On Asymptotic Series for Functions in the Theory of Diffraction of Light

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For Sommerfeld's exact solution of the problem of the diffraction of light by a wedge, limited by two perfectly reflecting planes, new asymptotic formulas are given; these also hold near one of the boundaries between light and shadow, at a distance from the diffracting body which is large compared with the wave-length of light.

1. INTRODUCTION. STATEMENT OF THE PROBLEM

As is well known, A. Sommerfeld, who is now celebrating his seventieth birthday, was the first to succeed in establishing an exact solution of the wave equation for a particular class of two-dimensional problems of diffraction.[1] The problem in question is the diffraction by a wedge, limited by two perfectly reflecting planes. The diffraction by a half-plane is contained therein as a particular case, when the angle θ between the limiting planes of the wedge tends to zero.

For the sake of simplicity we shall assume that the monochromatic source of light is at infinity and that the direction of propagation of the incident plane wave is perpendicular to the edge of the wedge, which edge may coincide with the z axis of the coordinate system. By splitting off the factor $\exp(+i\omega t)$ from all field quantities and taking into consideration, that all field quantities become independent of z, it is easy to reduce the problem to the solution of the two-dimensional scalar wave equation:

$$(\partial^2/\partial x^2+\partial^2/\partial y^2)u+k^2u=0 \qquad (1)$$

with $k=\omega/c$. According to the two possible states of the polarization of the incident wave (whether the electric vector oscillates perpendicularly to the plane of incidence (A), or within this plane (B)) one has to fulfill the boundary conditions:

$$\text{(A)}\quad u=0; \qquad \text{(B)}\quad \partial u/\partial n=0,$$

where n means as usual the direction of the normal upon the surface of the wedge. In either case the boundary condition of a perfectly reflecting mirror calls for the vanishing of the tangential components of the electric field strength.

In the case (A) the nonvanishing components of E and H are H_x, H_y, E_z, and as a consequence of Maxwell's equations one can put

$$E_z=u,\quad H_x=(i/k)(\partial u/\partial y),\quad H_y=-(i/k)(\partial u/\partial x).$$

The boundary condition is fulfilled if $u=0$. In the case (B) the components E_x, E_y, H_z are different from zero and one can put

$$E_x=-(i/k)(\partial u/\partial y),\quad E_y=(i/k)(\partial u/\partial x),\quad H_z=u.$$

The postulate, that the tangential components of E vanish, is now equivalent to $\partial u/\partial n=0$.

Now we introduce cylindrical coordinates defined by

$$x=r\cos\psi,\quad y=r\sin\psi$$

in such a way, that the origin of the coordinate system coincides with the edge of the wedge and that the values $\psi=0$ and $\psi=2\pi-\theta$ of the azimuth ψ correspond to the boundary planes of the wedge. If $\psi=\psi_0$ is the azimuth of the radius vector drawn from the origin to the light source at infinity (so that $0\leqq\psi_0\leqq 2\pi-\theta$), the incident wave is represented by

$$u_0=e^{ikx\cos\psi_0}, \qquad (2)$$

where as always in the sequel the factor $e^{+i\omega t}$ is omitted and the amplitude of u_0 is normalized to unity. (Compare Fig. 1. The arrow indicates the direction of the incident wave and the shaded domain the shadow according to geometrical optics.)

[1] A. Sommerfeld, Math. Ann. **45**, 263 (1894) and **47**, 317 (1896). For the case of the wedge, H. M. Macdonald, *Electric Waves* (Cambridge, 1902), p. 186, and H. S. Carslaw, Proc. Lond. Math. Soc. **18**, 291 (1919) obtained an essential simplification of the solution. We follow here also, in the notation, a summarizing article written by Sommerfeld himself in the book edited by P. Frank and R. V. Mises, *Differential- u. Integralgleichungen der Physik*, 2nd edition (1935), Vol. **2**, Chap. 20.

By putting

$$u=v(r,\psi-\psi_0)\pm v(r,\psi+\psi_0), \qquad (3)$$

which corresponds to the separation of the total field into the incident and the reflected wave, the solution of the problem is reduced to the function $v(r, \varphi)$ of Sommerfeld. This is a solution of the wave equation (1), which has several branches in the (x, y) plane; it has the period $2\pi n$ in the azimuth φ, but is a one valued function of the variable $\cos \varphi/n$. One has therefore

$$v(r, \varphi+2\pi n)=v(r, \varphi), \qquad (4a)$$

$$v(-\varphi)=v(\varphi), \quad v(\pi n-\varphi)=v(\pi n+\varphi). \qquad (4b)$$

If the half-period πn of the function $v(r, \varphi)$ coincides with the azimuth $2\pi-\theta$ of the second limiting surface of the wedge

$$\pi n=2\pi-\theta, \qquad (5)$$

the boundary conditions (A) or (B) are fulfilled on both surface planes of the wedge, according to whether one assumes the upper or the lower sign in Eq. (3).

