Many-Body Interactions in Atomic and Nuclear Systems

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When particles interact with each other through the intervening mechanism of a field, the description of their dynamical behavior by means of action-at-a-distance potentials is only of an approximate nature. Two-body, three-body, $\cdots$, m-body potentials may be regarded as successive stages of this approximation; their relative magnitudes are examined systematically for several types of classical and quantized fields, e.g., electromagnetic, mesotron, etc. It is found that the description of electrons in atomic systems by the customary two-body potentials is an excellent approximation; in nuclei, independent of the details of the field, one finds: three-body potentials $\cong (v_n/c)\times$(two-body potentials)$\cdots$, m-body potentials $\cong (v_n/c)^{m-2}\times$(two-body potentials), where v_n is the average velocity of the heavy particles in the nucleus. The usual description of nuclei in terms of two-body potentials cannot therefore be considered satisfactory, except in the case of the deuteron.

INTRODUCTION

IN field theories of particle interaction, an exact description of the dynamical behavior of the particles necessitates the explicit introduction of variables describing the state of the field. A description of particle dynamics in terms of interactions depending only on the instantaneous relative coordinates of the particles[1] (action-at-a-distance) is, therefore, necessarily of an approximate character. In the first stage of this approximation the interactions are of the so-called "two-body" type:

$$V(r_{12}, r_{13}, \cdots)=\sum_{i,\,j} V_{ij}(r_{ij}).$$

In the higher stages, as shown below, one obtains additional interactions of such a nature that the *force* between any *two* particles is dependent on the positions of some of the *others* ("many-body" forces). For example,

$$V=\sum_{i,\,j,\,k} V_{ijk}(r_{ij}, r_{jk}, r_{ik})$$

represents three-body interactions. It is the purpose of this paper to examine the conditions under which a set of two-body interactions constitute an adequate substitute for the explicit use of field variables. It will be found that the use of two-body interactions is an excellent approximation for electronic motions in atomic systems, but a relatively poor one for the heavy particles in nuclei.

CLASSICAL ELECTROMAGNETIC THEORY OF ELECTRON DYNAMICS

(a) Equations of motion

The equation of motion of a given electron in the presence of potentials, φ, $\mathbf{A}$ is

$$m\mathbf{a}_k=e_k\left\{-\text{grad.}_k\,\varphi(\mathbf{r}_k,t)-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}_k,t)+\frac{\mathbf{v}_k}{c}\times\text{curl}_k\,\mathbf{A}(\mathbf{r}_k,t)\right\},\quad k=1,\cdots,n. \quad (1)$$

Here φ, $\mathbf{A}$, arise from the other electrons, and describe the state of the field; they are determined by the equations:[2]

$$\nabla^2\varphi=-4\pi\sum_j e_j\delta(\mathbf{r}-\mathbf{r}_j(t))-\frac{1}{c}\,\text{div.}\,\frac{\partial\mathbf{A}}{\partial t}, \quad (2)$$

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-4\pi\sum_j e_j\frac{\mathbf{v}_j}{c}\delta(\mathbf{r}-\mathbf{r}_j(t))+\text{grad.}\left(\text{div.}\,\mathbf{A}+\frac{1}{c}\frac{\partial\varphi}{\partial t}\right). \quad (3)$$

To obtain an "action-at-a-distance" dynamical description of the particles, one attempts to express φ, $\mathbf{A}$ in terms of the instantaneous relative particle coordinates. Choosing the gauge so

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[1] One also admits interactions depending on the particle velocity into the action-at-a-distance dynamical description.

[2] W. Heitler, *Quantum Theory of Radiation* (Oxford Univ. Press, 1936), p. 2, Eqs. (8a), (8b).

that div. $\mathbf{A}=0$, one finds:[2a]

$$\varphi(\mathbf{r}_k, t)=\sum_j \frac{e_j}{r_{kj}}; \quad \text{exactly.} \tag{4}$$

$$\mathbf{A}(\mathbf{r}_k, t)=\sum_j \frac{e_j}{2c}\left(\frac{\mathbf{v}_j}{r_{kj}}+\frac{\mathbf{r}_{kj}(\mathbf{v}_j\cdot\mathbf{r}_{kj})}{r_{kj}^3}\right)$$

$$+\text{terms in }\frac{\mathbf{v}_j^2}{c^2}+\text{terms in }\frac{\mathbf{a}_j}{c^2}+\cdots;$$

$$\mathbf{a}_j\equiv\text{acceleration of } j\text{th particle.} \tag{5}$$

The forces acting on the kth particle will be correct to the second order in v_j/c, and the first order in a_j/c^2, if all terms in $\mathbf{A}$ except those in v_j/c, are neglected.[2b]

With this approximation, the equations of motion (1) for the system of electrons, become:

$$m\mathbf{a}_k=\sum_j \frac{e_k e_j \mathbf{r}_{kj}}{r_{kj}^3}-\sum_j \frac{e_k e_j}{2c^2}\left(-\frac{v_j^2\mathbf{r}_{kj}}{r_{kj}^3}+\frac{3(\mathbf{v}_j\cdot\mathbf{r}_{kj})^2\mathbf{r}_{kj}}{r_{kj}^5}\right)$$

$$+\sum_j \frac{e_k\mathbf{v}_k}{c}\times\left(\frac{e_j}{c}\,\frac{\mathbf{v}_j\times\mathbf{r}_{kj}}{r_{kj}^3}\right)$$

$$+\sum_j \frac{e_k e_j}{2c^2}\left(\frac{\mathbf{a}_j}{r_{kj}}+\frac{(\mathbf{a}_j\cdot\mathbf{r}_{kj})\mathbf{r}_{kj}}{r_{kj}^3}\right)$$

$$\equiv\mathbf{f}_c(k)+\mathbf{f}_v(k)+\mathbf{f}_m(k)+\mathbf{f}_a(k), \quad k=1,\cdots,n. \tag{6}$$

Here, $\mathbf{f}_c(k)$, $\mathbf{f}_v(k)$, $\mathbf{f}_m(k)$, represent, in order, the Coulomb force, a velocity-dependent correction to the Coulomb force, and the magnetic force. $\mathbf{f}_a(k)$ is the force on the kth particle determined by the acceleration of the others.

Since there are n equations of motion (4), the n unknown accelerations are given in terms of the positions and velocities. To the zeroth order in v/c,

$$\mathbf{a}_j=\frac{1}{m}\sum_l \frac{e_j e_l}{r_{jl}^3}\mathbf{r}_{jl}, \tag{7}$$

whence, to v^2/c^2

$$\mathbf{f}_a(k)=\sum_{j,l}\frac{e_k e_j}{2mc^2}\left(\frac{e_j e_l \mathbf{r}_{jl}}{r_{kj}r_{jl}^3}+\frac{e_j e_l(\mathbf{r}_{jl}\cdot\mathbf{r}_{kj})\mathbf{r}_{kj}}{r_{kj}^3 r_{jl}^3}\right)$$

$$\cong\frac{e^2}{mc^2}\frac{1}{r}\frac{e^2}{r^2}\cong\frac{e^2}{mc^2}\frac{1}{r}|\mathbf{f}_c|. \tag{8}$$

Thus $\mathbf{f}_a(k)$ is of the three-body type. The three-body forces become appreciable whenever the relative distances between the electrons are not large in comparison with their classical electromagnetic radii.

It is a general fact that "acceleration-dependent" forces, when expressed in terms of coordinates and velocities, are of the many-body type.

(b) Lagrangian and Hamiltonian of the particle system

The equations of motion, (6), are derivable from a two-body Lagrangian:[3]

$$L=\sum_k \tfrac{1}{2}m\mathbf{v}_k^2-\tfrac{1}{2}\sum_{k,j}\frac{e_k e_j}{r_{kj}}$$

$$+\tfrac{1}{2}\sum_{k,j}\frac{e_k e_j}{2c^2}\left(\frac{\mathbf{v}_k\cdot\mathbf{v}_j}{r_{kj}}+\frac{(\mathbf{v}_k\cdot\mathbf{r}_{kj})(\mathbf{v}_j\cdot\mathbf{r}_{kj})}{r_{kj}^3}\right)$$

$$\equiv\sum_k \tfrac{1}{2}m\mathbf{v}_k^2-\tfrac{1}{2}\sum_k e_k\varphi(\mathbf{r}_k)+\tfrac{1}{2}\sum_k \frac{e_k}{c}\mathbf{v}_k\cdot\mathbf{A}(\mathbf{r}_k), \tag{9}$$

from which one obtains the Hamiltonian:

$$H=\sum_k \tfrac{1}{2}m\mathbf{v}_k^2+\tfrac{1}{2}\sum_{k,j}\frac{e_k e_j}{r_{kj}}$$

$$+\tfrac{1}{2}\sum_{k,j}\frac{e_k e_j}{2c^2}\left(\frac{\mathbf{v}_k\cdot\mathbf{v}_j}{r_{kj}}+\frac{(\mathbf{v}_k\cdot\mathbf{r}_{kj})(\mathbf{v}_j\cdot\mathbf{r}_{kj})}{r_{kj}^3}\right)$$

$$\equiv\sum_k \tfrac{1}{2}m\mathbf{v}_k^2+\tfrac{1}{2}\sum_k e_k\varphi(\mathbf{r}_k)+\tfrac{1}{2}\sum_k \frac{e_k\mathbf{v}_k}{c}\cdot\mathbf{A}(\mathbf{r}_k), \tag{10}$$

where the $\mathbf{v}$'s are to be expressed in terms of the

[2a] The gauge div. $\mathbf{A}=0$, and the subsequent approximate solution for $\mathbf{A}$ from the field equations, has been used by Breit in his derivation of the velocity dependent interaction between two electrons. See Phys. Rev. **39**, 616 (1932).

[2b] For further discussion of this neglect, see Addendum.

[3] This Lagrangian was first given by C. G. Darwin, Phil. Mag. **39**, 537 (1920). All relativistic terms in the kinetic energy, such as $\Sigma_k m_k v_k^4/8c^2$, are consistently omitted in what follows, since they introduce no essential modification in the derivation of many-body potentials. A further discussion of the approximations involved in this Lagrangian is given in the Addendum.

$\mathbf{p}$'s, by the equations

$$\mathbf{v}_k=\frac{1}{m}\left[\mathbf{p}_k-\sum_j \frac{e_k e_j}{2c^2}\left(\frac{\mathbf{v}_j}{r_{kj}}+\frac{\mathbf{r}_{kj}(\mathbf{v}_j\cdot\mathbf{r}_{kj})}{r_{kj}^3}\right)\right]$$
$$\equiv\frac{1}{m}\left(\mathbf{p}_k-\frac{e_k}{c}\mathbf{A}(\mathbf{r}_k)\right),\quad k=1,\cdots,n. \quad (11)$$

Substituting into (10), the value of $\mathbf{v}_k$ given by (11), one obtains

$$H=\sum_k \frac{\mathbf{p}_k^2}{2m}+\tfrac{1}{2}\sum_k e_k\varphi(\mathbf{r}_k)-\tfrac{1}{2}\sum_k \frac{e_k}{c}\frac{\mathbf{p}_k}{m}\cdot\mathbf{A}(\mathbf{r}_k) \quad (12)$$

with the $\mathbf{A}$'s given in terms of the $\mathbf{p}$'s, by (cf. (5) and (11))

$$\mathbf{A}(\mathbf{r}_k)=\sum_j \frac{e_j}{2mc}\left[\frac{(\mathbf{p}_j-(e_j/c)\mathbf{A}(\mathbf{r}_j))}{r_{kj}}+\frac{\mathbf{r}_{kj}(\mathbf{p}_j-(e_j/c)\mathbf{A}(\mathbf{r}_j))\cdot\mathbf{r}_{kj}}{r_{kj}^3}\right]$$
$$\equiv\sum_j \frac{e_j}{2mc}\left(\frac{\mathbf{p}_j}{r_{kj}}+\frac{\mathbf{r}_{kj}(\mathbf{r}_{kj}\cdot\mathbf{p}_j)}{r_{kj}^3}\right)-\sum_j \frac{e_j^2}{2mc^2 r_{kj}}\left(\mathbf{A}(\mathbf{r}_j)+\frac{\mathbf{r}_{kj}(\mathbf{r}_{kj}\cdot\mathbf{A}(\mathbf{r}_j))}{r_{kj}^2}\right), \quad (13)$$
$$k=1,\cdots,n.$$

