The upper (lower) sign holds when approaching $\varphi = \pi$ from the illuminated (shadowy) side. The discontinuity of v_B on the boundary of the shadow is just compensated by the discontinuity of v^* in such a way that the sum $v = v^* + v_B$ is continuous. Further we can confirm a general result of Sommerfeld, according to which on the boundary of the shadow itself for large ρ the relation asymptotically holds

$$v = \tfrac{1}{2} v^* (\rho \gg 1, \ \varphi = \pm \pi + 2\pi nN). \qquad (37)$$

That not only for the function v_B, but also for all its derivatives the discontinuities on the boundary of the shadow are compensated for by those of v^*, can be confirmed by using Eq. (28). We do not want, however, to enter the details of this proof here.

Numerical estimates of $A_2(\varphi)$ gave the result that already the second term of the series (35) can be neglected in all practical cases so that the proposed task of finding a representation of the diffraction wave, which can be used for the transition from light to shadow and at large distances, can be considered as practically solved by the principal term of our series, as given in (35).

On the Anomalous Propagation of Phase in the Focus

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The problem of anomalous phase propagation of a spherical wave at the focus has been discussed for the case of a diffracting aperture of arbitrary shape. The solution given by Kirchhoff's integral has been split up into an "incident light wave," which shows the distribution of light to be expected according to geometrical optics and a "diffracted wave," which may be thought of as due to scattering of the incident wave at the diffracting edge. A sudden change of phase by π has been shown to occur already in the incident wave. Thus we may, in this sense, consider this phenomenon as a geometric optical one.

The case of a circular diffracting aperture, the focus lying on the normal through its center, which has been treated usually, appears to be not very suitable for an experimental investigation of the discussed phenomenon. It is this particular shape of the diffracting edge, which produces diffraction phenomena of considerable light intensity along the optical axis. These, however, are not because of the existence of a focus, but only because of the particular shape of the diffracting aperture.

I. Introduction

SOMMERFELD'S first great scientific achievement was the solution of the problem of diffraction at a perfectly conducting half-plane by the methods of exact analysis.[1] On the occasion of his jubilee it may thus be appropriate to present a note, which deals with a related problem and is based on a paper[2] I wrote while staying at his institute at Munich. The problem in question is the anomalous phase propagation of a spherical wave at the focus, which was dis-covered by Gouy in 1890. As appears from the very rich literature[3] on this subject, the phenomenon is completely describable by wave optics. Still, I do not believe that these papers satisfy as yet our want for a simple, plausible interpretation of the problem.[4] An attempt, therefore, will be made in this note to fill this gap by following a new method of approach, starting from Kirchhoff's theory of diffraction

[1] A. Sommerfeld, Math. Ann. **47**, 317 (1896).

[2] A. Rubinowicz, Ann. d. Physik **53**, 257 (1917) and **73**, 339 (1924), to be referred to as I and II in the text.

[3] For extensive literature see F. Reiche, Ann. d. Physik **29**, 65, 401 (1909) and J. Picht, *Optische Abbildung* (Braunschweig, 1931).

[4] An elementary suggestive representation by means of Fresnel's zones has, however, been given by A. D. Fokker, Physica **3**, 334 (1923); **4**, 166 (1924).
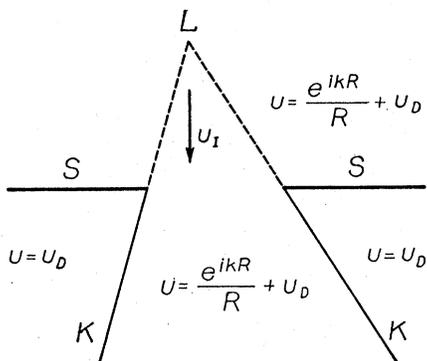
FIG. 1. Diffraction of a divergent spherical wave. L source of light, S diffracting screen, K boundary of the shadow. The arrow u_I shows the direction of propagation of the incident light wave.

and, besides, generalizing the treatment so as to apply to an arbitrary diffracting aperture.

We start from Kirchhoff's theory of diffraction but generalize the treatment so as to apply it to an arbitrary diffracting aperture.