We restrict ourselves to the case

$$n>1, \quad \theta<\pi \qquad (6)$$

where, however, it is in no way assumed that n is an integer. Moreover we introduce the abbreviation

$$\rho=kr \qquad (7)$$

for the "numerical distance." The following further conditions have to be fulfilled, in order

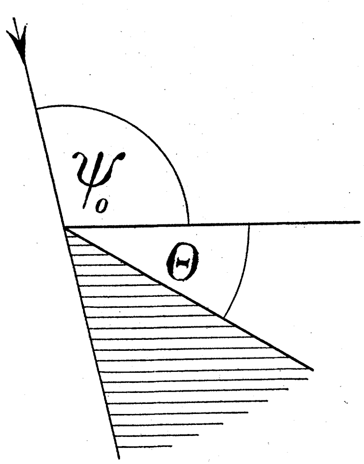

FIG. 1.

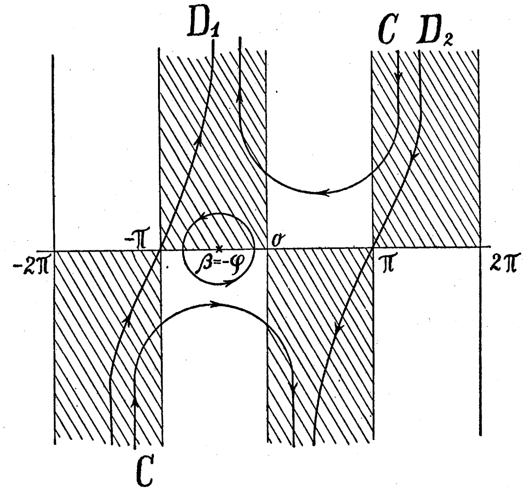

FIG. 2.

that (3) is really a solution of the problem. Firstly, at large distances from the diffracting body $\rho\rightarrow\infty$ the function $v(r,\psi-\psi_0)$ has to converge towards zero or towards the incident wave function u_0 according to whether the point under consideration is in the "illuminated" domain or in the "shadow" of geometrical optics. Secondly, the deviation of $v(r,\psi-\psi_0)$ from the terms of geometrical optics must contain for large ρ only outgoing and no incoming cylindrical waves ("emission-condition"). (If these conditions are fulfilled for the incident wave $v(r,\psi-\psi_0)$ they are automatically also fulfilled for the reflected wave $v(r,\psi+\psi_0)$ as a consequence of the boundary conditions on the surface of the wedge.) At last one has to postulate, that for $\rho\rightarrow 0$ v remains finite and $\rho\, \partial v/\partial\rho\rightarrow 0$.

Sommerfeld was able to fulfill all these conditions by making the following decisive Ansatz

$$v(\rho, \varphi)=\frac{1}{2\pi n}\int_C \frac{e^{i\rho\cos\beta}}{1-e^{-i(\beta+\varphi)/n}}d\beta. \qquad (I)$$

Here C means the path in the plane of the complex variable. As indicated in the following Fig. 2, it consists of two parts. The shaded domains of the figure are those, where $\exp(i\rho\cos\beta)\rightarrow 0$ for $\rho\rightarrow\infty$.

We do not wish to repeat here the complete proof that $v(\rho, \varphi)$ fulfills all necessary conditions; instead we shall discuss only the behavior of this function for large values of ρ. To this purpose it is important to realize that the integrand has poles at the places

$$\beta=-\varphi+2\pi nN;\quad (N=0,\pm1,\pm2,\cdots)$$

and that the behavior of v for large ρ is quite different, according to whether one of these poles falls into the interval $(-\pi, +\pi)$ or not. This fact becomes obvious, if we substitute for the path C of the integration the path D_1+D_2 of Fig. 2 and then add the residue of the pole, in case such a one is situated in the interval $(-\pi, \pi)$. In this way one gets

$$v=v^*+v_B \tag{8}$$

with

$$v^*=\begin{cases}\exp\left[i\rho\cos(\varphi+2\pi nN)\right], & \\ & \text{if}\quad -\pi<\varphi+2\pi nN<+\pi,\\ 0 & \text{otherwise}\end{cases} \tag{9}$$

$$v_B=\frac{1}{2\pi n}\int_{D_1+D_2}\frac{e^{i\rho\cos\beta}}{1-e^{-i(\beta+\varphi)/n}}d\beta. \tag{10}$$