The n linear inhomogeneous vector equations (13), may be solved, either exactly, with determinants, or approximately, with $e_j^2/2mc^2r_{kj}$ as an expansion parameter. The latter method leads to

$$\mathbf{A}(\mathbf{r}_k)=\sum_j \frac{e_j}{2mc}\left(\frac{\mathbf{p}_j}{r_{kj}}+\frac{\mathbf{r}_{kj}(\mathbf{r}_{kj}\cdot\mathbf{p}_j)}{r_{kj}^3}\right)$$
$$-\sum_{j,\,l}\frac{e_j^2}{2mc^2r_{kj}}\frac{e_l}{2mc}\left[\left(\frac{\mathbf{p}_l}{r_{jl}}+\frac{\mathbf{r}_{jl}(\mathbf{p}_l\cdot\mathbf{r}_{jl})}{r_{jl}^3}\right)+\frac{\mathbf{r}_{kj}}{r_{kj}^2}\mathbf{r}_{kj}\cdot\left(\frac{\mathbf{p}_l}{r_{jl}}+\frac{\mathbf{r}_{jl}(\mathbf{p}_l\cdot\mathbf{r}_{jl})}{r_{jl}^3}\right)\right]$$
$$+\sum_{j,\,l,\,s}\text{terms in}\left(\frac{e^2}{mc^2}\frac{1}{r}\right)^2\frac{e}{c}\frac{\mathbf{p}}{mr}$$
$$+\sum_{j,\,l,\,s,\,t}\text{terms in}\left(\frac{e^2}{mc^2}\frac{1}{r}\right)^3\frac{e}{c}\frac{\mathbf{p}}{mr}+\cdots, \quad (14)$$

whence, neglecting terms in $[(e^2/mc^2)(1/r)]^2$, $[(e^2/mc^2)(1/r)]^3$, $\cdots$, and substituting for $\mathbf{A}(\mathbf{r}_k)$ into the Hamiltonian (12), one obtains

$$H=\sum_k \frac{\mathbf{p}_k^2}{2m}+\tfrac{1}{2}\sum_{k,\,j}\frac{e_ke_j}{r_{kj}}-\tfrac{1}{2}\sum_{k,\,j}\frac{e_ke_j}{2m^2c^2}\left(\frac{\mathbf{p}_k\cdot\mathbf{p}_j}{r_{kj}}+\frac{(\mathbf{p}_k\cdot\mathbf{r}_{kj})(\mathbf{p}_j\cdot\mathbf{r}_{kj})}{r_{kj}^3}\right)$$
$$+\sum_{k,\,j,\,l}\frac{e_ke_j^2e_l}{8m^3c^4}\left[\frac{\mathbf{p}_k\cdot\mathbf{p}_l}{r_{kj}r_{jl}}+\frac{(\mathbf{p}_k\cdot\mathbf{r}_{jl})(\mathbf{p}_l\cdot\mathbf{r}_{jl})}{r_{kj}r_{jl}^3}+\frac{(\mathbf{p}_k\cdot\mathbf{r}_{kj})(\mathbf{p}_l\cdot\mathbf{r}_{kj})}{r_{kj}^3r_{jl}}+\frac{(\mathbf{r}_{kj}\cdot\mathbf{r}_{jl})(\mathbf{p}_k\cdot\mathbf{r}_{kj})(\mathbf{p}_l\cdot\mathbf{r}_{jl})}{r_{kj}^3r_{jl}^3}\right]$$
$$\equiv H(\text{kinetic})+H(\text{Coulomb})+H(\text{Darwin})+H'. \quad (15)$$

The interaction H' contains "velocity-dependent" three-body potentials (for $j\neq l$). These are:

H'(three-body vel.-depend) $\cong$

$$\sum_{k,\,j,\,l}\frac{e_ke_j^2e_l}{8m^3c^4}\frac{\mathbf{p}_k\cdot\mathbf{p}_l}{r_{kj}r_{jl}}\cong\left(\frac{e^2}{mc^2}\frac{1}{r}\right)\left(\frac{v^2}{c^2}\right)\frac{e^2}{r}. \quad (16)$$

Thus one obtains a three-body Hamiltonian from the two-body Lagrangian because of the peculiar relation between the $\mathbf{p}$'s and $\mathbf{v}$'s, in (11), and the consequent peculiar relation between the $\mathbf{A}$'s and $\mathbf{p}$'s (Eq. (13)). While the use of the three-body Hamiltonian is somewhat artificial in classical mechanics, the same equations of motion (6), being derivable from the two-body Lagrangain (9), its introduction is essential in quantum mechanics, since the $\mathbf{p}$'s, not the $\mathbf{v}$'s, enter into the fundamental commutation relationships.

The terms in $\mathbf{A}$, of order

$$[(e^2/mc^2)(1/r)]^2e\mathbf{p}/mcr,\quad [(e^2/mc^2)(1/r)]^3e\mathbf{p}/mcr,\quad \cdots,\quad [(e^2/mc^2)(1/r)]^{m-2}e\mathbf{p}/mcr,$$

when substituted into the

$$-\tfrac{1}{2}\Sigma_k(e_k/c)(\mathbf{p}_k/m)\cdot\mathbf{A}(\mathbf{r}_k)$$

term of the Hamiltonian give 4, 5, $\cdots$, m-body-potentials. The ratio of the magnitudes of the individual terms in the m-body and Coulomb potentials is $\cong[(e^2/mc^2)(1/r)]^{m-2}v^2/c^2$.

If one considers the classical electron to have a spin $\mathbf{s}_k$ and a magnetic moment $\boldsymbol{\mu}_k$, proportional to $\mathbf{s}_k$, then[3a]

$$\mathbf{A}(\mathbf{r}_k)=\sum_j \frac{e_j}{2c}\left(\frac{\mathbf{v}_j}{r_{kj}}+\frac{\mathbf{r}_{kj}(\mathbf{v}_j\cdot\mathbf{r}_{kj})}{r_{kj}^3}\right)+\sum_j \frac{\boldsymbol{\mu}_j\times\mathbf{r}_{kj}}{r_{kj}^3}, \quad (17)$$

$$L=\sum_k \frac{m\mathbf{v}_k^2}{2}-\tfrac{1}{2}\sum_{k,j}\frac{e_ke_j}{r_{kj}}+\left[\tfrac{1}{2}\sum_{k,j}\frac{e_ke_j}{2c^2}\left(\frac{\mathbf{v}_k\cdot\mathbf{v}_j}{r_{kj}}+\frac{(\mathbf{v}_k\cdot\mathbf{r}_{kj})(\mathbf{v}_j\cdot\mathbf{r}_{kj})}{r_{kj}^3}\right)+\tfrac{1}{2}\sum_{k,j}\frac{e_k\mathbf{v}_k}{c}\cdot\frac{\boldsymbol{\mu}_j\times\mathbf{r}_{kj}}{r_{kj}^3}\right]$$
$$+\left[\tfrac{1}{2}\sum_{k,j}\boldsymbol{\mu}_k\cdot\left(\frac{e_j\mathbf{v}_j\times\mathbf{r}_{kj}}{cr_{kj}^3}\right)+\tfrac{1}{2}\sum_{k,j}\boldsymbol{\mu}_k\cdot\mathrm{curl}_k\left(\frac{\boldsymbol{\mu}_j\times\mathbf{r}_{kj}}{r_{kj}^3}\right)\right]$$
$$\equiv\sum_k\frac{m\mathbf{v}_k^2}{2}-\tfrac{1}{2}\sum_k e_k\varphi(\mathbf{r}_k)+\tfrac{1}{2}\sum_k\frac{e_k\mathbf{v}_k}{c}\cdot\mathbf{A}(\mathbf{r}_k)+\tfrac{1}{2}\sum_k\boldsymbol{\mu}_k\cdot\mathrm{curl}_k\,\mathbf{A}(\mathbf{r}_k), \quad (18)$$

$$\mathbf{v}_k=\frac{1}{m}\left(\mathbf{p}_k-\sum_j\frac{e_j}{2c}\left(\frac{\mathbf{v}_j}{r_{kj}}+\frac{\mathbf{r}_{kj}(\mathbf{v}_j\cdot\mathbf{r}_{kj})}{r_{kj}^3}\right)-\sum_j\frac{\boldsymbol{\mu}_j\times\mathbf{r}_{kj}}{r_{kj}^3}\right)\equiv\frac{1}{m}\left(\mathbf{p}_k-\frac{e_k}{c}\mathbf{A}(\mathbf{r}_k)\right), \quad (19)$$

$$H=\sum_k\frac{m\mathbf{v}_k^2}{2}+\tfrac{1}{2}\sum_k e_k\varphi(\mathbf{r}_k)+\tfrac{1}{2}\sum_k\frac{e_k}{c}\mathbf{v}_k\cdot\mathbf{A}(\mathbf{r}_k)-\tfrac{1}{2}\sum_k\boldsymbol{\mu}_k\cdot\mathrm{curl}_k\,\mathbf{A}(\mathbf{r}_k)$$
$$=\sum_k\frac{\mathbf{p}_k^2}{2m}+\tfrac{1}{2}\sum_k e_k\varphi(\mathbf{r}_k)-\tfrac{1}{2}\sum_k\frac{e_k}{c}\frac{\mathbf{p}_k}{m}\cdot\mathbf{A}(\mathbf{r}_k)-\tfrac{1}{2}\sum_k\boldsymbol{\mu}_k\cdot\mathrm{curl}_k\,\mathbf{A}(\mathbf{r}_k). \quad (20)$$

Using (19), one may express $\mathbf{A}$ in (17), in terms of the $\mathbf{p}$'s, just as before. Thus

$$\mathbf{A}=\sum_j\frac{e_j}{2mc}\left(\frac{\mathbf{p}_j}{r_{kj}}+\frac{\mathbf{r}_{kj}(\mathbf{p}_j\cdot\mathbf{r}_{kj})}{r_{kj}^3}\right)+\sum_j\frac{\boldsymbol{\mu}_j\times\mathbf{r}_{kj}}{r_{kj}^3}-\sum_{j,l}\frac{e_j^2}{2mc^2r_{kj}}\frac{e_l}{2mc}\left[\left(\frac{\mathbf{p}_l}{r_{jl}}+\frac{\mathbf{r}_{jl}(\mathbf{p}_l\cdot\mathbf{r}_{jl})}{r_{jl}^3}\right)+\frac{\mathbf{r}_{kj}}{r_{kj}^2}\mathbf{r}_{kj}\cdot\left(\frac{\mathbf{p}_l}{r_{jl}}+\frac{\mathbf{r}_{jl}(\mathbf{p}_l\cdot\mathbf{r}_{jl})}{r_{jl}^3}\right)\right]-\sum_{j,l}\frac{e_j^2}{2mc^2r_{kj}}\times\left[\frac{\boldsymbol{\mu}_l\times\mathbf{r}_{jl}}{r_{jl}^3}+\frac{\mathbf{r}_{kj}}{r_{kj}^2}\mathbf{r}_{kj}\cdot\left(\frac{\boldsymbol{\mu}_l\times\mathbf{r}_{jl}}{r_{jl}^3}\right)\right]. \quad (21)$$

[3a] A moving magnetic dipole gives rise to an electric dipole with moment $\cong\mathbf{v}_j/c\times\boldsymbol{\mu}_j$. The interaction of this electric dipole with the electric field due to the other particles is of the same order in v/c as the spin-orbit term appearing in the Lagrangian (18). It is omitted for reasons of simplicity since its inclusion introduces nothing essentially new.

Upon substitution of this expression for $\mathbf{A}$ into the Hamiltonian (20), the term:

$$-\tfrac{1}{2}\sum_k(e_k/c)(\mathbf{p}_k/m)\cdot\mathbf{A}(\mathbf{r}_k)$$

gives the Darwin two-body potential, and the three-body potential obtained before, and in addition, a "spin-orbit" two-body potential and a "spin-orbit" three-body potential. The term: $-\tfrac{1}{2}\sum_k\boldsymbol{\mu}_k\cdot\mathrm{curl}_k\,\mathbf{A}(\mathbf{r}_k)$ gives "spin-orbit" and "spin-spin" two-body potentials; and "spin-orbit" and "spin-spin" three-body potentials. The "spin-spin" three-body potential is

H'(three-body spin-spin)

$$=\sum_{k,j,l}\frac{e_j^2}{4mc^2}\boldsymbol{\mu}_k\cdot\mathrm{curl}_k\left[\frac{1}{r_{kj}}\left(\frac{\boldsymbol{\mu}_l\times\mathbf{r}_{jl}}{r_{jl}^3}\right)+\frac{\mathbf{r}_{kj}}{r_{kj}^3}\mathbf{r}_{kj}\cdot\left(\frac{\boldsymbol{\mu}_l\times\mathbf{r}_{jl}}{r_{jl}^3}\right)\right]\cong\frac{e^2}{mc^2r}\frac{\mu^2}{r^3}. \quad (23)$$

It will be seen below that a three-body potential, with the same spin dependence, as in (23), is of considerable importance in the mesotron field theory of nuclear forces.