At all those points where light should be present according to the laws of geometrical optics, the incident wave u_I is given by the solution of $\Delta u + k^2 u = 0$ for direct propagation of light, that is, if the time factor is $e^{-i\nu t}$, e^{ikR}/R for a point source; R is the distance from the source. It is supposed to vanish everywhere else. The wave function u_I will thus decrease discontinuously by e^{ikR}/R while passing from the "light cone" across the "boundary of the shadow" into the "shadow" (Fig. 1).

Concerning the shape of the geometrical boundary of the shadow we note that it is given by the surface of the cone K, which is generated by the lines joining the source of light L and the points of the diffracting edge D. The part of the cone between L and D has to be regarded as cut off.

The diffraction wave u_D can be considered as due to scattering of the incident light wave u_I at the different points of the diffracting edge D; it is composed of the secondary elementary waves produced there. Since we expect $u = u_I + u_D$ taken as a whole to be continuous on the boundary of the shadow, u_D must be discontinuous in such a way as to compensate the discontinuity of u_I. It ought thus to increase suddenly by e^{ikR}/R at the passage from the light cone into the shadow.

Let us suppose that the diffraction of a convergent spherical wave may be treated just like that of a divergent spherical wave considered above by representing the corresponding wave function u as a sum of u_I and u_D. Before determining these functions we have again to ascertain the boundary of the geometrical shadow. Obviously this will be formed by the double cone $K_1 + K_2$, generated, as above, by joining its vertex, now the focus F, with D (Fig. 2), which simultaneously cuts off the remaining part of the cone K_1.

Let us represent the incident light wave by $u_{I_1} = e^{-ikR}/R$ (R now denoting the distance from the focus F) in that portion of space which is bounded by the cone K_1 and the screen S and contains also the interior of the cone K_1. The minus sign had to be put into the exponent because the wave is now convergent; the time factor $e^{i\nu t}$ is the same as above. When passing through the cone K_1 into the geometrical shadow u_{I_1} is again supposed to disappear.

Let us now further assume that a diffracted wave u_D is produced by scattering of the incident wave u_{I_1} at the diffracting edge D. If $u_{I_1} + u_D$ is to be continuous at the boundary of the shadow, u_D ought to increase suddenly by e^{-ikR}/R when passing through the cone K_1 into the geometrical shadow. Now let us still make the plausible assumption that the elementary wavelets scattered at the different points of the diffracting edge, and with them also the diffracted wave u_D consti-
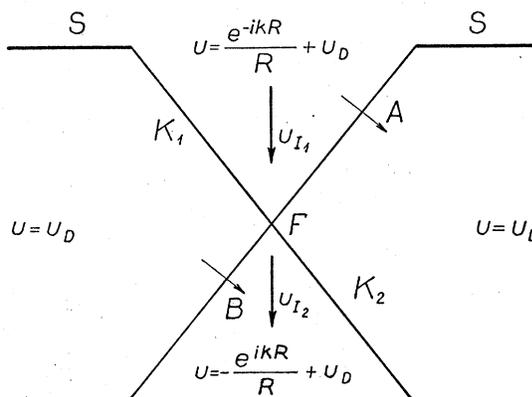

FIG. 2. Diffraction of a convergent spherical wave. F focus of the incident spherical wave, S diffracting screen, K_1 and K_2 boundaries of the shadow. The two arrows u_{I_1} and u_{I_2} show the direction of propagation of the incident light wave in the cones K_1 and K_2.

tuted by them, do not show any peculiar irregularity in their behavior while passing through the focus F. Then the discontinuity of u_D ought to be continued regularly on K_2. Since R is always positive according to our notation, a sudden increase of u_D by e^{-ikR}/R on the cone K_1 will correspond to that by e^{+ikR}/R on K_2. The direction of the passages through the surfaces is that indicated by the arrows A and B in Fig. 2. Otherwise the diffracted wave would be discontinuous at the focus F and travel along the cone K_2 towards the focus, that is, in a direction opposite to that in which it should go.