As a consequence of $n>1$, the inequality $-\pi<\varphi+2\pi nN<\pi$ can be fulfilled by not more than one single integer value of N, and it is easily seen that it is fulfilled for intervals of the length 2π and not fulfilled for intervals of the length $2\pi(n-1)$, which follow each other alternately. The former intervals we can call the illuminated ones, whereas the intervals of the latter kind indicate the shadow. Using this notation one has:

$$\text{light for}\quad -\pi+2\pi nN<\varphi<\pi+2\pi nN, \tag{11a}$$

shadow for

$$+\pi+2\pi nN<\varphi<\pi+2\pi nN+2\pi(n-1). \tag{11b}$$

The places $\varphi=-\pi+2\pi nN$ and $\varphi=+\pi+2\pi nN$ correspond to the *boundaries of the shadow*, with which we shall occupy ourselves particularly. A simple geometrical consideration shows indeed that $v^*(\rho,\psi-\psi_0)$ and $v^*(\rho,\psi+\psi_0)$ correspond in the physical angle interval exactly to the incident and to one or two reflected waves, respectively, as one expects them to occur according to geometrical optics.[2]

For the following it is important that the functions v^* and v_B on the boundaries of the shadow $\varphi=\pm\pi+2\pi nN$ are both discontinuous, whereas their sum, as one recognizes from the path C in Fig. 1, is completely regular.

One can transform the expression (10) for the diffraction wave v_B by the substitution $\beta=\eta-\pi$ for D_1 and $\beta=\eta+\pi$ for D_2 and by adding the resulting integrands. In this way one obtains

$$v_B=\frac{1}{2\pi i}\cdot\frac{1}{n}\sin\frac{\pi}{n}\int_{-i\infty-\epsilon}^{+i\infty+\epsilon}\times\frac{e^{-i\rho\cos\eta}}{\cos(\pi/n)-\cos\left[(\eta+\varphi)/n\right]}d\eta, \tag{12}$$

where ϵ is an angle between zero and π. Physically interesting is the case of large values of ρ. The application of the method of steepest descent to the integral (12) gives in this case at once as contribution of the saddle-point $\eta=0$ the following asymptotic formula for v_B:

$$v_B=(2\pi\rho)^{-\frac{1}{2}}e^{-i(\rho+\pi/4)}\frac{n^{-1}\sin\pi/n}{\cos\pi/n-\cos\varphi/n}. \tag{13}$$

One can interpret this result as a cylindrical wave going out from the edge of the diffracting wedge with a characteristic dependence of the amplitude from the azimuth φ. The result is correct under the supposition

$$\rho(\cos\pi/n-\cos\varphi/n)^2\gg1,$$

it fails therefore in the neighborhood of the boundary of the shadow, where

$$\cos\pi/n=\cos\varphi/n.$$

Hence the problem arises to obtain such an asymptotic formula for the diffraction wave v_B for large ρ, which will be valid also in the vicinity of the boundary of the shadow.

This problem was attacked already by Sommerfeld[3] who tried to get an improved expression for v_B by a modification of the path of integration in (12). His results, however, were not satisfactory, because the neglected terms were not sufficiently small in comparison to the retained terms. We propose here a different treatment, which is based on a transformation of the

[2] As an example we point to the case $\pi-\theta<\psi_0<\pi$ where no boundary of the shadow of the incident wave, but two boundaries of the shadow of the reflected wave are situated in the physical angle interval $0<\psi<2\pi-\theta$.

[3] A. Sommerfeld, Crelles J. **158**, 199 (1928).

integrand without changing the path of the integration.

We get an indication of this method by a previous result of Sommerfeld concerning the particular case of a half-plane ($\theta=0$, $n=2$). In this case he was able to transform the expression (12) into

$$v_B=-e^{i\pi/4}(2/\pi a)^{\frac{1}{2}}e^{i\rho\cos\varphi}\cdot\cos\varphi/2\cdot\int_{(a\rho)^{\frac{1}{2}}}^{\infty}e^{-i\tau^2}d\tau \quad (14)$$

with

$$a=1+\cos\varphi, \quad (15)$$

where the square root $a^{\frac{1}{2}}$ is always to be taken as positive. For the sum v^*+v_B therefore the expression results:

$$v(\rho,\varphi)=e^{i\pi/4}\pi^{-\frac{1}{2}}e^{i\rho\cos\varphi}\int_{-\infty}^{(2\rho)^{\frac{1}{2}}\cos\varphi/2}e^{-i\tau^2}d\tau, \quad (14a)$$

which is also regular for $\varphi=\pi$.

