The Electrodynamics of Material Media

CARL ECKART
Ryerson Laboratory, The University of Chicago, Chicago, Illinois

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The general equations of a gauge invariant, classical theory of the electrodynamics of material media are obtained. The gauge invariance is insured by taking the equations

$$\nabla\cdot\mathbf{D}=\rho,\quad \nabla\times\mathbf{H}-\mathbf{D}'=\mathbf{J}$$

as conditions auxiliary to the variation principle

$$\delta\int\int\int\int\{L+\Sigma_n\theta_n[N_n'+\nabla\cdot(N_n\mathbf{V}_n)]dvdt\}=0.$$

The Lagrangian function, L, depends on $\mathbf{D}$, $\mathbf{H}$, N_n, $\mathbf{V}_n$, θ_n and possibly their derivatives; here N_n is the numerical density of atoms in the state n, $\mathbf{V}_n$ their macroscopic or average velocity, and θ_n is a variable that functions as the velocity potential in some cases and has the dimensions of action. The electromagnetic potentials enter the theory as Lagrangian multipliers only.

It is shown that if there is only one state and

$$L=\tfrac{1}{2}Nm\mathbf{V}^2-(\hbar^2/8m)(\nabla N)^2/N+\tfrac{1}{2}(\mathbf{H}^2-\mathbf{D}^2),$$

then the Schrödinger wave equation is obtained on making the substitution

$$\psi=N^{\frac{1}{2}}\exp(-i\theta/\hbar).$$

If the atoms are stationary (so that terms in $\mathbf{V}_n$ may be neglected), and

$$L=\Sigma_nN_nW_n+\mathbf{D}\cdot\mathbf{P}+\tfrac{1}{2}(\mathbf{H}^2-\mathbf{D}^2),$$

where W_n is the energy of the nth state, and

$$\mathbf{P}=\Sigma\Sigma_{(m,\,n)}(N_mN_n)^{\frac{1}{2}}\mathbf{P}_{mn}\cos[(\theta_m-\theta_n)/\hbar+\alpha_{nm}]$$

is the polarization of the medium, an adequate theory of dispersion results. However, the spontaneous transitions are not correctly accounted for by the equations.

If the electromagnetic fields are neglected and

$$L=\tfrac{1}{2}Nm\mathbf{V}^2-U(N),$$

the equations are those for the irrotational motion of a gas, with θ/m as the velocity potential.

INTRODUCTION

ELECTRODYNAMICS has never been developed in a manner analogous to analytic mechanics. Some years ago, G. Mie[1] made an important attempt in this direction, and deduced the equations of his theory from a single Lagrangian function. One consequence of this attempt was a formulation of the requirement of gauge invariance[2] by H. Weyl; unfortunately, Mie's theory did not meet this requirement, nor can Weyl's geometrical resolution of the difficulty be considered as satisfactory at the present time. It is the purpose of this paper to indicate the possibility of a classical theory that is in some respects similar to Mie's and satisfies the requirement of gauge invariance.

In addition to its use of a Lagrangian function, analytic mechanics is characterized by the use of a great variety of coordinates. In any special case, these are chosen so as to make the introduction of empirical data into the calculations as simple as possible. No formal attempt to define the coordinates in terms of atomic quantities is made, nor to deduce their equations of motion from those of the ultimate particles. On the other hand, the terms and suggestions of atomic theory are often used freely in the informal explanations that accompany the mathematical theory.

A similar procedure was suggested by W. Heisenberg[3] as a basis for the development of quantum theory. It is interesting to note the close relation between the initial stages of the development of quantum theory and the classical theory outlined below. However, this does not make the later parts of quantum theory superfluous; instead, the need for them becomes more apparent.

THE GENERAL EQUATIONS

The medium may be composed of various kinds of particles, capable of existing in different states. For simplicity, only a finite number of states will be considered. Let N_n be the numerical density or concentration of particles in the state[4] n, and $\mathbf{V}_n$ their macroscopic or average velocity. Another scalar function of position and time, θ_n,

[1] G. Mie, Ann. d. Physik **37**, 511 (1912); **39**, 1 (1912); **40**, 1 (1913).

[2] H. Weyl, Sitzungsber. d. Preuss. Akad. (1918).

[3] W. Heisenberg, Zeits. f. Physik **33**, 879 (1925).