And now to our decisive remark: Outside the double cone K_1+K_2, that is, within the geometrical shadow, we have $u=u_D$, since for a passage through K_1 out of the light cone, like that shown by the arrow A, the discontinuity of u_{I_1} is just compensated by that of u_D. Now, u_D increases discontinuously at the passage through K_2 indicated by arrow B. To compensate this discontinuity on K_2 we shall have to put $u=u_D-e^{+ikR}/R$ inside K_2. Hence we see that the incident light must be represented by $u_{I_2}=-e^{+ikR}/R$ inside the cone K_2. The minus sign of u_{I_2} indicates that u_{I_1} and u_{I_2} differ in phase by π. We may thus summarize our results in the statement: The incident light wave is represented by a convergent spherical wave inside K_1 and by a divergent spherical wave inside K_2, which differ in phase by half a period.

Since, in general, the diffraction wave vanishes in the limiting case of infinitely small wavelengths, i.e., for $k\rightarrow\infty$, while the incident wave still persists, the phase anomaly will have to be considered as a phenomenon, which has to be taken into account in the geometric optical treatment of the problem as a sudden change in phase by half a period. It must, however, be said that this statement cannot apply at all those points of space, where, independently of the value of k, the intensity of the diffracted wave is of the same order of magnitude as that of the incident light wave. This is only possible at points, where the secondary waves from at least some finite portion of D arrive with all their phases equal. In general the focus F will be such a point, but in some particular cases still other points of this kind exist. In the case exclusively considered hitherto, for instance, where the diffracting aperture is circu-

lar and the focus lies on the normal through its center, all the points on the optical axis thus formed will be of that kind.

In the following paragraphs a mathematical formulation and some extension of the propositions made will be given.

II. INCIDENT AND DIFFRACTED WAVE

The distribution of light, the case of diffraction of a convergent spherical wave, at those points of space which lie in front of the screen S and the surface, extended over the diffracting edge D, say f, is given by Kirchhoff's integral:

$$u_K=\frac{1}{4\pi}\int_f\left(\frac{e^{-ik\rho}}{\rho}\frac{\partial}{\partial n}\frac{e^{ikr}}{r}-\frac{e^{ikr}}{r}\frac{\partial}{\partial n}\frac{e^{-ik\rho}}{\rho}\right)df, \quad (1)$$

where ρ is the distance of the surface element df from the focus F, r the distance of df, from the point P, at which we calculate u_K and n the normal of f in the direction away from the focus F. No restricting assumptions will be made at present concerning the shape of the diffracting edge D.

To transform this expression we shall first represent the incident light wave, say u_I, by means of surface integrals. Making use of the same notation as in Section I, we put

$$u_I=u_{I_1}+u_{I_2}, \quad (2a)$$

where u_{I_1} and u_{I_2} are defined as follows:

$u_{I_1}=e^{-ikR}/R$ inside the region bounded

      by K_1 and f,

$=0$ at all other points.   (2b)

$u_{I_2}=-e^{+ikR}/R$ inside the cone K_2,

$=0$ at all other points.   (2c)

To obtain integral representations for (2b) and (2c) we make use of the following well-known theorem: Let u^* be a regular solution of $\Delta u+k^2u=0$ inside the region enclosed by the surface Φ. Then the integral

$$\frac{1}{4\pi}\int_\Phi\left(u^*\frac{\partial}{\partial n}\frac{e^{ikr}}{r}-\frac{e^{ikr}}{r}\frac{\partial u^*}{\partial n}\right)df \quad (3)$$

represents the function u^* at the point P (whose

distance from df is denoted by r) if P lies inside Φ, and vanishes for P outside Φ. n is the external normal of Φ.

Using (3) we get immediately·

$$u_{I_1} = \frac{1}{4\pi} \int_{f+K_1+\Omega_{\mathrm{I}}} \left(\frac{e^{-ik\rho}}{\rho} \frac{\partial}{\partial n} \frac{e^{ikr}}{r} - \frac{e^{ikr}}{r} \frac{\partial}{\partial n} \frac{e^{-ik\rho}}{\rho} \right) df, \quad (4a)$$

$$u_{I_2} = -\frac{1}{4\pi} \int_{K_2+\Omega_2} \left(\frac{e^{ik\rho}}{\rho} \frac{\partial}{\partial n} \frac{e^{ikr}}{r} - \frac{e^{ikr}}{r} \frac{\partial}{\partial n} \frac{e^{ik\rho}}{\rho} \right) df, \quad (4b)$$

where Ω_1 and Ω_2 are spherical surfaces round F as their center. They had to be introduced on account of the singularity of $e^{\pm ik\rho}/\rho$ at F, which is thus cut off.