In the following section we shall define a system of functions $S_m(w)$ of the argument $w=\rho a$ in such a way that

$$S_0(w)=e^{iw}\cdot 2\int_{w^{\frac{1}{2}}}^{\infty}e^{-i\tau^2}d\tau. \quad (16)$$

$S_m(w)$ is for large w of the order of magnitude $w^{-\frac{1}{2}}$ whereas for small w, if $m>0$, $S_m(w)$ is of the order $w^{\frac{1}{2}}$. In Section 3 we shall finally establish for v_B a semi-convergent development, which holds also near the boundary of the shadow $\varphi=\pi$; namely,

$$v_B=e^{-i(\rho-\pi/4)}(2a)^{-\frac{1}{2}}\times\sum_{m=0}^{M}C_m(\varphi)S_m(\rho a)\rho^{-m}+R_M(\rho,\varphi). \quad (17)$$

The functions $C_m(\varphi)$ are regular for $\varphi=\pi$, but infinite for the other boundaries of the shadow $\varphi=\pi+2\pi nN$; ($N\neq 0$). For every boundary of the shadow $\varphi=\pi+2\pi nN$ one can construct a development of the form (16); one has only to replace φ by $\varphi-2\pi nN$, with a given integer N, in the definition of a and in $C_m(\varphi)$. But in the general case there exists no development of this type which holds for *all* boundaries of the shadow simultaneously as well as for large ρ. This seems only to be possible for $n=2$, in which case the series (16) breaks off with the first term and reduces to (14).

2. The Functions $S_m(w)$

We shall develop Sommerfeld's solution into a series proceeding according to the system of functions $S_m(w)$, which is directly defined as a generalization of Fresnel's integral by

$$S_m(w)=e^{iw}w^m\int_w^{\infty}e^{-it}t^{-(m+\frac{1}{2})}dt$$
$$=e^{iw}w^m\cdot 2\int_{w^{\frac{1}{2}}}^{\infty}e^{-i\tau^2}\tau^{-2m}d\tau. \quad (18)$$

In the physical application the variable w has a positive value and also the square root $w^{\frac{1}{2}}$ on the lower limit of the second integral is to be defined as positive. We shall prove that these functions have a simple connection with the confluent hypergeometric function $F(\alpha,\beta,x)$.[4] The latter can be defined as the solution of the differential equation

$$xF''+(\beta-x)F'-\alpha F=0, \quad (19)$$

which is given by the integral

$$F(\alpha,\beta,\gamma)=(2\pi i)^{-1}\Gamma(\beta)\int_C e^t t^{\alpha-\beta}(t-x)^{-\alpha}dt. \quad (20)$$

The powers of t are unambiguously defined by the statement that arg t and arg $(t-x)$ are running from $-\pi$ to $+\pi$ during the circulation around the point $t=0$ or $t=x$ along the path of integration. The path of integration can be chosen in different ways (Fig. 3). If it includes both singularities $t=x$ and $t=0$ (path C), one obtains a particular solution of (19), which can be represented also by the power series

$$F(\alpha,\beta,x)=1+\frac{1}{1!}\frac{\alpha}{\beta}x+\frac{1}{2!}\frac{\alpha(\alpha+1)}{\beta(\beta+1)}x^2+\cdots. \quad (21)$$

If, on the other hand, the path of the integration includes only one point (either the point $t=x$,

[4] Cf. Whittaker-Watson, *Modern Analysis*, 4th Edition (1927), Chap. 16; Frank-Mises, reference 1, Vol. **2**, Chap. 26, 5. W. Gordon, Zeits. f. Physik **48**, 180 (1928). We omit on the left side of (20a) the factor $\frac{1}{2}$ in the definition of F_1 and F_2 used in the latter paper.

path C_1; or the point $t=0$, path C_2) whereas the other singularity is lying outside, one gets the solution F_1 or F_2, respectively:

$$(k=1,\,2)\quad F_k(\alpha,\,\beta,\,x)=(2\pi i)^{-1}\Gamma(\beta)\times\int_{C_k}e^t t^{\alpha-\beta}(t-x)^{-\alpha}dt. \tag{20a}$$