In concluding this section it may be pointed out that when the $\mathbf{A}$'s in the Hamiltonian are expressed in terms of $\mathbf{p}_k$, $\mathbf{r}_k$, $\mathbf{s}_k$ one need only apply the commutation relations to these variables, to obtain a quantum-mechanical action-at-a-distance theory of electronic motion.

Quantum Electromagnetic Theory of Electron Dynamics

In the preceding section the classical electromagnetic field was eliminated, to a certain degree of approximation, from the equations of motion of the electrons. The resulting action-at-a-distance Hamiltonian (15), could then be quantized. In the following section the procedure will be reversed: one first quantizes the electromagnetic field, as well as the matter, and then proceeds to an approximate action-at-a-distance description of electronic motion, by means of a quantum-mechanical perturbation method. This method will give not only the same many-body potentials found above ("classical"), but also, many-body potentials explicitly dependent on $\hbar$ ("specifically quantum-mechanical").

The Hamiltonian for the total system, field and matter, is: (Fourier representation of the quantized field variables[4])

$$H=\{\tfrac{1}{2}\sum_\rho P_\rho{}^2+\nu_\rho{}^2Q_\rho{}^2\}+\left\{\sum_k \frac{\mathbf{p}_k{}^2}{2m}+\sum_{k,\,j}\tfrac{1}{2}\frac{e_ke_j}{r_{kj}}\right\}$$
$$+\left\{-\sum_k \frac{e_k}{c}\frac{\mathbf{p}_k}{m}\cdot\mathbf{A}(\mathbf{r}_k)+\sum_k\frac{e_k{}^2}{2mc^2}(\mathbf{A}(\mathbf{r}_k))^2\right.$$
$$\left.-\sum_k \boldsymbol{\mu}_k\cdot\mathrm{curl}_k\,\mathbf{A}(\mathbf{r}_k)\right\}$$
$$\equiv H(\text{field})+H(\text{matter})+H(\text{interaction}) \quad (24)$$

with

$$\mathbf{A}(\mathbf{r})=\sum_\rho(4\pi c^2)^{\frac{1}{2}}\mathbf{a}_\rho\left(Q_\rho\cos(\mathbf{k}_\rho\cdot\mathbf{r})-\frac{P_\rho}{\nu_\rho}\sin(\mathbf{k}_\rho\cdot\mathbf{r})\right)$$
$$=\sum_\rho(4\pi c^2)^{\frac{1}{2}}\mathbf{a}_\rho(q_\rho\exp(i\mathbf{k}_\rho\cdot\mathbf{r}) \quad (25)$$
$$+q_\rho{}^*\exp(-i\mathbf{k}_\rho\cdot\mathbf{r}))$$
$$=\sum_\rho q_\rho\mathbf{A}_\rho+q_\rho{}^*\mathbf{A}_\rho{}^*,$$
$$\text{div.}\,\mathbf{A}=0;\quad |\mathbf{a}_\rho|^2=1. \quad (26)$$

In the eigenstates $\Psi(Q_\rho,\mathbf{r}_j)$ of H, the number of light quanta is not specified, i.e.

$$\Psi(Q_\rho,\mathbf{r}_j)=\sum_{a;\,n_\rho} c_{a,\,n_\rho}\psi_a(\mathbf{r}_j)\varphi_{n_\rho}(Q_\rho) \quad (27)$$

with

$$H(\text{matter})\ \psi_a(\mathbf{r}_j)=\epsilon_a\psi_a(\mathbf{r}_j),$$
$$H(\text{field})\ \varphi_{n_\rho}(Q_\rho)=[\sum_\rho(n_\rho+\tfrac{1}{2})\hbar\nu_\rho]\varphi_{n_\rho}(Q_\rho). \quad (28)$$

The elimination of the field variables, and the consequent action-at-a-distance dynamical description of the particles, corresponds, in quantum mechanics, to the consideration of transitions of the system among states, where the number of light quanta is always zero, i.e., states of type

$$\Psi(Q_\rho,\mathbf{r}_j)=\{\sum_{a',\,0_\rho} c_{a',\,0_\rho}\psi_{a'}(\mathbf{r}_j)\}\varphi_{0_\rho}(Q_\rho). \quad (29)$$

In particular, one may limit the discussion to transitions among the matter eigenstates, i.e., $c_{a',\,0_\rho}=\delta_{a',\,a}$. The transitions among the states ψ_a, caused by H(interaction), i.e., by the virtual emission and absorption of light quanta, are now to be regarded as caused by an equivalent action-at-a-distance potential H'. The dependence of H' on the particle coordinates is to be such that the transition probabilities between the matter states ψ_a, with H' as perturbing potential, are identical to any desired approximation, with those arising from H(interaction) in the complete field theory.

It is to be noted that the quantum-mechanical method of eliminating the field variables, is necessarily of a perturbation character of successive approximations, since one must consider transitions between states in which no quanta are present. Such a treatment, as already observed above, leads to two types of many-body potentials, "classical" and "specifically quantum-mechanical," which are discussed in order.

(a) Classical many-body potentials

In a system, consisting, for simplicity, of only *three* electrons and the radiation field, the following type of transition takes place: The first electron, say, *simultaneously* emits two virtual light quanta, by means of the $(e^2/2mc^2)(\mathbf{A}(\mathbf{r}_1))^2$ term in the perturbation; one of these virtual light quanta is absorbed by the second electron by means of the $-(e\mathbf{p}_2/cm)\cdot\mathbf{A}(\mathbf{r}_2)$ term, while the other is absorbed by the third electron by means of the $-(e\mathbf{p}_3/cm)\cdot\mathbf{A}(\mathbf{r}_3)$ term. The matrix element for this transition, which depends on the instantaneous coordinates of all the three electrons, is given by

$$\sum_{n,\,n'}\frac{\left(i\left|\dfrac{e^2}{2mc^2}(\mathbf{A}(\mathbf{r}_1))^2\right|n\right)\left(n\left|-\dfrac{e}{c}\dfrac{\mathbf{p}_2}{m}\cdot\mathbf{A}(\mathbf{r}_2)\right|n'\right)\left(n'\left|-\dfrac{e}{c}\dfrac{\mathbf{p}_3}{m}\cdot\mathbf{A}(\mathbf{r}_3)\right|f\right)}{(E_i-E_n)(E_i-E_{n'})}. \quad (30)$$

The initial and final states i, and f, are characterized by no virtual light quanta present and matter wave functions ψ_i and ψ_f; the intermediate states n, n' have, respectively, two and one virtual light quanta present, and matter wave functions ψ_n, $\psi_{n'}$.

[4] W. Heitler, reference 1, paragraphs 6 and 7.

Employing the customary expressions for the matrix elements of emission and absorption of the virtual light quanta,[5] neglecting recoil energies of the electrons, using the completeness relations for the matter wave functions, and adding over the possible polarization directions of the virtual light quanta, as well as over all possible permutations of the order of their emission and absorption, one obtains for the matrix element (30):

$$\int \psi_i^* \left\{ \sum_{\rho\rho'} \frac{16\pi^2 e^4}{m^3} \frac{\exp(i\mathbf{k}_\rho \cdot \mathbf{r}_{12}) \exp(i\mathbf{k}_{\rho'} \cdot \mathbf{r}_{13})(\mathbf{p}_2 - \mathbf{k}_\rho(\mathbf{p}_2\cdot\mathbf{k}_\rho)/k_\rho^2)(\mathbf{p}_3 - \mathbf{k}_{\rho'}(\mathbf{p}_3\cdot\mathbf{k}_{\rho'})/k_{\rho'}^2)}{\nu_\rho^2 \nu_{\rho'}^2} \right\} \psi_f d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3. \quad (31)$$

Thus, the equivalent action-at-a-distance three-body potential, H', becomes

$$\left(\nu_\rho = ck_\rho;\quad \sum_\rho f(\mathbf{k}_\rho) \rightarrow \frac{1}{(2\pi)^3}\int f(\mathbf{k}_\rho) d\mathbf{k}_\rho\right)$$

$$H'(r_{12}, r_{13}) = \sum_{\rho,\rho'} \frac{16\pi^2 e^4}{m^3} \frac{\exp(i\mathbf{k}_\rho \cdot \mathbf{r}_{12}) \exp(i\mathbf{k}_{\rho'} \cdot \mathbf{r}_{13})(\mathbf{p}_2 - \mathbf{k}_\rho(\mathbf{p}_2\cdot\mathbf{k}_\rho)/k_\rho^2)(\mathbf{p}_3 - \mathbf{k}_{\rho'}(\mathbf{p}_3\cdot\mathbf{k}_{\rho'})/k_{\rho'}^2)}{\nu_\rho^2 \nu_{\rho'}^2}$$

$$= \frac{1}{4\pi^4} \frac{e^4}{m^3 c^4} \int \frac{\exp(i\mathbf{k}_\rho \cdot \mathbf{r}_{12})}{k_\rho^2} \left(\mathbf{p}_2 - \frac{\mathbf{k}_\rho(\mathbf{p}_2\cdot\mathbf{k}_\rho)}{k_\rho^2}\right) d\mathbf{k}_\rho \int \frac{\exp(i\mathbf{k}_{\rho'} \cdot \mathbf{r}_{13})}{k_{\rho'}^2} \left(\mathbf{p}_3 - \frac{\mathbf{k}_{\rho'}(\mathbf{p}_3\cdot\mathbf{k}_{\rho'})}{k_{\rho'}^2}\right) d\mathbf{k}_{\rho'}. \quad (32)$$

Now

$$\int \frac{\exp(i\mathbf{k}_\rho \cdot \mathbf{r}_{12})}{k_\rho^2} \left(\mathbf{p}_2 - \frac{\mathbf{k}_\rho(\mathbf{p}_2\cdot\mathbf{k}_\rho)}{k_\rho^2}\right) d\mathbf{k}_\rho = \pi^2 \frac{1}{r_{12}} \left(\mathbf{p}_2 + \frac{\mathbf{r}_{12}(\mathbf{p}_2\cdot\mathbf{r}_{12})}{r_{12}^2}\right),$$

whence

$$H'(r_{12}, r_{13}) = \frac{e^4}{4m^3c^4} \left(\frac{\mathbf{p}_2}{r_{12}} + \frac{\mathbf{r}_{12}(\mathbf{p}_2\cdot\mathbf{r}_{12})}{r_{12}^3}\right) \left(\frac{\mathbf{p}_3}{r_{13}} + \frac{\mathbf{r}_{13}(\mathbf{p}_3\cdot\mathbf{r}_{13})}{r_{13}^3}\right), \quad (33)$$

which is just the term obtained in the classical treatment, above.[6] (Cf. Eq. (15).)

If one considers three-body potentials arising from the same type of transition as before, except that $-\boldsymbol{\mu}_2\cdot\text{curl}_2\,\mathbf{A}(\mathbf{r}_2)$, and $-\boldsymbol{\mu}_3\cdot\text{curl}_3\,\mathbf{A}(\mathbf{r}_3)$ replace $-(e\mathbf{p}_2/cm)\cdot\mathbf{A}(\mathbf{r}_2)$, and $-(e\mathbf{p}_3/cm)\cdot\mathbf{A}(\mathbf{r}_3)$, one obtains "spin-spin" three-body potentials, which again, are just the same as those obtained in classical theory. (Cf. Eq. (23).)

(b) Specifically quantum-mechanical many-body potentials

Consider, again, a system of three electrons and the radiation field. In addition to the transitions discussed above, the following type of process also takes place: one electron emits two virtual light quanta, *in succession;* one of these quanta is absorbed by the second electron, the other by the third. The perturbation energy responsible for any one of these single absorptions or emissions is either

$$-\boldsymbol{\mu}_k\cdot\text{curl}_k\,\mathbf{A}(\mathbf{r}_k),\quad \text{or}\quad -(e\mathbf{p}_k/cm)\cdot\mathbf{A}(\mathbf{r}_k).$$

Only the first of these perturbations will be treated explicitly, since in the mesotron field theory of nuclear interaction, which has a formalism almost identical with that of electromagnetic theory, the term analogous to

$$-\boldsymbol{\mu}_k\cdot\text{curl}_k\,\mathbf{A}(\mathbf{r}_k),$$

gives by far the largest effect.

The matrix element corresponding to the transition described in the preceding paragraph, is

$$\sum_{n,n',n''} \frac{(i|\boldsymbol{\mu}_1\cdot\text{curl}_1\,\mathbf{A}(\mathbf{r}_1)|n)(n|\boldsymbol{\mu}_1\cdot\text{curl}_1\,\mathbf{A}(\mathbf{r}_1)|n')(n'|\boldsymbol{\mu}_2\cdot\text{curl}_2\,\mathbf{A}(\mathbf{r}_2)|n'')(n''|\boldsymbol{\mu}_3\cdot\text{curl}_3\,\mathbf{A}(\mathbf{r}_3)|f)}{(E_i-E_n)(E_i-E_{n'})(E_i-E_{n''})}, \quad (34)$$

[5] W. Heitler, reference 1, pp. 95 and 96, Eqs. (12) and (13).
[6] A symmetrical expression in the indices 1, 2, 3, is obtained by interchanging the roles which the particles play in the transition process.

