. Roy. Soc. **A142**, 215 (1933).

In the second part we derive a formula applicable to low atomic numbers, neglecting the K binding energy with respect to mc^2 and to the γ-ray energy. These formulae are very simple.

A. Nonrelativistic—electric multipoles

The number of transitions per electron per second from a state Ψ_0 in the K shell to a state Ψ_f in the continuum is given by

$$N_e=\frac{1}{\hbar^2}\left|\Lambda-\int dv\bar{\Psi}_f\frac{e\hbar}{mic}\mathbf{A}'\cdot\mathbf{grad}\,\Psi_0\right|^2, \quad (11)$$

where Ψ_f is normalized to unit flux through the boundary at infinity. Λ, the surface integral contributed by the gauge transformation, is calculated from (9). Also

$$\Psi_0=(a^3/\pi)^{\frac{1}{2}}e^{-ar},$$

$$\bar{\Psi}_f=Y_l^{-m}\left(\frac{m\pi p}{\hbar}\right)^{\frac{1}{2}}\frac{[\Gamma(l+1+in)\Gamma(l+1-in)]^{\frac{1}{2}}}{\Gamma(l+1)\Gamma(l+\frac{3}{2})}\times e^{\pi n/2}(pr/2)^l e^{-ipr}F(l+1+in,\,2l+2,\,2ipr),$$

where $a=Z\alpha(mc/\hbar)$, p is $1/\hbar$ times the momentum of the ejected electron and $n=a/p=Z\alpha/[2\nu-(Z\alpha)^2]^{\frac{1}{2}}$ where ν is the γ-ray energy over mc^2. We require the expression $\mathbf{A}'\cdot\mathbf{grad}\,\Psi_0$, which, using (3), is found to be simply

$$-a\sigma_l^m[l(l+1)]^{\frac{1}{2}}Y_l^m\frac{f_l(kr)}{(kr)}\Psi_0.$$

Both $\mathbf{A}'\cdot\mathbf{grad}$ and λ contain Y_l^m, hence both contribute to transitions to a final state $\Psi_f\sim Y_l^m$.

We make the indicated substitutions and carry out the trivial angular integrations, obtaining

$$N_e=\frac{M^2}{\hbar^2}\left\{[-kr^{l+2}f_{l+1}]_{r=0}+2a(l+1)\int_0^\infty drf_lr^{l+1}\times e^{-\rho r}F(l+1+in,\,2l+2,\,2ipr)\right\}^2,$$

where $\rho=a+ip$ and

$$M=\frac{\sqrt{2}}{\Gamma(-l+\frac{1}{2})}\left(-\frac{2}{k}\right)^l\times M'\ \text{(given below)}.$$

In the radial integral we replace f_l by its most singular term, namely

$$\frac{i(kr/2)^{-l-\frac{1}{2}}(-)^{l+1}}{(kr)^{\frac{1}{2}}\Gamma(-l+\frac{1}{2})}.$$

It can be shown that the contributions of less singular terms are smaller by at least $k^2/(a^2+p^2)\sim v^2/c^2$. With this substitution the radial integral can be carried out in series, and we obtain

$$N_e=\frac{M'^2}{\hbar^2}\left[1-\frac{a}{\rho}\frac{l+1}{l+\frac{1}{2}}\times F\left(1,\,l+1+in,\,2l+2,\,\frac{2ip}{\rho}\right)\right]^2, \quad (13)$$

where

$$M'=\frac{e\sigma_l^m}{\pi c}\left(\frac{2a^3\hbar}{m}\right)^{\frac{1}{2}}\frac{p^{l+\frac{1}{2}}}{k^{l+2}}\left(\frac{l}{l+1}\right)^{\frac{1}{2}}\times\frac{[\Gamma(l+1+in)\Gamma(l+1-in)]^{\frac{1}{2}}}{\Gamma(l+1)}e^{\pi n/2}.$$