As one sees in the figure, the relation holds:

$$F=F_1+F_2. \tag{22}$$

For large x there exist the, in general semi-convergent, developments:

$$F_1=\Gamma(\beta)/\Gamma(\alpha)\cdot e^x x^{\alpha-\beta}\times[1+(1-\alpha)(\beta-\alpha)/x+\cdots] \tag{23.1}$$

$$F_2=\Gamma(\beta)/\Gamma(\beta-\alpha)\cdot(-x)^{-\alpha}\times[1-\alpha(\alpha-\beta+1)/x+\cdots]. \tag{23.2}$$

The series for F_1 breaks off, if $\alpha=1,\,2,\,\cdots$ or $\beta-\alpha=0,\,-1,\,-2\,\cdots$, whereas the series for F_2 breaks off for $\alpha=0,\,-1,\,-2\,\cdots$ and for $\beta-\alpha=1,\,2,\,\cdots$.

We are interested here particularly in the case $\alpha=1$, $\beta=-m+\frac{3}{2}$, $x=iw$ with $m=0,\,1,\,2,\,\cdots$; that is to say in:

$$F_2(1,\,-m+\tfrac{3}{2},\,iw)=(2\pi i)^{-1}\Gamma(-m+\tfrac{3}{2})\times\int_{C_2}e^t t^{m-\frac{1}{2}}(t-iw)^{-1}dt. \tag{24}$$

In this case one can contract the path of integration to the negative real t axis and one obtains after the substitution $t\rightarrow -t$ and $t\rightarrow -\tau^2$ by taking care of $\Gamma(-m+\frac{3}{2})\Gamma(m-\frac{1}{2})=\pi(-1)^{m+1}$:

$$F_2(1,\,-m+\tfrac{3}{2},\,iw)=\Gamma(m-\tfrac{1}{2})\int_0^\infty e^{-t}t^{m-\frac{1}{2}}(t+iw)^{-1}dt=\Gamma(m-\tfrac{1}{2})\int_{-\infty}^{+\infty}e^{-\tau^2}\tau^{2m}(\tau^2+iw)^{-1}d\tau. \tag{25}$$

On the other hand the path C_1 can in the case in question be replaced by a circle around the point $t=iw$ and gives (in accordance with the fact that the series (23.1) reduces here to the first term)

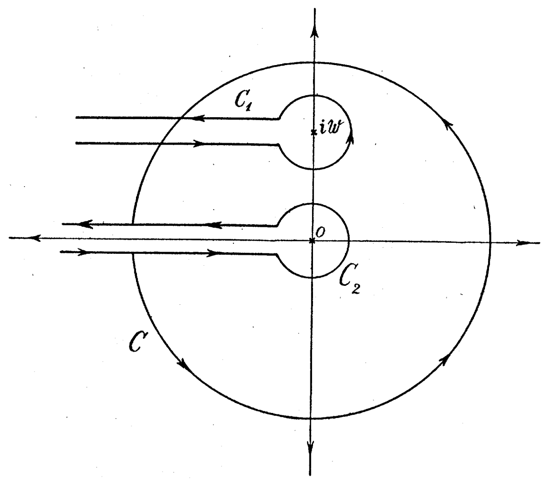

FIG. 3.

$$F_1(1,\,-m+\tfrac{3}{2},\,iw)=\Gamma(-m+\tfrac{3}{2})\times e^{i(m\pi/2-\pi/4)}w^{m-\frac{1}{2}}e^{iw}. \tag{26}$$

From (23.2) there follows the asymptotic expansion for large w

$$F_2(1,\,-m+\tfrac{3}{2},\,iw)=(m+\tfrac{1}{2})(iw)^{-1}\times[1-(m+\tfrac{1}{2})(iw)^{-1}+\cdots]. \tag{27}$$

Further one gets from (21) and (26) the development

$$F_2(1,\,-m+\tfrac{3}{2},\,iw)=[1+(-m+\tfrac{3}{2})^{-1}iw+(-m+\tfrac{3}{2})^{-1}(-m+(5/2))^{-1}(iw)^2+\cdots]+\pi\Gamma(m-\tfrac{1}{2})^{-1}e^{-i(m\pi/2-\pi/4)}w^{m-\frac{1}{2}}e^{iw}, \tag{28}$$

which is especially practical for small values of w and shows that $F_2=1$ for $w=0$ when $m>0$, whereas for $m=0$ F_2 tends to infinity as $w^{-\frac{1}{2}}$ when w tends to zero.