The initial and final states, i and f, are characterized by no virtual light quanta present and matter wave functions ψ_i and ψ_f; the intermediate states have, respectively, one, two, and one virtual light quanta present, and matter wave functions ψ_n, $\psi_{n'}$, $\psi_{n''}$.

The matrix element of $-\mathbf{\mu}_1\cdot\mathrm{curl}_1\,\mathbf{A}(\mathbf{r}_1)$ for the emission of a light quantum of frequency ν_ρ, and the transition of the three electrons, from ψ_i to ψ_n, is:

$$-\int\psi_i^*\mathbf{\mu}_1\cdot\mathrm{curl}_1\left((4\pi c^2)^{\frac{1}{2}}(\hbar/2\nu_\rho)^{\frac{1}{2}}\mathbf{a}_\rho\exp(i\mathbf{k}_\rho\cdot\mathbf{r}_1)\right)\times\psi_n d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \quad (35)$$

and similarly for the other emissions and absorptions.

Substituting (35) into (34), neglecting recoil energies, and making use of the completeness relations for the matter wave functions, one obtains for (34):

$$-\int\psi_i^*\left\{\frac{4\pi^2c^4}{\hbar}\left(\mathbf{\mu}_1\cdot\mathrm{curl}_1\sum_\rho\mathbf{a}_\rho\exp(i\mathbf{k}_\rho\cdot\mathbf{r}_1)\mathbf{\mu}_2\cdot\mathrm{curl}_2(\mathbf{a}_\rho\exp(-i\mathbf{k}_\rho\cdot\mathbf{r}_2)\right)\right.$$
$$\left.\times\left(\mathbf{\mu}_1\cdot\mathrm{curl}_1\sum_{\rho'}\frac{\mathbf{a}_{\rho'}\exp(i\mathbf{k}_{\rho'}\cdot\mathbf{r}_1)\mathbf{\mu}_3\cdot\mathrm{curl}_3(\mathbf{a}_{\rho'}\exp(-i\mathbf{k}_{\rho'}\cdot\mathbf{r}_3))}{(\nu_\rho\nu_{\rho'})(\nu_\rho(\nu_\rho+\nu_{\rho'})\nu_{\rho'})}\right)\right\}\psi_f d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3. \quad (36)$$

Summing over the possible polarization directions of the virtual light quanta, one obtains

$$-\int\psi_i^*\left\{\frac{4\pi^2c^4}{\hbar}\mathbf{\mu}_1\cdot\mathrm{curl}_1\sum_\rho\mathrm{curl}_2\,(\mathbf{\mu}_2\cdot\exp(i\mathbf{k}_\rho\cdot\mathbf{r}_{12})\mathbf{\mu}_1\cdot\mathrm{curl}_1\sum_{\rho'}\frac{\mathrm{curl}_3\,(\mathbf{\mu}_3\exp(i\mathbf{k}_{\rho'}\cdot\mathbf{r}_{13})}{\nu_\rho^2\nu_{\rho'}^2(\nu_\rho+\nu_{\rho'})}\right\}\psi_f d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3. \quad (37)$$

The equivalent action-at-a-distance three-body potential, corresponding to a process where ν_ρ is the first to be emitted, and first to be absorbed, is thus given by the quantity inside the curly brackets in (37). Taking into account all possible orders of emission and absorption of the virtual light quanta ν_ρ, $\nu_{\rho'}$, one obtains, for the three-body potential, $H''(r_{12}, r_{13})$:

$$H''(r_{12},r_{13})=-\frac{4\pi^2}{\hbar c}\left[\mathbf{\mu}_1\cdot\mathrm{curl}_1\sum_\rho\frac{\mathrm{curl}_2\,(\mathbf{\mu}_2\exp(i\mathbf{k}_\rho\cdot\mathbf{r}_{12}))}{\nu_\rho^3/c^3}\right]\times\left[\mathbf{\mu}_1\cdot\mathrm{curl}_1\sum_{\rho'}\frac{\mathrm{curl}_3\,(\mathbf{\mu}_3\exp(i\mathbf{k}_{\rho'}\cdot\mathbf{r}_{13}))}{\nu_{\rho'}^2/c^2}\right] \quad (38)$$

+same term with indices 2 and 3 interchanged.

With $\nu_\rho/c=k_\rho$, and $\sum_\rho f(\mathbf{k}_\rho)\rightarrow\frac{1}{(2\pi)^3}\int f(\mathbf{k}_\rho)d\mathbf{k}_\rho$ the first bracket becomes

$$\frac{1}{(2\pi)^2}\mathbf{\mu}_1\cdot\mathrm{curl}_1\left(\frac{\mathbf{r}_{12}\times\mathbf{\mu}_2}{r_{12}^2}\right), \quad (38a)$$

while the second bracket is equal to

$$-\frac{1}{4\pi}\mathbf{\mu}_1\cdot\mathrm{curl}_1^2\left(\frac{\mathbf{\mu}_3}{r_{13}}\right), \quad (38b)$$

being, apart from the numerical factor $1/4\pi$, just the spin-spin two-body potential between the first and third particle.

Thus,

$$H''(r_{12},r_{13})=\frac{1}{2\pi\hbar c}\left[\mathbf{\mu}_1\cdot\mathrm{curl}_1\left(\frac{\mathbf{r}_{12}\times\mathbf{\mu}_2}{r_{12}^2}\right)\right]\times\left[\mathbf{\mu}_1\cdot\mathrm{curl}_1^2\left(\frac{\mathbf{\mu}_3}{r_{13}}\right)\right] \quad (39)$$

+same term with indices 2 and 3 interchanged $\cong\mu^4/\hbar cr^5$.

Orders of Magnitude of Electromagnetic Many-Body Potentials in Atomic Systems

The magnitudes of the many-body potentials, in particular of the three-body potentials, will now be estimated. Suppose the three electrons are orbital electrons in an atom. Then, on the average, the distances $r_{kj}\cong r$, between the electrons are of the order of Bohr radii, and are related to the electronic velocities, v, by: $r\cong\hbar/mv$; the electronic magnetic moments μ are $\cong e\hbar/mc\cong erv/c$. Thus, from Eqs. (15), (16), (33), the relation between the magnitudes of the individual terms in the velocity-dependent three-body potential and in the Coulomb two-body potential, is

$$H'(\text{three-body vel.-depend.}) \cong \frac{e^2}{mc^2r}\frac{v^2}{c^2}\frac{e^2}{r}$$

$$\cong \frac{1}{137}\left(\frac{v}{c}\right)^3 H(\text{Coulomb}). \quad (40)$$

Further, from Eq. (23),

$$H'(\text{three-body spin-spin}) \cong \frac{e^2}{mc^2r}\frac{\mu^2}{r^3}$$

$$\cong \frac{e^2}{mc^2r}\left(\frac{\hbar/mc}{r}\right)^2\frac{e^2}{r} \cong \frac{1}{137}\left(\frac{v}{c}\right)^3 H(\text{Coulomb}). \quad (41)$$

Finally, for the specifically quantum three-body potential (cf. Eq. (39)),

$$H'' \cong \frac{\mu^4}{\hbar c}\frac{1}{r^5} \cong \frac{e^2}{\hbar c}\left(\frac{\hbar/mc}{r}\right)^4\frac{e^2}{r}$$

$$\cong \frac{1}{137}\left(\frac{v}{c}\right)^4 H(\text{Coulomb}). \quad (42)$$

It is therefore clear that the description of the electromagnetic interaction of electrons in atomic systems, by means of action-at-a-distance two-body potentials, is an extraordinarily good approximation,[6a] since $v/c \cong Z/137$.[6b]

Mesotron Field Theory of Nuclear Interaction

(a) Elimination of field variables for the "classical" mesotron field

The mesotron theory of the interaction of nuclear heavy particles[7] (protons and neutrons) is closely modeled upon the electromagnetic theory of electron interaction. Here the variables of the system also fall into two categories: those describing the motion of the heavy particles and those describing the state of the field. The dynamical relations between these two sets of variables are given by equations analogous to the Maxwell equations and the Lorentz force formula; the heavy particles act as sources of the field, and the field in turn affects the motion of the heavy particles. The characteristic difference between this theory and that of Maxwell is the appearance of a fundamental length, R, in the field equations; a consequence of this new constant in the quantized form of the theory, is the existence of "quanta" of rest mass, $m_0 = \hbar/cR$.

The mesotron field due to mesotron-charge, mesotron-current, mesotron-magnetization and mesotron-polarization densities of the heavy particles,[8] ρ, $\boldsymbol{I}$, $\boldsymbol{M}$, $\boldsymbol{P}$ is specified by two six-vectors[9] $\boldsymbol{E}$, $\boldsymbol{B}$; $\boldsymbol{E}^*$, $\boldsymbol{B}^*$, and two four-vectors Φ, $\boldsymbol{A}$; Φ^*, $\boldsymbol{A}^*$ satisfying the equations:

$$\boldsymbol{B} = \text{curl } \boldsymbol{A}, \quad (43)$$

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \text{grad. } \Phi, \quad (44)$$

$$\text{curl } \boldsymbol{B} = \frac{4\pi \boldsymbol{I}}{c} + \frac{4\pi}{c}\frac{\partial \boldsymbol{P}}{\partial t} + 4\pi \text{ curl } \boldsymbol{M} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} - \frac{1}{R^2}\boldsymbol{A}, \quad (45)$$

$$\text{div. } \boldsymbol{E} = 4\pi\rho - 4\pi \text{ div. } \boldsymbol{P} - \frac{1}{R^2}\Phi. \quad (46)$$

Taking the div. of (45) and the time derivative of (46), one obtains:

$$0 = \text{div. } \boldsymbol{I} + \frac{\partial \rho}{\partial t} - \frac{1}{R^2}\left(\frac{\partial \Phi}{\partial t} + \text{div. } \boldsymbol{A}\right) \quad (47)$$

[6a] See, however, the Addendum.

[6b] Z is the effective nuclear charge for the electron in question.

[7] The "vector" mesotron field theory will be used, since it gives the correct sign for the neutron-proton two-body potential. For a general discussion of mesotron theory, see H. Yukawa, Proc. Phys.-Math. Soc. Jap. **17**, 48 (1935); **19**, 1084 (1937); **20**, 320 (1938); also H. Bhabha, Proc. Roy. Soc. **166**, 501 (1938); N. Kemmer, Proc. Roy. Soc. **166**, 127 (1938); Frohlich, Heitler and Kemmer, Proc. Roy. Soc. **166**, 154 (1938). The use of a linear combination of the "scalar," "vector," "pseudo vector" and "pseudo scalar" charged mesotron fields, and/or of uncharged mesotrons, would not essentially alter any of the results on many-body interactions obtained below. (See section .n present paper on general field theory.)

It may be pointed out that the singularities in the mesotron field equations are exactly of the same type as the singularities in the electromagnetic field equations. Hence, the mesotron theory convergence difficulties are no better, and no worse, than those of electromagnetism; cf, H. Bhabha, Nature **143**, 276 (1939).

[8] $\rho = |\rho|\Pi$; $\rho^* = |\rho|\Pi^*$ where the Π and Π^* are operators (complex conjugate to each other) changing the heavy particle from a proton to a neutron, and from a neutron to a proton, respectively. Similarly for $\boldsymbol{I}$, $\boldsymbol{M}$, $\boldsymbol{P}$.

[9] Here the asterisks denote complex conjugates.

and so, if charge-current is to be conserved, the potentials must satisfy the "Lorentz" condition.

$$\frac{\partial\Phi}{\partial t}+\text{div. }A=0. \tag{48}$$

Second-order equations for the potentials may be obtained in the usual way:

$$\nabla^2\Phi-\frac{1}{R^2}\Phi-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}=-4\pi(\rho-\text{div. }P), \tag{49}$$

$$\nabla^2 A-\frac{1}{R^2}A-\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}=-4\pi\left(\frac{I}{c}+\text{curl }M+\frac{1}{c}\frac{\partial P}{\partial t}\right). \tag{50}$$

Finally, the equation of motion of a heavy particle in presence of potentials Φ, A is[10]

$$M\mathbf{a}_k=\Bigg\{g_k^*\left(-\text{grad. }\Phi-\frac{1}{c}\frac{\partial A}{\partial t}\right)+g_k^*\frac{\mathbf{v}_k}{c}\times\text{curl }A+\text{grad. }(\mathfrak{u}_k^*\cdot\text{curl }A)+\text{grad.}\left(p_k^*\cdot\left(-\text{grad. }\Phi-\frac{1}{c}\frac{\partial A}{\partial t}\right)\right)+\text{comp. conj.}\Bigg\},\quad k=1,\cdots,n, \tag{51}$$

where $g_k^*=g\Pi_k^*$, $\mathfrak{u}_k^*=\mathfrak{u}\Pi_k^*$, $p^*=p\Pi_k^*$ (see footnote 8) with g, $\mathfrak{u}$, p, the mesotron-charge, mesotron-magnetic moment, mesotron-electric moment of the heavy particle. Terms involving $p\cong\mathbf{v}/c\times\mathfrak{u}$ will be consistently omitted in what follows, since they do not essentially affect any of the subsequent results.