We can express the hypergeometric function F in (13) in terms of two terminating hypergeometric series by using the familiar analytical continuation. If we do so and make use of the formulae:

$$\rho/2ip=(1-in)/2, \quad p^2/k^2=2/\nu(1+n^2),$$

$$\Gamma(l+1+in)\Gamma(l+1-in)=\frac{\pi n}{\sinh \pi n}(l^2+n^2)[(l-1)^2+n^2]\cdots(1+n^2)$$

we obtain for the conversion coefficient for two K electrons:[13]

$$\alpha_K{}^l=2N_e\cdot\pi^2\hbar k/(\sigma_l^m)^2, \quad \{v/c\ll1\}^{14}$$

$$=16\alpha\frac{l}{l+1}[\Gamma(l+\tfrac{1}{2})]^2\left(\frac{2}{\nu}\right)^{l+1}\cdot\frac{n^4}{(1+n^2)^{l-2}}\cdot\frac{[(l+1)(1+n^2)^{l-2}e^{-2n\,\mathrm{ctn}^{-1}n}-V_l]^2}{(l^2+n^2)[(l-1)^2+n^2]\cdots(1+n^2)(1-e^{-2\pi n})}, \quad (14)$$

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[13] These formulae, up to $l=5$, have been derived by Hebb and Uhlenbeck, reference 3, with whose results we are in agreement. Their parameter η is the reciprocal of our n.

[14] The inequalities in braces give the conditions of applicability of our final formulae.

where

$$V_l=\frac{4^l\prod_{i=1}^{l}(i^2+n^2)}{(1+n^2)^3(2l)!}\times\left[in\frac{l+1}{l+in}F\left(1,\,-2l,\,1-l-in,\,\frac{1-in}{2}\right)-1\right].$$

It satisfies the recursion relation:

$$V_{l+1}=V_l(1+n^2)\frac{l+2}{l+1}+\frac{2^{2l+1}l}{(2l+2)!}\cdot\frac{1}{(1+n^2)}\cdot\prod_{i=1}^{l}(i^2+n^2)$$

so that

$$V_0=0,$$
$$V_1=0,$$
$$V_2=\tfrac{1}{3},$$
$$V_3=4(3+2n^2)/15,\quad\text{etc.}$$

Limit $n\rightarrow 0$:

In this limit of low Z, $\mathbf{A}'\cdot\mathbf{grad}\,\Psi_0\rightarrow 0$, and only Λ contributes. We obtain,[15] again for two electrons

$$\alpha_K{}^l=Z^3\alpha^4\frac{l}{l+1}\left(\frac{2}{\nu}\right)^{l+5/2}\quad\begin{matrix}\{v/c\ll 1\},\\ \{Ze^2/\hbar v\ll 1\}.\end{matrix}\tag{15}$$

The convergence to this limit, however, is not uniform in l.[16]

Asymptotic formula for large l:

This can be most easily arrived at by using the contour integral representation:

$$-\rho\frac{\Gamma(2l+2)}{4^l\Gamma(l+1+in)\Gamma(l+1-in)}\times\int_{-1}^{1}\frac{(1+u)^{l+in}(1-u)^{l-in}}{u+in}du$$

for the hypergeometric function in (13). There is a saddle point on the imaginary axis quite close to the origin:

$$\bar{u}=-\frac{in}{2}+\frac{i}{2}(n^2+2/l)^{\frac{1}{2}}.$$

Carrying out the integration, one obtains a formula closely related to (15), the formula for vanishing Z.

For $l\gg 1$:

$$\alpha_K{}^l=Z^3\alpha^4\frac{l}{l+1}\left(\frac{2}{\nu}\right)^{l+5/2}\left[1-2nl^{\frac{1}{2}}\left\{\frac{(1+n^2+1/2l)-n\mu/2)^l}{(2+l\mu^2)^{\frac{1}{2}}}+O(1/l)\right\}\right]^2,\tag{16}$$

where $\mu=n+(n^2+2/l)^{\frac{1}{2}}$.

The formula can be further simplified for $n^2l\gg 1$ (i.e., l sufficiently high, not too near threshold). In this case $\mu\rightarrow 2n+1/ln$. The principal terms in the bracket cancel and one is left simply with the formula for vanishing Z reduced roughly by a factor l.

$$\alpha_K{}^l=Z^3\alpha^4\frac{l}{l+1}\left(\frac{2}{\nu}\right)^{l+5/2}\cdot O\left(\frac{1}{l}\right)\quad\begin{matrix}\{n^2l\gg 1\},\\ \{v\ll c\}.\end{matrix}\tag{17}$$

It can be shown that the term $O(1/l)$ will not vanish identically in n.

For $n^2l\ll 1$, (16) goes over into (15), a convergence non-uniform in l.