The previously announced connection between the function $S_m(w)$, defined by (18), and the function $F_2(1,\,-m+\frac{3}{2},\,iw)$ defined by (25), is produced by the following transformation, which is a direct generalization of a transformation used by Sommerfeld in the case $m=0$. We substitute in (25) $t\rightarrow wt$ and get

$$F_2(1,\,-m+\tfrac{3}{2},\,iw)=\Gamma(m-\tfrac{1}{2})^{-1}w^{m-\frac{1}{2}}\times\int_0^\infty e^{-wt}t^{m-\frac{1}{2}}(t+i)^{-1}dt, \tag{25a}$$

hence

$$\frac{d}{dw}\left[e^{-iw}w^{-(m-\frac{1}{2})}F_2\right] = -\Gamma(m-\tfrac{1}{2})^{-1}e^{-iw}\int_0^\infty e^{-wt}t^{m-\frac{1}{2}}dt = -(m-\tfrac{1}{2})e^{-iw}w^{-(m+\frac{1}{2})}.$$

By integration with respect to w we obtain from this fixing the constant of integration by means of the requirement that F_2 vanish for $w=\infty$:

$$F_2(1, -m+\tfrac{3}{2}, iw) = (m-\tfrac{1}{2})w^{m-\frac{1}{2}}e^{iw} \times \int_w^\infty e^{-it}t^{-(m+\frac{1}{2})}dt,$$

therefore according to (18)

$$S_m(w) = (m-\tfrac{1}{2})^{-1}w^{\frac{1}{2}}F_2(1, -m+\tfrac{3}{2}, iw). \quad \text{(II)}$$

From (27) and (28) there follows now immediately the behavior of $S_m(w)$ for small and for large values of w; in the latter case:

$$S_m(w) = -iw^{-\frac{1}{2}}[1-(m+\tfrac{1}{2})(iw)^{-1}+\cdots] \quad \text{for } w\gg 1, \quad \text{(27a)}$$

whereas in the former case one has

$$S_m(w) \sim (m-\tfrac{1}{2})^{-1}w^{\frac{1}{2}} \text{ for } w\sim 0 \text{ and } m>0. \quad \text{(28a)}$$

Finally from (25a) the useful recursion formula results:

$$\frac{d}{dw}\left[F_2(1, -m+\tfrac{3}{2}, iw)w^{-(m-\frac{1}{2})}\right] = -w^{-(m+\frac{1}{2})}(m-\tfrac{1}{2})F_2(1, -m+\tfrac{1}{2}, iw),$$

hence

$$\frac{d}{dw}[w^{-m}S_m(w)] = -(m+\tfrac{1}{2})w^{-(m+1)}S_{m+1}(w). \quad (29)$$

3. The Asymptotic Series

We start from the expression (12) for

$$v_B = \frac{1}{2\pi i}\cdot\frac{1}{n}\sin\frac{\pi}{n}\int_{-i\infty-\epsilon}^{+i\infty+\epsilon} \times \frac{e^{-i\rho\cos\eta}}{\cos(\pi/n)-\cos[(\eta+\varphi)/n]}d\eta. \quad (12)$$

The method of steepest descent, mentioned at the end of Section 1 consists of the introduction of the variable

$$s^2 = -i(1-\cos\eta), \quad s = e^{-i\pi/4}2^{\frac{1}{2}}\sin\eta/2, \quad (30)$$

so that

$$e^{-i\rho\cos\eta} = e^{-i\rho}e^{-\rho s^2}$$

$$\frac{d\eta}{ds} = e^{i\pi/4}2^{\frac{1}{2}}\left(1-\frac{i}{2}s^2\right)^{-\frac{1}{2}}. \quad \text{(30a)}$$

As path of integration, the real s axis from $-\infty$ to $+\infty$ may be chosen. The obvious procedure would be to develop the whole integrand, except the exponential function into powers of s. Corresponding to the poles of the integrand at $\eta=-\varphi\pm\pi+2\pi nN$ with $N=0, \pm 1, \pm 2\cdots$, the result obtained in this way would, however, become infinite for the values $\varphi=\pm\pi+2\pi nN$ which correspond to the boundaries of the shadow.

We can avoid the singularity of the result, at least at one given boundary of the shadow, by not developing the whole integrand (except the exponential function), but by also extracting explicitly the factor $(\cos\eta+\cos\varphi)^{-1}$ in such a way, that we put

$$f(s,\varphi) = \frac{\cos\eta+\cos\varphi}{\cos\pi/n-\cos[(\eta+\varphi)/n]}\cdot\frac{1}{\cos\eta/2} \quad (31.1)$$

and

$$v_B = \frac{1}{2\pi n}\sin\frac{\pi}{n}\cdot 2^{\frac{1}{2}}e^{-i(\rho-\pi/4)} \times \int_{-\infty}^{+\infty} e^{-\rho s^2}f(s,\varphi)\frac{ds}{s^2+ia} \quad (31.2)$$

with the abbreviation $a=1+\cos\varphi$ introduced in (15), and with $\cos\eta+\cos\varphi=a-is^2$.