The potentials Φ, A arise from the other heavy particles. As in electromagnetic theory, to obtain an action-at-a-distance description, one attempts to express Φ, A in terms of the instantaneous relative heavy-particle coordinates. Approximate solutions of (49), (50), for point sources, are[11]

$$\Phi(\mathbf{r}_k,t)=\sum_j g_j\frac{e^{-r_{kj}/R}}{r_{kj}}+\text{terms in }\frac{\mathbf{v}_j^2}{c^2},\ \frac{\mathbf{a}_j}{c^2},\ \text{etc.} \tag{52}$$

$$A(\mathbf{r}_k,t)=\sum_j\frac{g_j\mathbf{v}_j}{c}\frac{e^{-r_{kj}/R}}{r_{kj}}+\sum_j\mathfrak{u}_j\times\text{grad.}_j\left(\frac{e^{-r_{kj}/R}}{r_{kj}}\right)+\text{terms in }\frac{\mathbf{v}_j^2}{c^2},\ \frac{\mathbf{a}_j}{c^2},\ \frac{d\mathfrak{u}_j}{dt},\ \frac{d^2\mathfrak{u}_j}{dt^2},\ \text{etc.} \tag{53}$$

The equations of motion (51), with Φ and A expressed by means of the sums in (52), (53), are derivable from the following Lagrangian and Hamiltonian:

$$L=\sum_k\frac{M\mathbf{v}_k^2}{2}+\Bigg\{-\tfrac{1}{2}\sum_k g_k^*\Phi(\mathbf{r}_k)+\tfrac{1}{2}\sum_k\frac{g_k^*}{c}\mathbf{v}_k\cdot A(\mathbf{r}_k)+\tfrac{1}{2}\sum_k\mathfrak{u}_k^*\cdot\text{curl}_k\,A(\mathbf{r}_k)+\text{c.c.}\Bigg\}, \tag{54}$$

$$H=\sum_k\frac{p_k^2}{2M}+\Bigg\{\tfrac{1}{2}\sum_k g_k^*\Phi(\mathbf{r}_k)-\tfrac{1}{2}\sum_k\frac{g_k^*}{c}\frac{p_k}{M}\cdot A(\mathbf{r}_k)-\tfrac{1}{2}\sum_k\mathfrak{u}_k^*\cdot\text{curl}_k\,A(\mathbf{r}_k)+\text{c.c.}\Bigg\}. \tag{55}$$

In H, the A's are to be expressed in terms of the $\mathbf{p}$'s by means of the equations:

$$A(\mathbf{r}_k)=\sum_j\frac{g_j}{cM}\frac{(\mathbf{p}_j-((g_j^*/c)A(\mathbf{r}_j)+\text{c.c.}))e^{-r_{kj}/R}}{r_{kj}}+\sum_j\mathfrak{u}_j\times\text{grad.}_j\left(\frac{e^{-r_{kj}/R}}{r_{kj}}\right)=\sum_j\frac{g_j}{c}\frac{\mathbf{p}_j}{M}\frac{e^{-r_{kj}/R}}{r_{kj}}+\sum_j\mathfrak{u}_j\times\text{grad.}_j\left(\frac{e^{-r_{kj}/R}}{r_{kj}}\right)-\sum_{j,l}\frac{g_jg_j^*e^{-r_{kj}/R}}{Mc^2r_{kj}}\frac{g_l\mathbf{p}_l}{Mc}\frac{e^{-r_{jl}/R}}{r_{jl}}-\sum_{j,l}\frac{g_jg_j^*e^{-r_{kj}/R}}{Mc^2r_{kj}}\mathfrak{u}_l\times\text{grad.}_l\left(\frac{e^{-r_{jl}/R}}{r_{jl}}\right)+\text{terms of higher order in }g^2/Mc^2r_{kj}. \tag{56}$$

[10] The classical equation of motion (51) can be considered in quantum mechanics, as an equation of motion between operators; in this sense it is equivalent to the wave equation (11) given by Bhabha, reference 7.

[11] These solutions, including explicit expressions for the terms in $\mathbf{v}_j^2/c^2$, $\mathbf{a}_j/c^2$, $d\mathfrak{u}_j/dt$, $d^2\mathfrak{u}_j/dt^2$ are obtained in Appendix I. The only terms which are treated in the text are those explicitly written out in Eqs, (52), (53), For a discussion of the acceleration-dependent terms, see the Addendum.

Substituting for $A(\mathbf{r}_k)$, one obtains for the Hamiltonian (55):

$$
\begin{aligned}
H=\sum_k \frac{\mathbf{p}_k^2}{2M}+\Bigg\{&\left[\tfrac{1}{2}\sum_{k,\,j}\frac{g_k^* g_j}{r_{kj}}e^{-r_{kj}/R}\right]\\
&+\left[-\tfrac{1}{2}\sum_{k,\,j}\boldsymbol{\mu}_k^*\cdot\operatorname{curl}_k\left(\boldsymbol{\mu}_j\times\operatorname{grad.}_j\left(\frac{e^{-r_{kj}/R}}{r_{kj}}\right)\right)\right]\\
&+\left[-\tfrac{1}{2}\sum_{k,\,j}\boldsymbol{\mu}_k^*\cdot\operatorname{curl}_k\left(\frac{g_j}{c}\frac{\mathbf{p}_j}{M}\frac{e^{-r_{kj}/R}}{r_{kj}}\right)\right.\\
&\left.-\tfrac{1}{2}\sum_{k,\,j}\frac{g_k^*}{c}\frac{\mathbf{p}_k}{M}\cdot\boldsymbol{\mu}_j\times\operatorname{grad.}_j\left(\frac{e^{-r_{kj}/R}}{r_{kj}}\right)\right]\\
&+\left[-\tfrac{1}{2}\sum_{k,\,j}\frac{g_k^* g_j}{M^2c^2}\frac{\mathbf{p}_k\cdot\mathbf{p}_j}{r_{kj}}e^{-r_{kj}/R}\right]\\
&+\left[\tfrac{1}{2}\sum_{k,\,j,\,l}\frac{g_j g_j^*}{Mc^2}\boldsymbol{\mu}_k^*\cdot\operatorname{curl}_k\left(\frac{e^{-r_{kj}/R}}{r_{kj}}\boldsymbol{\mu}_l\right.\right.\\
&\left.\left.\times\operatorname{grad.}_l\left(\frac{e^{-r_{jl}/R}}{r_{jl}}\right)\right)\right]\\
&+\left[\tfrac{1}{2}\sum_{k,\,j,\,l}\frac{g_k^* g_j g_j^* g_l}{M^3c^4}\frac{\mathbf{p}_k\cdot\mathbf{p}_l e^{-r_{kj}/R}e^{-r_{jl}/R}}{r_{kj}r_{jl}}\right]\\
&+\left[\tfrac{1}{2}\sum_{k,\,j,\,l}\frac{g_k^* g_j g_j^*}{M^2c^3}\mathbf{p}_k\left(\frac{e^{-r_{kj}/R}}{r_{kj}}\right)\cdot\boldsymbol{\mu}_l\times\operatorname{grad.}_l\left(\frac{e^{-r_{jl}/R}}{r_{jl}}\right)\right.\\
&\left.+\tfrac{1}{2}\sum_k\frac{g_j g_j^* g_l}{M^2c^3}\boldsymbol{\mu}_k^*\cdot\operatorname{curl}_k\left(\mathbf{p}_l\frac{e^{-r_{kj}/R}}{r_{kj}}\frac{e^{-r_{jl}/R}}{r_{jl}}\right)\right]\\
&+\text{terms of higher order in }\frac{g^2}{Mc^2r}+\text{c.c.}\Bigg\} \qquad (57)
\end{aligned}
$$

$\equiv H$(kinetic) $+H$(two-body "Coulomb")

$+H$(two-body spin-spin)

$+H$(two-body spin orbit)

$+H$(two-body vel.-depend.)

$+H$(three-body spin-spin)

$+H$(three-body vel.-depend)

$+H$(three-body spin-orbit).

The two-body "Coulomb" and two-body spin-spin interaction Hamiltonians have already been discussed by the authors of reference 7.[11a] The order of magnitude of the other two-body interactions, as well as of all of the three-body interactions, will be given in a later section.

(b) Quantum mesotron theory

In the preceding section, the unquantized mesotron field was approximately eliminated, and an action-at-a-distance dynamical description of the heavy particles obtained in terms of a Hamiltonian (Eq. (57)) which could then be quantized. The elimination of the *quantized* mesotron field gives results similar to the corresponding treatment in electromagnetic theory; on the one hand the classical Hamiltonian (57) is confirmed, and on the other hand, additional specifically quantum many-body potentials are obtained.

Since the Hamiltonian formalism of the quantized mesotron theory is somewhat different from the formalism of electrodynamics (due to the fundamental length), it will be discussed briefly.

In the Fourier representation of the field:

$$
\begin{aligned}
\Phi(\mathbf{r},t)&=\sum_\sigma a_\sigma(t)\Phi_\sigma(\mathbf{r})\\
&=\sum_\sigma a_\sigma(t)(4\pi c^2)^{\frac{1}{2}}\exp(i\mathbf{k}_\sigma\cdot\mathbf{r}), \qquad (58)
\end{aligned}
$$

$$
A_L(\mathbf{r},t)=\sum_\sigma q_\sigma(t)A_\sigma(r)=\sum_\sigma q_\sigma(t)\frac{1}{k_\sigma}\operatorname{grad.}\Phi_\sigma(\mathbf{r});
$$

$$
\operatorname{curl} A_L=0, \qquad (59)
$$

$$
\begin{aligned}
A_T(\mathbf{r},t)&=\sum_\rho q_\rho(t)A_\rho(r)\\
&=\sum_\rho q_\rho(t)(4\pi c^2)^{\frac{1}{2}}\mathbf{a}_\rho\exp(i\mathbf{k}_\rho\cdot\mathbf{r});
\end{aligned}
$$

$$
\operatorname{div.} A_T=0,\ |\mathbf{a}_\rho|^2=1, \qquad (60)
$$

$$
A(\mathbf{r},t)=A_L(\mathbf{r},t)+A_T(\mathbf{r},t). \qquad (61)
$$

The potential equations (49), (50), and the Lorentz condition (48), become:

$$
\ddot{a}_\sigma+\nu_\sigma^2 a_\sigma=\sum_k g_k\Phi_\sigma^*(\mathbf{r}_k), \qquad (62)
$$

[11a] The procedure of solving the equations of the mesotron potentials in order to obtain two-body Coulomb and two-body spin-spin interactions was introduced by Yukawa, reference 7. Recently these interactions have been derived by Stueckelberg by a method of canonical transformation; Phys. Rev. **54**, 889 (1938). A treatment similar to Stueckelberg's has been reported by E. Feenberg, December meeting of the American Physical Society, 1938.

$$\ddot{q}_\sigma+\nu_\sigma^2 q_\sigma=\sum_k \frac{g_k}{c}\mathbf{v}_k\cdot \mathbf{A}_\sigma^*(\mathbf{r}_k), \tag{63}$$

$$\ddot{q}_\rho+\nu_\rho^2 q_\rho=\sum_k \frac{g_k}{c}\mathbf{v}_k\cdot \mathbf{A}_\rho^*(\mathbf{r}_k)+\sum_k \boldsymbol{\mu}_k^*\cdot \mathrm{curl}_k\, \mathbf{A}_\rho^*(\mathbf{r}_k), \tag{64}$$

$$\dot{a}_\sigma=ck_\sigma q_\sigma, \tag{65}$$

$$\nu^2=c^2k^2+\frac{c^2}{R^2}. \tag{66}$$

The equations of motion of the heavy particles (51), are:

$$M\mathbf{a}_k=\left\{g_k^*\left(-\mathrm{grad.}_k \sum_\sigma a_\sigma\Phi_\sigma(\mathbf{r}_k)-\frac{1}{c}\sum_\sigma \dot{q}_\sigma \mathbf{A}_\sigma(\mathbf{r}_k)-\frac{1}{c}\sum_\rho \dot{q}_\rho \mathbf{A}_\rho(\mathbf{r}_k)\right)+\frac{g_k^*}{c}\mathbf{v}_k\times \mathrm{curl}_k \sum_\rho q_\rho \mathbf{A}_\rho(\mathbf{r}_k),\right.$$
$$\left.+\mathrm{grad.}_k\left(\boldsymbol{\mu}_k^*\cdot \mathrm{curl}_k \sum_\rho q_\rho \mathbf{A}_\rho(\mathbf{r}_k)\right)+\mathrm{c.c.}\right\}, \tag{67}$$
$$k=1,\cdots,n.$$