B. Nonrelativistic magnetic multipoles

In nonrelativistic theory, there is no conversion of magnetic multipole radiation by K electrons, or by any electron with zero orbital angular momentum. The transition from a state with zero to one with l units of orbital angular momentum can be accompanied only by the absorption of a 2^l pole quantum. But since the electron's parity change is $(-)^l$, magnetic radiation is forbidden, since magnetic 2^l pole has parity $(-)^{l+1}$.

Physically this means there are no radial forces on such an electron. The electric field of a magnetic multipole is tangential and the magnetic force is perpendicular to the velocity, i.e., also tangential.

Conversion of magnetic radiation by S electrons thus depends essentially on their spin.

C. Relativistic calculations, neglecting binding

The number of electronic transitions per second from a state 0 in the K shell to a state f in the continuum where the electron is ejected into the

[15] This formula was given by the authors in Phys. Rev. **54**, 149 (1938) where through error it was given as for one K electron.

[16] For increasing l, the tangential component of the magnetic field, and hence the radial Lorentz force, becomes more important.

solid angle $d\Omega$ is given for two electrons in the K shell:

$$N_e d\Omega = \frac{2\pi e^2}{\hbar} d\Omega |H|^2,$$

where $$H = \sum_{\text{spin}} \int dv \bar{\Psi}_f (\boldsymbol{\alpha}\cdot\mathbf{A}+\varphi)\Psi_0,$$

$\boldsymbol{\alpha}$ is the vector whose components are the first three Dirac matrices for $\mathbf{v}/c$, and Ψ_f is normalized to unit energy. The integral is over all space, and the sum over final and initial spin states. If now we make the gauge transformation (8), we obtain

$$H = \sum_{\text{spin}} \left[\int dv \bar{\Psi}_f \mathbf{A}'\cdot\boldsymbol{\alpha}\Psi_0 + \int d\sigma\lambda\bar{\Psi}_f \boldsymbol{\alpha}\cdot\mathbf{n}\Psi_0 \right],$$

where the second term is the analog of (9) in the nonrelativistic case, $\mathbf{n}$ is a unit vector along $\mathbf{r}$, and the surface integral is over a small sphere around the origin. In this approximation, where the influence of the Coulomb field is neglected, we use plane waves for the space-dependent part of Ψ_f. Then the wave functions factor into spin- and space-dependent parts. If we introduce the projection operators,

$$2\tau_0 = 1+\beta,$$

$$2\tau_f = 1 + \frac{\beta mc^2 + \boldsymbol{\alpha}\cdot\mathbf{p}_f\hbar c}{E_f},$$

we can replace the wave functions by $\tau_0\Psi_0$ and $\tau_f\Psi_f$ in H, and the $\sum_{\text{spin}} H\bar{H}$ can be replaced by the sum over both the spin states and the states of positive and negative energy in the well-known way.[17]

We get

$$|H|^2 = \text{Trace}\,[(\mathbf{A}\cdot\boldsymbol{\alpha})\tau_0(\mathbf{A}\cdot\boldsymbol{\alpha})^*\tau_f + (\boldsymbol{\alpha}\cdot\mathbf{N})\tau_0(\mathbf{A}\cdot\boldsymbol{\alpha})^*\tau_f + (\mathbf{A}\cdot\boldsymbol{\alpha})\tau_0(\mathbf{N}\cdot\boldsymbol{\alpha})^*\tau_f + (\boldsymbol{\alpha}\cdot\mathbf{N})\tau_0(\boldsymbol{\alpha}\cdot\mathbf{N})^*\tau_f]$$

with $$\mathbf{A} = C_f \int \mathbf{A}' \exp(i\mathbf{p}\cdot\mathbf{r})dv,$$

$$\mathbf{N} = C_f \int d\sigma\lambda \exp(i\mathbf{p}\cdot\mathbf{r})\mathbf{n},$$

C_f = normalizing constant.

[17] Cf. W. Heitler, *Quantum Theory of Radiation*, p. 150.

Taking the trace and evaluating the space integrals we find the matrix element. Dividing by the rate of radiation, and normalizing, we obtain for the internal conversion coefficient for two K electrons, neglecting binding energy:

$$\alpha_K{}^l = \frac{2Z^3\alpha^4}{\nu^3}\left(\frac{\nu+2}{\nu}\right)^{l-\frac{1}{2}}\left[\frac{(l+1)\nu^2+4l}{l+1}\right]. \qquad \{Ze^2/\hbar v \ll 1\}, \quad (18)$$

where ν = γ-ray energy in units mc^2.