The essential effect of the addition of the factor $\cos\eta+\cos\varphi$ in the definition of $f(s,\varphi)$ is the fact that this function is now regular for $\varphi=\pm\pi$ and $\eta=0$ and becomes only infinite for $\eta=0$, if $\varphi=\pi+2\pi nN$, with N integer, but Nn not integer (particularly $N\neq 0$). Therefore our result can be used for $\varphi=\pm\pi$ and fails only on the other boundaries of the shadow. If one replaces in the definition of a and of $f(s, \varphi)$ and, in all following equations the variable φ by $\varphi-2\pi n N_0$, one gets for every given integer N_0 a development of v_B, which holds at $\varphi=\pm\pi+2\pi n N_0$ instead of at $\varphi=\pm\pi$.

If we consider now $f(s, \varphi)$ as developed in powers of s according to

$$f(s, \varphi)=\sum_{m=0}^{\infty} e^{im\pi/4}A_m(\varphi)s^m \tag{32}$$

and substitute this development in (31.2), the terms with odd m cancel when integrating and the terms with even m give after the substitution $s=\tau\rho^{-\frac{1}{2}}$ integrals of the type which occurred in (25). By using formula (II) of Section 2 one gets finally (compare (17)):

$$v_B=(2\pi a)^{-\frac{1}{2}}n^{-1}\sin\frac{\pi}{n}e^{-i(\rho-\pi/4)} \cdot\sum_m i^m\Gamma(m+\tfrac{1}{2})\pi^{-\frac{1}{2}}A_{2m}(\varphi)S_m(\rho a)\rho^{-m}. \tag{33}$$

The circumstance, however, that the series (32) converges only on a part of the path of integration, has the consequence that the series (33) in general has the meaning of a semi-convergent development for large values of ρ.

We did not calculate the rather complicated expression for $A_{2m}(\varphi)$ for a general m; but we wish to emphasize that these functions have a finite value for $\varphi=\pm\pi$. One recognizes this most simply by putting $\varphi=\pm\pi$ on the left side of (32) and developing in powers of s afterwards. For the first two functions $A_0(\varphi)$ and $A_2(\varphi)$ one gets from (30) and (32)

$$A_0(\varphi)=\frac{1+\cos\varphi}{\cos\pi/n-\cos\varphi/n};\qquad (2a)^{-\frac{1}{2}}A_0(\varphi)=\frac{|\cos\varphi/2|}{\cos\pi/n-\cos\varphi/n} \tag{34.1}$$

$$A_2(\varphi)=\frac{1}{4}A_0(\varphi)-\frac{1}{n^2}\cos\frac{\varphi}{n}\frac{1+\cos\varphi}{(\cos\pi/n-\cos\varphi/n)^2} +\frac{2[(1/n)\sin\varphi/n]^2(1+\cos\varphi)-(\cos\pi/n-\cos\varphi/n)^2}{(\cos\pi/n-\cos\varphi/n)^3}, \tag{34.2}$$

hence

$$[(2a)^{-\frac{1}{2}}A_0(\varphi)]_{\varphi=\pi\mp\epsilon}=\mp\tfrac{1}{2}(1/n\cdot\sin\pi/n)^{-1} \tag{34.3}$$

$$A_2(\pi)=\frac{1}{2}\frac{\cos\pi/n}{\sin^2\pi/n}. \tag{34.4}$$

The finiteness of the series for v_B at $\varphi=\pm\pi$ (where $a=0$), follows from the fact that, when $m>0$, the factor $a^{-\frac{1}{2}}$ is compensated by $S_m(\rho a)$ because of $S_m(\rho a)\sim(m-\frac{1}{2})^{-1}(\rho a)^{\frac{1}{2}}$ and when $m=0$, is compensated by $A_0(\varphi)$.

In the particular case $n=2$ one has

$$\tfrac{1}{2}[f(s)+f(-s)]=-2\cos\varphi/2$$

and therefore *all* $A_{2m}(\varphi)$ *vanish in this case identically for* $m>0$ and the whole series (33) reduces to its first term.