Inspection indicates that the above equations of motion, for field and matter, are derivable from the following Lagrangian and Hamiltonian:[12]

$$L=\sum_\rho(\dot{q}_\rho^*\dot{q}_\rho-\nu_\rho^2 q_\rho^* q_\rho)+\sum_\sigma(\dot{q}_\sigma^*\dot{q}_\sigma-\nu_\sigma^2 q_\sigma^* q_\sigma)-\sum_\sigma(\dot{a}_\sigma^*\dot{a}_\sigma-\nu_\sigma^2 a_\sigma^* a_\sigma)+\sum_k\frac{M\mathbf{v}_k^2}{2}$$
$$+\left\{\sum_k\left(-g_k^*\sum_\sigma a_\sigma\Phi_\sigma(\mathbf{r}_k)+\frac{g_k^*}{c}\mathbf{v}_k\cdot\sum_\sigma q_\sigma\mathbf{A}_\sigma(\mathbf{r}_k)+\frac{g_k^*}{c}\mathbf{v}_k\cdot\sum_\rho q_\rho\mathbf{A}_\rho(\mathbf{r}_k)+\boldsymbol{\mu}_k^*\cdot\mathrm{curl}_k\sum_\rho q_\rho\mathbf{A}_\rho(\mathbf{r}_k)\right)+\mathrm{c.c.}\right\}, \tag{68}$$

$$H=\sum_\rho(p_\rho^*p_\rho+\nu_\rho^2 q_\rho^* q_\rho)+\sum_\sigma(p_\sigma^*p_\sigma+\nu_\sigma^2 q_\sigma^* q_\sigma)-\sum_\sigma(b_\sigma^*b_\sigma+\nu_\sigma^2 a_\sigma^* a_\sigma)+\sum_k\frac{1}{2M}$$
$$\times\left(\mathbf{p}_k-\left[\frac{g_k^*}{c}\left(\sum_\sigma q_\sigma\mathbf{A}_\sigma(\mathbf{r}_k)+\sum_\rho q_\rho\mathbf{A}_\rho(\mathbf{r}_k)\right)+\mathrm{c.c.}\right]\right)^2+\left(\sum_k g_k^*\sum_\sigma a_\sigma\Phi_\sigma(\mathbf{r}_k)+\mathrm{c.c.}\right)$$
$$-\left(\sum_k\boldsymbol{\mu}_k^*\cdot\mathrm{curl}_k\sum_\rho q_\rho\mathbf{A}_\rho(\mathbf{r}_k)+\mathrm{c.c.}\right). \tag{69}$$

Now, a peculiarity of the system of Eqs. (62)–(67) is that the variables q_σ and a_σ are not independent, the connection between them being given by the Lorentz condition (65). Making use of this relation and of (62), one can completely eliminate the variables a_σ from the equation of motion (67), obtaining:[13]

$$M\mathbf{a}_k=\left\{g_k^*\left(-\mathrm{grad.}_k\sum_j g_j\frac{e^{-r_{kj}/R}}{r_{kj}}-\frac{1}{c}\sum_\sigma\frac{c^2}{R^2\nu_\sigma^2}\dot{q}_\sigma\mathbf{A}_\sigma(\mathbf{r}_k)-\frac{1}{c}\sum_\rho\dot{q}_\rho\mathbf{A}_\rho(\mathbf{r}_k)\right)+\frac{g_k^*}{c}\mathbf{v}_k\times\mathrm{curl}_k\sum_\rho q_\rho\mathbf{A}_\rho(\mathbf{r}_k)\right.$$
$$\left.+\mathrm{grad.}_k\left(\boldsymbol{\mu}_k^*\cdot\mathrm{curl}_k\sum_\rho q_\rho\mathbf{A}_\rho(\mathbf{r}_k)\right)+\mathrm{c.c.}\right\},$$
$$k=1,\cdots,n. \tag{70}$$

$$\ddot{q}_\sigma+\nu_\sigma^2 q_\sigma=\sum_k\frac{g_k}{c}\mathbf{v}_k\cdot\mathbf{A}_\sigma^*(\mathbf{r}_k), \tag{71}$$

$$\ddot{q}_\rho+\nu_\rho^2 q_\rho=\sum_k\frac{g_k}{c}\mathbf{v}_k\cdot\mathbf{A}_\rho^*(\mathbf{r}_k)+\sum_k\boldsymbol{\mu}_k\cdot\mathrm{curl}_k\,\mathbf{A}_\rho^*(\mathbf{r}_k). \tag{72}$$

The new set of equations of motion: (70), (71), (72), as may be directly verified, are derivable from the following Lagrangian and Hamiltonian:[14]

[12] In deriving the equations of motion (62)–(67) from the Lagrangian (68), q_ρ, q_ρ^*, q_σ, q_σ^*, a_σ, a_σ^*, $\mathbf{r}_k$ are to be treated as independent variables. Eq. (65) must then be assumed as an auxiliary relation. Also $p_\rho\equiv\partial L/\partial\dot{q}_\rho$, etc.

[13] Equation (70) is derived in Appendix II.

[14] In the limit $R\rightarrow\infty$, the variables describing the longitudinal field disappear completely from the Lagrangian (73), and the equations of motion (70). Since in this limit, the mesotron field equations are identical with the Maxwell equations (apart from the complex character of the field variables) one obtains the well-known result of electro-

$$L_1=\sum_\rho(\dot{q}_\rho{}^*\dot{q}_\rho-\nu_\rho{}^2q_\rho{}^*q_\rho)+\sum_\sigma\frac{c^2}{R^2\nu_\sigma{}^2}(\dot{q}_\sigma{}^*\dot{q}_\sigma-\nu_\sigma{}^2q_\sigma{}^*q_\sigma)$$

$$+\sum_k\frac{M\mathbf{v}_k{}^2}{2}+\Bigg\{-\frac{1}{2}\sum_{k,j}g_k{}^*g_j\frac{e^{-r_{kj}/R}}{r_{kj}}$$

$$+\sum_k\frac{g_k{}^*}{c}\mathbf{v}_k\cdot\sum_\sigma\frac{c^2}{R^2\nu_\sigma{}^2}q_\sigma\boldsymbol{A}_\sigma(\mathbf{r}_k)+\sum_k\frac{g_k{}^*}{c}\mathbf{v}_k\cdot\sum_\rho q_\rho\boldsymbol{A}_\rho(\mathbf{r}_k)$$

$$+\sum_k\boldsymbol{\mu}_k{}^*\cdot\mathrm{curl}_k\sum_\rho q_\rho\boldsymbol{A}_\rho(\mathbf{r}_k)+\mathrm{c.c.}\Bigg\},\quad(73)$$

$$H_1=\Bigg\{\sum_\rho(p_\rho{}^*p_\rho+\nu_\rho{}^2q_\rho{}^*q_\rho)+\sum_\sigma\bigg(\frac{p_\sigma{}^*p_\sigma}{c^2/R^2\nu_\sigma{}^2}$$

$$+\left(\frac{c^2}{R^2\nu_\sigma{}^2}\right)\nu_\sigma{}^2q_\sigma{}^*q_\sigma\bigg)\Bigg\}+\Bigg\{\sum_k\frac{\mathbf{p}_k{}^2}{2M}$$

$$+\left(\frac{1}{2}\sum_{k,j}g_k{}^*g_j\frac{e^{-r_{kj}/R}}{r_{kj}}+\mathrm{c.c.}\right)\Bigg\}$$

$$+\Bigg\{\Bigg[-\sum_k\frac{g_k{}^*}{c}\frac{\mathbf{p}_k}{M}\cdot\bigg(\sum_\sigma\frac{c^2}{R^2\nu_\sigma{}^2}q_\sigma\boldsymbol{A}_\sigma(\mathbf{r}_k)$$

$$+\sum_\rho q_\rho\boldsymbol{A}_\rho(\mathbf{r}_k)\bigg)+\mathrm{c.c.}\Bigg]+\sum_k\frac{1}{2Mc^2}$$

$$\times\left[g_k{}^*\sum_\sigma\frac{c^2}{R^2\nu_\sigma{}^2}q_\sigma\boldsymbol{A}_\sigma(\mathbf{r}_k)+g_k{}^*\sum_\rho q_\rho\boldsymbol{A}_\rho(\mathbf{r}_k)+\mathrm{c.c.}\right]^2$$

$$-\left[\sum_k\boldsymbol{\mu}_k{}^*\cdot\mathrm{curl}_k\sum_\rho q_\rho\boldsymbol{A}_\rho(\mathbf{r}_k)+\mathrm{c.c.}\right]\Bigg\}$$

$$\equiv H(\text{mesotron-field})+H(\text{heavy-particles})+H(\text{interaction}).\quad(74)$$

Numerically,

$H_1=H=$[Volume Integral of Energy Density of Field+Heavy Particle Kinetic Energy+Magnetic Moment Interaction Energy].

The Hamiltonian H_1, plays a role in the quantized mesotron theory, completely analogous to the role of the Hamiltonian (24) in quantum electrodynamics. It is to be noted that quantum perturbation calculations, in particular, the derivation of action-at-distance two-body and many-body potentials, will not involve that part of the longitudinal interaction already exactly expressed by the "Coulomb" term:

$$\frac{1}{2}\sum_{k,j}g_k{}^*g_j\frac{e^{-r_{kj}/R}}{r_{kj}}+\mathrm{c.c.}$$

magnetism, *viz:* The effect of the longitudinal waves may be completely expressed by means of action-at-a-distance two-body Coulomb potentials.

(c) Many-body potentials obtained in classical mesotron theory

Three-body potentials can now be obtained just as in electrodynamics, by a perturbation treatment of transitions in which one has *simultaneous* emission (by means of the g^2A^2/Mc^2 interaction term) of two virtual mesotrons by the first heavy particle, one of the mesotrons being absorbed by the second heavy particle and the other by the third. The absorptions take place by means of either the $-(g\mathbf{p}/cM)\cdot\boldsymbol{A}$ or the $-\boldsymbol{\mu}\cdot\mathrm{curl}\,\boldsymbol{A}$ interaction terms.

In this way, one obtains exactly the three-body potentials already found in the Hamiltonian of Eq. (57).[15]

(d) Specifically quantum-mechanical many-body potentials

The largest specifically quantum-mechanical three-body potential is obtained by a transition of the following type:

The first heavy particle emits two virtual mesotrons *in succession*; one of these is absorbed by the second heavy particle, and the other by the third. All individual absorptions and emissions are due to the $-\boldsymbol{\mu}\cdot\mathrm{curl}\,\boldsymbol{A}$ interaction term.

Since the expression for the transverse field energy: $\Sigma_\rho(p_\rho{}^*p_\rho+\nu_\rho{}^2q_\rho{}^*q_\rho)$, and for the magnetic moment interaction term:

$$-[\Sigma_k\boldsymbol{\mu}_k{}^*\cdot\mathrm{curl}_k\,\Sigma_\rho q_\rho\boldsymbol{A}_\rho(\mathbf{r}_k)+\mathrm{c.c.}]$$

are formally alike in the electromagnetic and mesotron theories,[16] the calculation of the three-body potential proceeds in the same way as the calculation of the corresponding three-body potential in electrodynamics. (Cf. Eqs. (34) to (38) and accompanying text.)

Thus the specifically quantum three-body potential arising from the transition outlined above

[15] One can also obtain H (two-body spin-spin) and H (two-body spin-orbit) by a second-order quantum perturbation calculation with neglect of heavy particle recoil energies. The results, naturally, are identical with those obtained in Eq. (57).