Concerning the integral representations (4a, b) we have to make the following remarks:

First the integral over f in (4a) is identical with u_{K}.

Secondly the integrals over K_1 and K_2 in (4a, b), which we shall denote by $-u_{D_1}$ and $-u_{D_2}$ yield the expressions

$$u_{D_1} = -\frac{1}{4\pi} \int_{K_1} \frac{e^{ik(r-\rho)}}{\rho} \left(\frac{ik}{r} - \frac{1}{r^2} \right) \cos (n, r) df, \quad (5a)$$

$$u_{D_2} = \frac{1}{4\pi} \int_{K_2} \frac{e^{ik(r+\rho)}}{\rho} \left(\frac{ik}{r} - \frac{1}{r^2} \right) \cos (n, r) df, \quad (5b)$$

since $(\partial/\partial n)(e^{\pm ik\rho}/\rho)$ vanishes on K_1 and K_2 and further

$$\frac{\partial}{\partial n} \frac{e^{ikr}}{r} = e^{ikr} \left(\frac{ik}{r} - \frac{1}{r^2} \right) \cos (n, r).$$

n is here the external normal of K_1 or K_2. Its direction is reversed as we pass along a generator of our cone through the focus F.

Finally we note that the two integrals over Ω_1 and Ω_2 yield $-$ or $+$ $\omega e^{ikR}/4\pi R$, where ω is the solid angle subtended by these surfaces.

Now, on the basis of these remarks and Eq. (2a) we obtain adding (4a) and (4b):

$$u_{\mathrm{K}} = u_1 + u_{D_1} + u_{D_2},$$

where the sum $u_{D_1} + u_{D_2}$ can be reduced to a line integral over the diffraction edge D. To perform this transformation we shall introduce on our double cone $K_1 + K_2$, an orthogonal coordinate system, given by the distance ξ from the focus F and the lines of intersection σ of $\xi = $ const. with K_1 and K_2. The sign of ξ may be chosen so that $\xi = -\rho$ on K_1 and $\xi = +\rho$ on K_2.

To obtain a suitable expression for df we consider the area subtended on our cone $K_1 + K_2$ by an element ds of the diffracting edge D. Let us denote by ρ_s the distance of ds from the focus F. Then the line element of $d\sigma$ (Fig. 3) on our area is given by $d\sigma = (\xi/\rho_s) \sin (\rho_s, ds) ds$ and therefore $df = d\xi d\sigma$ by $df = (\xi/\rho_s) \sin (\rho_s, ds) d\xi ds$. The sign of df represented in this way will be positive on K_2 and negative on K_1, while it is always positive in the integrals (5a, b) for u_{D_1} and u_{D_2}. But in these integrals also $\cos (n, r)$ changes its sign at the focus F, where, as we have said, the direction of n is suddenly reversed. Choosing now that direction of n, which coincides with its former direction on K_1, opposite to it on K_2, we shall have to put, with these new assumptions concerning n

$$-\cos (n, r)(\xi/\rho_s \sin (\rho_s, ds) ds d\xi$$

instead of $\cos (n, r) df$ in our integrals. Considering further that now

$$\cos (n, r) = (r_s/r) \cos (n, r_s),$$

where r_s is the distance of the point P from ds, we obtain the expression

$$u_D = -\frac{1}{4\pi} \int_D ds \sin (\rho_s, ds) \cos (n, r_s)$$

$$\times \frac{r_s}{\rho_s} \int_{-\rho_s}^{+\infty} d\xi e^{ik(\xi+r)} \left(\frac{ik}{r^2} - \frac{1}{r^3} \right)$$

for the sum $u_D - u_{D_1} + u_{D_2}$. Since now

$$r^2 = r_s^2 + (\xi + \rho_s)^2 - 2r_s(\rho_s + \xi) \cos (r_s, \rho_s)$$

we get finally, on following the method given in I (p. 261)