Writing out explicitly the terms of (33) for $m=0$ and $m=1$ we get according to (16)

$$v_B=\pi^{-\frac{1}{2}}e^{i\pi/4}(1/n\sin\pi/n) \times\left\{\frac{2|\cos\varphi/n|}{\cos\pi/n-\cos\varphi/n}e^{i\rho\cos\varphi}\int_{(a\rho)^{\frac{1}{2}}}^{\infty}e^{-i\tau^2}d\tau +\frac{i}{2}(2a)^{-\frac{1}{2}}A_2(\varphi)S_1(\rho a)\frac{e^{-i\rho}}{\rho}+\cdots\right\}. \tag{35}$$

For large values of ρa one gets from (27a)

$$v_B=(2\pi\rho)^{-\frac{1}{2}}e^{-i(\rho+\pi/4)} \times\frac{(1/n)\sin\pi/n}{\cos\pi/n-\cos\varphi/n}[1+(*)\rho^{-1}+\cdots]$$

in accordance with (13). On the other hand one has for $\varphi=\pi$ because of

$$\int_0^{\infty}e^{-i\tau^2}d\tau=\tfrac{1}{2}\pi^{\frac{1}{2}}e^{-i\pi/4}:$$

$$v_B=\mp\tfrac{1}{2}e^{-i\rho}+(i/2)(2\rho)^{-\frac{1}{2}} \times e^{-i\rho}(1/n)\cot\pi/n+\cdots. \tag{36}$$

The upper (lower) sign holds when approaching $\varphi=\pi$ from the illuminated (shadowy) side. The discontinuity of v_B on the boundary of the shadow is just compensated by the discontinuity of v^* in such a way that the sum $v=v^*+v_B$ is continuous. Further we can confirm a general result of Sommerfeld, according to which on the boundary of the shadow itself for large ρ the relation asymptotically holds

$$v=\tfrac{1}{2}v^*(\rho\gg 1,\ \varphi=\pm\pi+2\pi nN). \tag{37}$$

That not only for the function v_B, but also for all its derivatives the discontinuities on the boundary of the shadow are compensated for by those of v^*, can be confirmed by using Eq. (28). We do not want, however, to enter the details of this proof here.

Numerical estimates of $A_2(\varphi)$ gave the result that already the second term of the series (35) can be neglected in all practical cases so that the proposed task of finding a representation of the diffraction wave, which can be used for the transition from light to shadow and at large distances, can be considered as practically solved by the principal term of our series, as given in (35).

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On the Anomalous Propagation of Phase in the Focus

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The problem of anomalous phase propagation of a spherical wave at the focus has been discussed for the case of a diffracting aperture of arbitrary shape. The solution given by Kirchhoff's integral has been split up into an "incident light wave," which shows the distribution of light to be expected according to geometrical optics and a "diffracted wave," which may be thought of as due to scattering of the incident wave at the diffracting edge. A sudden change of phase by π has been shown to occur already in the incident wave. Thus we may, in this sense, consider this phenomenon as a geometric optical one.

The case of a circular diffracting aperture, the focus lying on the normal through its center, which has been treated usually, appears to be not very suitable for an experimental investigation of the discussed phenomenon. It is this particular shape of the diffracting edge, which produces diffraction phenomena of considerable light intensity along the optical axis. These, however, are not because of the existence of a focus, but only because of the particular shape of the diffracting aperture.

I. INTRODUCTION

SOMMERFELD'S first great scientific achievement was the solution of the problem of diffraction at a perfectly conducting half-plane by the methods of exact analysis.[1] On the occasion of his jubilee it may thus be appropriate to present a note, which deals with a related problem and is based on a paper[2] I wrote while staying at his institute at Munich. The problem in question is the anomalous phase propagation of a spherical wave at the focus, which was discovered by Gouy in 1890. As appears from the very rich literature[3] on this subject, the phenomenon is completely describable by wave optics. Still, I do not believe that these papers satisfy as yet our want for a simple, plausible interpretation of the problem.[4] An attempt, therefore, will be made in this note to fill this gap by following a new method of approach, starting from Kirchhoff's theory of diffraction

[1] A. Sommerfeld, Math. Ann. **47**, 317 (1896).

[2] A. Rubinowicz, Ann. d. Physik **53**, 257 (1917) and **73**, 339 (1924), to be referred to as I and II in the text.

[3] For extensive literature see F. Reiche, Ann. d. Physik **29**, 65, 401 (1909) and J. Picht, *Optische Abbildung* (Braunschweig, 1931).

[4] An elementary suggestive representation by means of Fresnel's zones has, however, been given by A. D. Fokker, Physica **3**, 334 (1923); **4**, 166 (1924).