[16] Compare the Hamiltonians (24) and (74). The complex nature of the field variables in mesotron theory introduces only trivial modifications in this connection.

is: (cf. the analogous electrodynamical Eq. (38)).

$$H''(r_{12}, r_{13}) = \left\{ \frac{-4\pi^2}{\hbar c} \left[\boldsymbol{\mu}_1^* \cdot \text{curl}_1 \sum_\rho \frac{\text{curl}_2 (\boldsymbol{\mu}_2 \exp (i\mathbf{k}_\rho \cdot \mathbf{r}_{12}))}{\nu_\rho^3/c^3} \right] \times \left[\boldsymbol{\mu}_1 \cdot \text{curl}_1 \sum_{\rho'} \frac{\text{curl}_3 (\boldsymbol{\mu}_3^* \exp (i\mathbf{k}_{\rho'} \cdot \mathbf{r}_{13}))}{\nu_{\rho'}^2/c^2} \right] + \text{same term with indices 2 and 3 interchanged} \right\} + \text{c.c.} \quad (75)$$

The characteristic difference between the two theories (the existence of the fundamental length R), now makes itself felt in the circumstance that the frequency ν_ρ is no longer proportional to the wave number k_ρ; the relation between them is in this case:

$$\nu_\rho^2/c^2 = k_\rho^2 + 1/R^2. \quad (76)$$

Converting the sums to integrals by the relation

$$\sum_\rho f(\mathbf{k}_\rho) \rightarrow \frac{1}{(2\pi)^3} \int f(\mathbf{k}_\rho) d\mathbf{k}_\rho$$

one observes that the second square bracket in (75) is equal to:

$$-\frac{1}{4\pi} \boldsymbol{\mu}_1 \cdot \text{curl}_1^2 \left(\boldsymbol{\mu}_3^* \frac{e^{-r_{13}/R}}{r_{13}} \right) \quad (76)$$

and is, apart from the numerical factor $1/4\pi$ just the spin-spin two-body potential between the first and third particles. The first square bracket becomes:

$$\frac{1}{(2\pi)^3} \boldsymbol{\mu}_1^* \cdot \text{curl}_1 \, \boldsymbol{\mu}_2 \times \int \frac{i\mathbf{k}_\rho \exp (i\mathbf{k}_\rho \cdot \mathbf{r}_{12})}{(k_\rho^2 + 1/R^2)^{\frac{3}{2}}} d\mathbf{k}_\rho. \quad (77)$$

Introducing the dimensionless variable, $\mathbf{Q} = \mathbf{k}_\rho r_{12}$, one obtains

$$\frac{1}{(2\pi)^3} \boldsymbol{\mu}_1^* \cdot \text{curl}_1 \frac{\boldsymbol{\mu}_2}{r_{12}} \times \int \frac{i\mathbf{Q} \exp (i\mathbf{Q} \cdot \mathbf{r}_{12}/r_{12})}{[Q^2 + (r_{12}/R)^2]^{\frac{3}{2}}} d\mathbf{Q}.$$

The integral is finite and is of the order of $\mathbf{r}_{12}/r_{12}$ for $r_{12} \cong R$. (For $r_{12} \ll R$, we have the electrodynamic case and the integral is $-4\pi(\mathbf{r}_{12}/r_{12})$; for $r_{12} \gg R$ the integral is very small.) Thus the specifically quantum three-body potential becomes for $r_{12} \cong R$

$$H''(r_{12}, r_{13}) \cong \left\{ \frac{1}{\hbar c} \left[\boldsymbol{\mu}_1^* \cdot \text{curl}_1 \left(\frac{\mathbf{r}_{12} \times \boldsymbol{\mu}_2}{r_{12}^2} \right) \right] \times \left[\boldsymbol{\mu}_1 \cdot \text{curl}_1^2 \left(\frac{\boldsymbol{\mu}_3^* e^{-r_{13}/R}}{r_{13}} \right) \right] + \text{same term with indices 2 and 3 interchanged} \right\} + \text{c.c.} \quad (78)$$

Order of Magnitude of Nuclear Potentials, from Mesotron Field Theory

The orders of magnitude of the nuclear potentials obtained in the mesotron theory will now be estimated. The average distances, $r_{kj} \cong r$, between the nuclear heavy particles, are $\cong R \equiv \hbar/m_0 c$, the average nuclear heavy particle velocities, $v_n \cong \hbar/MR \cong m_0 c/M$. The mesotron charge g determines the magnitude of the various interactions. The mesotron-magnetic moment μ, is $\cong gR \cong g\hbar/m_0 c$. It is to be noted that the mass occurring in the Compton wave-length in μ, is the mass of the *emitted* mesotron, and not of the *emitting* heavy particle. The resulting anomalously large value of μ, is needed if one is to have in Eq. (57): H(two-body "Coulomb") $\cong H$(two-body spin-spin);[17] which is necessary for a proper description of the strong spin-dependence of the total two-body potential.[18]

The heavy particle mesotron charge g, is related to the mesotron rest mass m_0. Thus:

$$M v_n \frac{\hbar}{MR} \cong \frac{M v_n^2}{2} \cong \text{Average Potential Energy} \cong \frac{g^2}{R},$$

whence

$$v_n/c \cong g^2/\hbar c. \quad (79)$$

Thus

$$m_0/M \cong g^2/\hbar c. \quad (80)$$

The order of magnitude of the individual terms in the various potentials in Eq. (57), and of the specifically quantum three-body potential H''

[17] See Eqs. (81), (82). Also, compare papers cited in reference 7.

[18] Direct experimental evidence for a strong spin-dependence of the total two-body potential follows from the transparency of para-H_2 and the opaqueness of ortho-H_2 for slow neutrons. See E. Teller, Phys. Rev. **49**, 420 (1936); J. Schwinger and E. Teller, Phys. Rev. **52**, 286 (1937); F. G. Brickwedde *et al.*, Phys. Rev. **54**, 266 (1938); W. F. Libby and E. A. Long, Phys. Rev, **55**, 339 (1939).

(Eq. (78)) can now be estimated. Thus:

$$H(\text{two-body "Coulomb"}) \cong g^2/R, \tag{81}$$

H(two-body spin-spin)

$$\cong \frac{\mu^2}{R^3} \cong \frac{g^2}{R} \cong H(\text{two-body "Coulomb"}), \tag{82}$$

H(two-body spin-orbit)

$$\cong g\frac{v_n}{c}\frac{\mu}{R^2} \cong \frac{v_n}{c} H(\text{two-body "Coulomb"}), \tag{83}$$

H(two-body vel.-dep.)

$$\cong \frac{v_n^2}{c^2}\frac{g^2}{R} \cong \frac{v_n^2}{c^2} H(\text{two-body "Coulomb"}), \tag{84}$$

$$H(\text{three-body spin-spin}) \cong \frac{g^2}{Mc^2}\frac{\mu^2}{R^4} \cong \frac{g^2}{Mc^2R}\frac{g^2}{R}$$
$$\cong \frac{v_n^2}{c^2} H(\text{two-body "Coulomb"}), \tag{85}$$

$$H(\text{three-body vel.-dep.}) \cong \frac{g^4}{Mc^4}\frac{v_n^2}{R^2} \cong \frac{g^2}{Mc^2R}\frac{v_n^2}{c^2}\frac{g^2}{R}$$
$$\cong \frac{v_n^4}{c^4} H(\text{two-body "Coulomb"}), \tag{86}$$

$$H(\text{three-body spin-orbit}) \cong \frac{g^3}{Mc^3}\frac{\mu v_n}{R^3} \cong \frac{g^2}{Mc^2R}\frac{v_n}{c}\frac{g^2}{R}$$
$$\cong \frac{v_n^3}{c^3} H(\text{two-body "Coulomb"}), \tag{87}$$

$$H''(\text{three-body quant.-mech.}) \cong \frac{\mu^4}{\hbar c R^5} \cong \frac{g^2}{\hbar c}\frac{g^2}{R}$$
$$\cong \frac{v_n}{c} H(\text{two-body "Coulomb"}). \tag{88}$$

It is especially to be noted that the anomalously large mesotron-magnetic moment $\mu \cong gR$ and the approximate equality of the average distances between the particles, and the range R, gives rise to a specifically quantum three-body potential and a spin-orbit two-body potential[19] both $\cong (v_n/c)H$(two-body "Coulomb").

[19] A two-body interaction of the spin of the particle with its own orbit, arises from the mesotron-polarization term:

$$\tfrac{1}{2}\sum_k p_k^* \cdot \text{grad.}_k\, \Phi(\mathbf{r}_k) + \text{c.c.}$$
$$= \tfrac{1}{2}\sum_{k,j}\left(\frac{\mathbf{v}_k}{c}\times \boldsymbol{\mu}_k^*\right)\cdot \text{grad.}_k\left(\frac{g_j e^{-r_{kj}/R}}{r_{kj}}\right) + \text{c.c.}$$
$$\cong \frac{v_n}{c}\mu\frac{g}{R^2} \cong \frac{v_n}{c} H\ (\text{two-body "Coulomb"}).$$

The numerical value of v_n/c is determined by the details of the magnitude and spatial dependence of the interactions and may be estimated to lie within the following limits: (for light nuclei)

$$\frac{1}{3} \geqq \frac{v_n}{c} \geqq \frac{1}{10}. \tag{89}$$

Many-Body Potentials for a General Field Theory[20]

It will now be shown that three-body potentials $\cong v_n/c \times$(two-body potentials), etc., are obtained in a general field theory satisfying the conditions:

(a) The Hamiltonian H of the total system may be written as

$$H(\text{field}) + H(\text{heavy-particles}) + H(\text{interaction}), \tag{90}$$

where, H(interaction) can be considered as a perturbation.

(b) Two-body potentials obtained from H(interaction) in the second approximation of the quantum perturbation calculation, are, at least of the same order of magnitude, as any two-body potentials which may already be contained in H(heavy-particles), due to an elimination procedure analogous to that performed in the electromagnetic and mesotron field theories.

The matrix element for the emission of a virtual light particle of mass m_0 and wave number k_ρ by one of the heavy particles will be denoted by

$$\int \psi_i^*(\mathbf{r}_j)\varphi_{0\rho}^*(q_\rho)H(\mathbf{r}_j, q_\rho; \text{interaction}) \times \varphi_{1\rho}(q_\rho)\psi_n(\mathbf{r}_j)d\mathbf{r}_j dq_\rho$$
$$\equiv \int \psi_i^*(\mathbf{r}_j)\, V_\rho \exp\,(i\mathbf{k}_\rho\cdot\mathbf{r}_j)\psi_n(\mathbf{r}_j)d\mathbf{r}_j, \tag{91}$$

where $\psi_i(\mathbf{r}_j)$, $\psi_n(\mathbf{r}_j)$ are material states, and $\varphi_{n\rho}(q_\rho)$ is a state of the field with n_ρ light particles of wave number $\mathbf{k}_\rho$ present.

The usual quantum perturbation method employed previously,[20a] now gives the following

[20] The contents of this section were first presented by one of the authors (H. P.) at the Washington meeting of the American Physical Society, April 28, 1938; Abstract No. 126, Phys. Rev. **53**, 938 (1938).

[20a] Again, with neglect of heavy particle recoil energies.

two-body and three-body action-at-a-distance potentials:

$$H(r_{12};\text{ two-body})=\sum_{\rho}\frac{V_\rho V_\rho^* \exp(i\mathbf{k}_\rho\cdot\mathbf{r}_{12})}{(\hbar^2c^2k_\rho^2+m_0^2c^4)^{\frac{1}{2}}},\quad(92)$$

$$H(r_{12},r_{13};\text{ three-body})=\sum_{\rho,\rho'}\frac{V_\rho V_\rho^* \exp(i\mathbf{k}_\rho\cdot\mathbf{r}_{12})}{[(\hbar^2c^2k_\rho^2+m_0^2c^4)^{\frac{1}{2}}]^2}\times\frac{V_{\rho'} V_{\rho'}^* \exp(i\mathbf{k}_{\rho'}\cdot\mathbf{r}_{13})}{(\hbar^2c^2k_{\rho'}^2+m_0^2c^4)^{\frac{1}{2}}}+\text{same terms}$$

with indices 2 and 3 interchanged

$$=\left[\frac{1}{\hbar c[k_\rho^2+(\hbar/m_0c)^{-2}]^{\frac{1}{2}}}\right]_{\text{Av}}H(r_{12};\text{ two-body})$$

$$\times H(r_{13};\text{ two-body})+\text{same term}$$

with indices 2 and 3 interchanged. (93)

Here, $\hbar/m_0c\equiv R$, must be the range of H(two-body), by a general argument due to Wick,[21] relating the rest mass of the virtually emitted particles and the range of the resulting interaction. Further, for $r_{12}\cong R$,

$$\left[\frac{1}{(k_\rho^2+1/R^2)^{\frac{1}{2}}}\right]_{\text{Av}}\cong R\quad(94)$$

since R is the only parameter characterizing the r space dependence of H(two-body), and consequently, the k space dependence of $V_\rho V_\rho^*/(\hbar^2c^2k_\rho^2+m_0^2c^4)^{\frac{1}{2}}$. Thus:

$$H(r_{12},r_{13};\text{ three-body})\cong(R/\hbar c)H(r_{12};\text{ two-body})\times H(r_{13};\text{ two-body}).\quad(95)$$

For nuclear systems, $(r_{12}\cong r_{13}\cong R)$

$$v_n(\hbar/R)\cong Mv_n^2/2\cong H(R;\text{ two-body}),\quad(96)$$

whence $H(\text{three-body})\cong\frac{v_n}{c}H(\text{two-body}).\quad(97)$

Similarly, higher order quantum perturbation calculations lead to:

$$H(m\text{-body})\cong[(R/\hbar c)(H(\text{two-body}))]^{m-2}$$

$$\cong(v_n/c)^{m-2}H(\text{two-body}).\quad(98)$$

[21] C. G. Wick, Nature **142**, 994 (1938).