Magnetic multipoles:

A similar, but simpler, calculation (no gauge term appears) gives for two K electrons:[18]

$$\beta_K{}^l = \frac{2Z^3\alpha^4}{\nu}\left(\frac{\nu+2}{\nu}\right)^{l+\frac{1}{2}} \qquad \{Ze^2/\hbar v \ll 1\}. \quad (19)$$

Limiting cases:

For low-energy γ-rays, $\nu \ll 1$:[15]

$$\alpha_K{}^l = \frac{l}{l+1}Z^3\alpha^4\left(\frac{2}{\nu}\right)^{l+5/2} \qquad \{v/c \ll 1\},$$
$$\beta_K{}^l = Z^3\alpha^4\left(\frac{2}{\nu}\right)^{l+\frac{3}{2}} \qquad \{Ze^2/\hbar v \ll 1\}. \quad (20)$$

For low energy γ-rays and for small binding energy, conversion of magnetic 2^l pole radiation is smaller than that of the electric 2^l pole by a factor

$$\frac{\beta_K{}^l}{\alpha_K{}^l} \rightarrow \frac{l+1}{4l}\left(\frac{v}{c}\right)^2.$$

For very high energy γ-rays, $\nu \gg 1$:[19]

$$\alpha_K{}^l = \beta_K{}^l = \frac{2Z^3\alpha^4}{\nu} \qquad \{\nu\rightarrow\infty\}, \; \{Ze^2/\hbar v \ll 1\}. \quad (21)$$

In this limit the conversion is independent of multipole type and order, as in the same limit for internal conversion by pair production.[20] Both

[18] This formula may be expected to give a fair estimate for low and middle Z, even though binding is neglected, as magnetic conversion essentially depends on spin. Compare Section 4*B*.

[19] This formula agrees, in the limit of low Z, with the electric dipole formula given by H. Casimir, Physik. Zeits. **32**, 665 (1931). The latter is applicable, for sufficiently large ν, to all Z.

[20] This result is more easily derived by the method suggested in reference 9.

electric and magnetic conversion formulae for negligible binding have the general properties:

(1) For a given l, the conversion increases as ν decreases, the more rapidly the greater l. For $\nu \rightarrow \infty$ the conversion for all multipoles approaches a common limit.

(2) For a given ν, conversion increases with l, more and more slowly as $\nu \rightarrow \infty$.

L shell:

For conversion from the L shell, (18) and (19) hold provided Z be replaced by $Z-\sigma/2$, where σ is the *energy* screening constant; for the L shell these results are valid for larger values of n than for the K shell.

5. Conclusion

Experiment has established internal conversion, either by detection of the actual electron groups or of the x-rays from the filling of the depleted K levels, in the cases of Rh,[21] Br,[22] Ma,[23] Ag[24] and Au;[25] probably in In[26] and Cd;[27] and, of course, in Zn. In the last, the data are sufficiently complete to assign the conversion tentatively to an electric quadripole radiation.[28] In the others it is not yet possible to classify the radiation. The general features suggested by the von Weizsäcker hypothesis are, however, observed: low energy and long lifetime (metastability) seem to be accompanied by large conversion—pointing to high angular momenta. An interesting check on the picture will be furnished by comparison of lifetime and conversion coefficient, both of which are fixed by multipole order for a given γ-ray energy.

We summarize our results: [Here l is multipole order, $n = Ze^2/\hbar v$].

(1) The smaller the γ-ray energy compared to mc^2, the more rapidly conversion increases with multipole order. For large γ-ray energies, the increase with multipole order is much less marked.

(2) The effect of binding is complicated, though in general conversion varies about as Z^3 for given l and energy. For a given value of n, there may be a minimum in the conversion coefficient as l is varied.

(3) For sufficiently large l, and low γ-ray energy the conversion (given by (17)) is of the order of $1/l\times$, the conversion neglecting binding.

(4) The magnetic multipole conversion will be small for low γ-ray energy and $Z<40$.

(5) Quantitative results for γ-ray energies under 0.2 Mev and $Z<40$ will be given by formula (14). For $Z<30$, high energies, and not extremely high multipole order, (18) and (19) give usable estimates. For $Z>50$ numerical calculation is necessary for accurate results.

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