$$u_D = \frac{1}{4\pi} \int_D \frac{e^{-ik\rho_s}}{\rho_s} \frac{e^{ikr_s}}{r_s} \frac{\cos (n, r_s)}{1 - \cos (r_s, \rho_s)}$$

$$\times \sin (\rho_s, ds) ds. \quad (6)$$

Kirchhoff's diffraction integral may thus be given the form

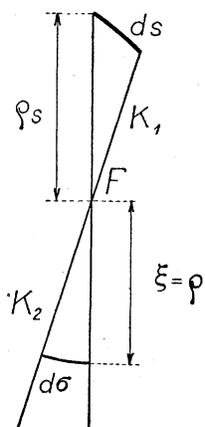$$u_{\mathrm{K}} = u_I + u_D, \quad (7)$$

FIG. 3. To the equation $d\sigma = (\xi/\rho_s) \sin (\rho_s, ds)$.

where u_I represents a wave of the kind to be expected according to geometrical optics and u_D a wave, which may be thought of as due to scattering of the incident wave by the diffracting edge. The secondary waves, which originate at the different elements ds of the edge, are seen to be asymmetric on account of the direction factor

$$\frac{\cos (n, r_s)}{1 - \cos (r_s, \rho_s)}. \tag{8}$$

This becomes infinite on that generator of the double cone $K_1 + K_2$, which belongs to the element ds, since there $\cos (r_s, \rho_s) = 1$.

III. APPROXIMATE REPRESENTATION OF THE DIFFRACTION WAVE

If we want to discuss the distribution of light for the particular case of diffraction of a convergent spherical wave, we have to resort to appropriate approximate formulae for the diffraction wave u_D (or for the complete wave function u_K). For all those points of space, at which the method of stationary phase can be applied to the integral (6) either directly or indirectly, such formulae are readily obtained in the same way as for a divergent spherical wave (cf. II). The necessary condition for this is that the phase factor $k(r_s - \rho_s)$ varied rapidly enough as we pass along the diffracting edge D. Then only the immediate vicinity of those points P_ν, where the relatively slowest variation of phase occurs,

that is, where

$$d(r_s - \rho_s)/ds = 0 \tag{9}$$

or

$$\cos (r_s, ds) = \cos (\rho_s, ds), \tag{9a}$$

will yield a noticeable contribution to the integral (6).

Hence it appears that the light scattered by a particular element ds of the edge is spread over a circular cone, whose vertex lies in that element. There is also the line joining the element ds and the focus F among the generators of this cone. Its direction coincides with that of the light ray of the convergent spherical wave, incident upon ds. It is in such half-cones that light is reflected in the points of a reflecting curve, e.g., a wire, according to the concepts of geometrical optics. We may thus call this half-cone the reflection cone.

If we want to apply directly the method of stationary phase to the integral (6), we have to assume that apart from the factor $\exp [ik(r_s - \rho_s)]$ the integrand varies so slowly that it may be taken out of the integral. This assumption is, however, not fulfilled in the points near the boundary of the shadow, where the direction factor (8) becomes infinite. Therefore we cannot apply this method directly at those very points in which we are most interested, at the treatment of diffraction problems. Bur fortunately there is an indirect way of deriving approximate formulae, which can be applied even at points in the boundary of the shadow itself. By following the method applied in II, (p. 352) we obtain for the contribution $u_D|_\nu$ of the effective region round the point P_ν to u_D the expression

$$u_D|_\nu = \frac{1}{2 r_s \rho_s (\zeta'')^{\frac{1}{2}}} \frac{(R + r_s - \rho_s)^{\frac{1}{2}}}{1 - \cos (r_s, \rho_s)} \cos (n, r_s)$$

$$\times \sin (\rho_s, ds) e^{i(-kR + 3\pi/4)}$$

$$\times \int_{+\infty}^{[(2k/\pi)(R + r_s - \rho_s)]^{\frac{1}{2}}} e^{i(\pi/2) v^2} dv, \tag{10}$$

if the points considered lie inside the cone K_1.[5]

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[5] The frequent appearance of Fresnel's integrals in Kirchhoff's theory of diffraction is due to the fact that they occur in all those cases where the incident light is reflected by the diffracting edge.