Conclusions

These investigations indicate that the replacement of field interactions by two-body action-at-a-distance potentials is a poor approximation in nuclear problems. The error made is at least of the order of v_n/c, if one compares the magnitudes, term by term, of two- and three-body potentials. Furthermore, the number of terms in the m-body interaction of an n-body nucleus: $n!/m!(n-m)!$, is, in general, many times larger than the number of two-body interaction terms: $n!/2!(n-2)!=n(n-1)/2$, but, a direct estimate of the magnitude of the total m-body interaction ((number of m-body terms) $\times$ (average magnitude of each)) is complicated by the fact that the m-body interactions are, at least in part, of an exchange and spin-dependent character. It seems therefore, that a satisfactory description of nuclei, other than the deuteron, can be obtained only by an explicit consideration from the very beginning of the role played by the field.[22]

Addendum

The electromagnetic two-body Lagrangian (9) is correct to the second order in v/c and to the zero order in time-derivatives of the velocity. If, however, one wishes to consider the acceleration terms previously neglected in the vector potential (5),[23] and hence terms in the time-derivative of the acceleration in the equations of motion,[24] there are two alternatives:

(1) To retain the time-derivative of the acceleration *explicitly* in the equations of motion. Now, if these equations are to be derivable from a variational principle, the "Lagrangian" must contain the accelerations explicitly. This involves an action-at-a-distance particle mechanics beyond the scope of the present canonical formalism, and, consequently renders impossible the transition to quantum-mechanics.

[22] It may be added that a description of nuclei in terms of many-body interactions, or, more hypothetically, in terms of an integrated field energy density is consistent with the nuclear model proposed by Bohr.

[23] A vector potential correct to the first order in v/c leads immediately to the equations of motion (6), which are, in turn, derivable from the two-body Lagrangian (9).

[24] The "self-force" term in the equations of motion is proportional to the time-derivative of the electron's *own* acceleration, and represents dissipation of its energy by radiation. Naturally such "self-force" terms (along with all other "self" terms) are beyond the scope of the mutual interaction problem, whether or not, the mutual interactions themselves are capable of Lagrangian-Hamiltonian description. This is formally indicated by the fact that the vector potential of the kth electron depends on the velocities, accelerations, $\cdots$, of the $n-1$ *others*. Thus the sum in Eq. (5) is extended over all j except $j=k$.)

(2) To express the accelerations and their time derivatives in the equations of motion by means of the coordinates and velocities of the particles. This can be done to a first approximation by setting the acceleration equal to the Coulomb force divided by the mass. If the resulting equations of motion are to be the basis of a quantum-mechanical treatment, they should be capable of reduction to a Lagrangian, and hence, Hamiltonian form. If such a reduction proves to be impossible, then the *canonical* action-at-a-distance description of acceleration-retardation effects must be renounced. On the other hand, if these equations of motion can be deduced from a Lagrangian and Hamiltonian, then additional many-body interactions will be obtained.

A counterpart of the explicit appearance of accelerations in the classical vector potential is the presence of particle recoil energies in the energy denominators of the quantum perturbation scheme involved in the elimination of the quantized field variables. An attempt to consider the effects of these recoil energies to a first approximation leads to problems in connection with the canonical action-at-a-distance description of the particles, similar to those arising in classical theory from the acceleration-retardation effects.

The questions raised in this addendum are being investigated both for the electromagnetic and the mesotron field theories, and it is intended to make a more detailed report in the near future.

Appendix I

To obtain a solution of (49), in the form of an expansion in v/c it is simplest to make a Fourier resolution of Φ and ρ.

$$\Phi(\mathbf{r},t)=\int\Phi_\nu(\mathbf{r})e^{i\nu t}d\nu, \tag{A1}$$

$$\rho(\mathbf{r},t)=\int\rho_\nu(\mathbf{r})e^{i\nu t}d\nu. \tag{A2}$$

Substituting into (49),

$$\nabla^2\Phi_\nu-\frac{1}{R^2}\Phi_\nu+\frac{\nu^2}{c^2}\Phi_\nu=-4\pi\rho_\nu, \tag{A3}$$

whence

$$\Phi_\nu(\mathbf{r})=\int\rho_\nu(\mathbf{r}')\frac{\exp(-\kappa|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}' \tag{A4}$$

with
$$\kappa\equiv(1/R^2-\nu^2/c^2)^{\frac{1}{2}}. \tag{A5}$$

For
$$\frac{v}{c}\ll 1,\quad \nu \text{ is } \ll\frac{c}{R},$$

for all $\rho_\nu(\mathbf{r})$ which contribute appreciably to $\rho(\mathbf{r},t)$. Expanding κ, one obtains,

$$\begin{aligned}\Phi_\nu(\mathbf{r})&\cong\int\rho_\nu(\mathbf{r}')\frac{\exp[-1/R(1-\nu^2R^2/2c^2)|\mathbf{r}-\mathbf{r}'|]}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\\&\cong\int\rho_\nu(\mathbf{r}')\frac{\exp(-1/R|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\\&\quad+\frac{R}{2c^2}\int\nu^2\rho_\nu(\mathbf{r}')\exp(-1/R|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}'\end{aligned} \tag{A6}$$

and

$$\begin{aligned}\Phi(\mathbf{r},t)&=\int\rho(\mathbf{r}',t)\frac{\exp(-1/R|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\\&\quad-\frac{R}{2c^2}\int\left(\frac{d^2}{dt^2}\rho(\mathbf{r}',t)\right)\exp(-1/R|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}'\\&\quad+\text{terms of higher order in the time derivatives of }\rho.\end{aligned} \tag{A7}$$

For point sources located at $\mathbf{r}_j(t)$ and having velocities, accelerations, $\mathbf{v}_j(t)$, $\mathbf{a}_j(t)$,

$$\rho(\mathbf{r}',t)=\sum_j g_j\delta(\mathbf{r}'-\mathbf{r}_j(t))$$

and $\Phi(\mathbf{r},t)$ becomes

$$\begin{aligned}\Phi(\mathbf{r},t)&=\sum_j g_j\frac{\exp(-1/R|\mathbf{r}-\mathbf{r}_j|)}{|\mathbf{r}-\mathbf{r}_j|}\\&\quad-\frac{R}{2c^2}\sum_j g_j\frac{d^2}{dt^2}\exp(-1/R|\mathbf{r}-\mathbf{r}_j|)+\text{terms of higher order in }\frac{\mathbf{v}_j}{c},\frac{\mathbf{a}_j}{c},\text{ etc.}\\&=\sum_j g_j\frac{\exp(-1/R|\mathbf{r}-\mathbf{r}_j|)}{|\mathbf{r}-\mathbf{r}_j|}\\&\quad-\frac{R}{2c^2}\Big\{\sum_j g_j(\mathbf{v}_j\cdot\text{grad.}_j)\times(\mathbf{v}_j\cdot\text{grad.}_j)\exp(-1/R|\mathbf{r}-\mathbf{r}_j|)\\&\quad+\sum_j g_j(\mathbf{a}_j\cdot\text{grad.}_j)\exp(-1/R|\mathbf{r}-\mathbf{r}_j|)\Big\}\\&\quad+\text{terms of higher order in }\frac{\mathbf{v}_j}{c},\frac{\mathbf{a}_j}{c},\text{ etc.}\end{aligned} \tag{A8}$$

One may solve Eq. (50) for A in an entirely similar manner. Thus:

$$A(\mathbf{r},t)=\sum_i \frac{g_j}{c}\mathbf{v}_j\frac{\exp(-1/R|\mathbf{r}-\mathbf{r}_j|)}{|\mathbf{r}-\mathbf{r}_j|}$$

$$+\sum_j \boldsymbol{\mu}_j\times \text{grad.}_j\left(\frac{\exp(-1/R|\mathbf{r}-\mathbf{r}_j|)}{|\mathbf{r}-\mathbf{r}_j|}\right)$$

$$+\frac{R}{2c^2}\left\{\sum_j \text{curl}_j[(\mathbf{v}_j\cdot\text{grad.}_j)\right.$$

$$\times(\mathbf{v}_j\cdot\text{grad.}_j)\boldsymbol{\mu}_j\exp(-1/R|\mathbf{r}-\mathbf{r}_j|)]$$

$$+\sum_j \text{curl}_j[(\mathbf{a}_j\cdot\text{grad.}_j)\boldsymbol{\mu}_j\exp(-1/R|\mathbf{r}-\mathbf{r}_j|)]$$

$$+\sum_j \text{curl}_j\left[\frac{d\boldsymbol{\mu}_j}{dt}(\mathbf{v}_j\cdot\text{grad.}_j)\exp(-1/R|\mathbf{r}-\mathbf{r}_j|)\right]$$

$$\left.+\sum_j \text{curl}_j\left[\frac{d^2\boldsymbol{\mu}_j}{dt^2}\exp(-1/R|\mathbf{r}-\mathbf{r}_j|)\right]\right\}$$

+terms of higher order in

$$\frac{\mathbf{v}_j}{c},\ \frac{\mathbf{a}_j}{c},\ \frac{d\boldsymbol{\mu}_j}{dt},\ \frac{d^2\boldsymbol{\mu}_j}{dt^2},\ \text{etc.}\qquad (A9)$$

Appendix II

To eliminate the variables a_σ from (67), in order to obtain (70), one makes use of (62) and (65). The latter two equations give:

$$a_\sigma=\frac{\sum_j g_j\Phi_\sigma^*(\mathbf{r}_j)-\ddot{a}_\sigma}{\nu_\sigma^2}=\frac{\sum_j g_j\Phi_\sigma^*(\mathbf{r}_j)-ck_\sigma\dot{q}_\sigma}{\nu_\sigma^2}.\qquad (A10)$$

The second expression for a_σ, is just the equation div. $\boldsymbol{E}=4\pi\rho-(1/R^2)\Phi$ written in terms of the Fourier components a_σ, q_σ. Substituting this value for a_σ into (67), one obtains:

$$M\mathbf{a}_k=\left\{g_k^*\left[-\text{grad.}_k\sum_j\sum_\sigma\frac{g_j\Phi_\sigma^*(\mathbf{r}_j)\Phi_\sigma(\mathbf{r}_k)}{\nu_\sigma^2}\right.\right.$$

$$\left.+\frac{1}{c}\sum_\sigma\dot{q}_\sigma\left(\frac{c^2k_\sigma}{\nu_\sigma^2}\text{grad.}_k\Phi_\sigma(\mathbf{r}_k)-A_\sigma(\mathbf{r}_k)\right)\right]$$

$$-\frac{g_k^*}{c}\sum_\rho\dot{q}_\rho A_\rho(\mathbf{r}_k)+\frac{g_k^*}{c}\mathbf{v}_k\times\text{curl}_k\sum_\rho q_\rho A_\rho(\mathbf{r}_k)$$

$$\left.+\text{grad.}_k\left(\boldsymbol{\mu}_k^*\cdot\text{curl}_k\sum_\rho q_\rho A_\rho(\mathbf{r}_k)\right)+\text{c.c.}\right\}.\qquad (A11)$$

From (59), (66)

$$\frac{c^2k_\sigma}{\nu_\sigma^2}\text{grad.}_k\Phi_\sigma(\mathbf{r}_k)-A_\sigma(\mathbf{r}_k)$$

$$=\left(\frac{c^2k_\sigma^2}{\nu_\sigma^2}-1\right)A_\sigma(\mathbf{r}_k)=-\frac{c^2}{R^2\nu_\sigma^2}A_\sigma(\mathbf{r}_k).\qquad (A12)$$

Also,

$$(\nabla_k^2-1/R^2)\sum_\sigma\frac{\Phi_\sigma^*(\mathbf{r}_j)\Phi_\sigma(\mathbf{r}_k)}{\nu_\sigma^2}$$

$$=-\sum_\sigma\frac{\Phi_\sigma^*(\mathbf{r}_j)(k_\sigma^2+1/R^2)\Phi_\sigma(\mathbf{r}_k)}{\nu_\sigma^2}=-4\pi\delta(\mathbf{r}_k-\mathbf{r}_j).$$

Thus

$$\sum_\sigma\frac{\Phi_\sigma^*(\mathbf{r}_j)\Phi_\sigma(\mathbf{r}_k)}{\nu_\sigma^2}=\frac{e^{-r_{kj}/R}}{r_{kj}}.\qquad (A13)$$

Substituting (A12) and (A13) into (A11), gives Eq. (70).