Here we have put

$$\zeta'' = \frac{d^2}{ds^2}(r_s - \rho_s) = \sin^2(\rho_s, ds)\left(\frac{1}{r_s} - \frac{1}{\rho_s}\right)$$

$$+ \frac{1}{H}(\cos(r_s, H) - \cos(\rho_s, H)), \quad (11)$$

where H is the radius of curvature of the diffract-ing edge in the point P, taken as a vector in the direction of the principal normal at P_ν.

If we want to calculate u_D by means of (10) at some point of space, we have first to ascertain from which points P_ν of the diffracting edge the incident light is reflected in cones, to the point of space considered. u_D will then be given as the sum of the respective $u_D|_\nu$.[6] For a circular diffracting aperture, the focus F lying on the normal through its center, formula (10) may be found in a paper by Schwarzschild.[7] Then only two points P_ν need be considered and u_K consists simply of the incident wave and the waves, reflected at these two points.

For the derivation of (10) it has been supposed that in the vicinity of P_ν, the phase may be represented with sufficient accuracy by

$$r_s - \rho_s = (r_s - \rho_s)_{P_\nu} + \tfrac{1}{2}\frac{d^2}{ds^2}(r_s - \rho_s)(s - s_v)^2$$

$$= (r_s - \rho_s)_{P_\nu} + \tfrac{1}{2}\zeta''(s - s_v)^2$$

in which use is made of (9). This approximation fails, however, for ζ'' vanishes on the reflection cone, as appears already, from the occurrence of $(\zeta'')^{\frac{1}{2}}$ in the denominator of (10). The points $\zeta'' = 0$ are distinguished as forming the intersec-tion lines of two "infinitely neighboring" reflec-tion cones. The images of the incident convergent light wave at its reflection from a definite ele-ment ds of the diffracting edge lie on these lines, which may thus be referred to as focal lines. They separate the reflection cone into two domains, one of positive and one of negative ζ''. It has,

however, been supposed that $\zeta'' > 0$ at the deriva-tion of (10). Now, on investigating the case $\zeta'' < 0$, we find that there is a change of phase by $\pi/2$ occurring at the passage across the focal line $\zeta'' = 0$. The phase anomaly is thus shown, not only by the incident, but also, eventually, so to say on a small scale, by the diffracted wave.[8]

The method of stationary phase fails near the focus F. In the focus itself, where $r_s = \rho_s$, all the wavelets originating at different points of the diffracting edge arrive with equal phase. Also near the focus their phases will differ little, so that it will be necessary to consider the diffract-ing edge as a whole and not only some small effec-tive regions. A general representation like that possible at those points of space, where the method of stationary phase applies, does no longer seem to be feasible. Besides, it will be advisable to use u_K and not u_D near F, where the latter becomes infinite according to (2) and (7).

We have still to say a few words about the special case of a circular diffracting aperture, the focus of the convergent spherical wave lying on the normal through the center of the circle. For every point on the optical axis thus formed not only the phase $r_s - \rho_s$, but the whole integrand in (6) will be constant. Then the amplitude of u_D does no longer depend upon the wave-length and will be comparable with that of the incident light. These peculiar conditions on the optical axis are due, however, to the particular shape of the diffracting edge and not to the existence of a focus. This large amplitude of u_D is also respon-sible for the fact that points on the axis are favored at all observations. The large focal length and the smallness of the aperture used at most observations will add to this peculiarity, since the phase variations of the elementary wavelets for a point at a given distance from the axis de-crease with the distance from and the smallness of the diffracting aperture. These two circumstances favor an extension of the conditions on the axis into the neighboring portions of space.

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[6] (10) enables us to treat approximately diffraction—though only if it is an extrafocal one—by more than one diaphragm, a problem rather difficult to solve by Fresnel-Kirchhoff's original methods.

[7] K. Schwarzschild, Sitzungsber. d. math. phys. Klasse d. k. bayerischen Akad. d. Wiss. **28**, 271 (1898).

[8] This anomaly of phase on a small scale occurs, of course, also at the diffraction of a divergent spherical wave, which has been treated in II.