[4] For the present, there is no need to distinguish between the various kinds of particles. The notation can thus be simplified by avoiding all reference to the kinds of particles and numbering the states in some consecutive order.

will also be associated to the state n. This variable is analogous to the velocity potential of the irrotational motion of a fluid; it is also analogous to Dirac's[5] angle variable that is canonically conjugate to the square root of the total number of particles.

These variables will not, in general, be sufficient to describe the state of the medium: Such other quantities as the electromagnetic field, the gravitational field, and the temperature, are also needed. For the present, all of these except the electromagnetic field will be ignored, and this will be described by the vectors $\mathbf{D}$ and $\mathbf{H}$, that satisfy the Maxwell equations

$$\nabla\cdot\mathbf{D}=\rho, \quad \nabla\times\mathbf{H}-\mathbf{D}'=\mathbf{J}. \tag{1}$$

Here ρ and $\mathbf{J}$ are the electric charge and current densities, so that if e_n is the net charge carried by a particle of kind and state n, then

$$\rho=\Sigma_n N_n e_n, \quad \mathbf{J}=\Sigma_n N_n e_n \mathbf{V}_n. \tag{2}$$

The dynamical properties of the medium are supposed to be summarized by a variation principle involving the Lagrangian, L, which depends on N_n, $\mathbf{V}_n$, θ_n, $\mathbf{D}$ and $\mathbf{H}$. It may also depend on the space and time derivatives of these quantities, but for simplicity it will be supposed independent of all derivatives except ∇N_n. The variation principle is

$$\delta\int\int\int\int\{L+\textstyle\sum_n\theta_n \times[N_n'-\nabla\cdot(N_n\mathbf{V}_n)]\}dvdt=0, \tag{3}$$

subject to Eqs. (1) and (2) as auxiliary conditions. If L has the dimension of energy per unit volume, this equation shows that θ_n has the dimension of action. As is customary in physics, not much will be said concerning the boundary of the region of integration, nor about the conditions there; however, there is good reason to believe that this constitutes an unnecessary limitation upon the power of the theory.

The Eulerian equations of this problem will be obtained by the method of Lagrangian multipliers. Let ϕ and $-\mathbf{A}$ be the multipliers associated to the Eq. (1): Then the variational integral becomes

$$\delta\int\int\int\int\{L+\textstyle\sum_n\theta_n[N_n'+\nabla\cdot(N_n\mathbf{V}_n)] +\phi[\nabla\cdot\mathbf{D}-\rho]-\mathbf{A}\cdot[\nabla\times\mathbf{H}-\mathbf{D}'-\mathbf{J}]\}dvdt=0;$$

ρ and $\mathbf{J}$ are treated as abbreviations for the sums in Eq. (2). It is convenient to use the following notation: if $\mathbf{Q}=\mathbf{i}Q_x+\mathbf{j}Q_y+\mathbf{k}Q_z$ is any vector, then

$$(\partial L/\partial\mathbf{Q})=\mathbf{i}(\partial L/\partial Q_x)+\mathbf{j}(\partial L/\partial Q_y)+\mathbf{k}(\partial L/\partial Q_z).$$

The equations obtained from the variation of $\mathbf{D}$ and $\mathbf{H}$ are

$$(\partial L/\partial\mathbf{D})-\nabla\phi-\mathbf{A}'=0,$$
$$(\partial L/\partial\mathbf{H})-\nabla\times\mathbf{A}=0.$$

With the abbreviations

$$\mathbf{E}=-(\partial L/\partial\mathbf{D}), \quad \mathbf{D}=(\partial L/\partial\mathbf{H}), \tag{4}$$

they become

$$\mathbf{E}=-\nabla\phi-\mathbf{A}', \quad \mathbf{D}=\nabla\times\mathbf{A} \tag{5}$$

so that ϕ and $\mathbf{A}$ are seen to be the electromagnetic potentials. They do not necessarily satisfy Maxwell's auxiliary equation. They may be eliminated from Eq. (5) by differentiation, and there result the remaining Maxwell equations

$$\nabla\cdot\mathbf{B}=0, \quad \nabla\times\mathbf{E}+\mathbf{B}'=0. \tag{6}$$

The equation obtained by varying θ_n is

$$(\partial L/\partial\theta_n)+N_n'+\nabla\cdot(N_n\mathbf{V}_n)=0, \tag{7}$$

so that

$$G=-(\partial L/\partial\theta_n) \tag{8}$$

is the net rate (per unit of time and volume) at which particles are entering the state n. The result of the variation of N_n and $\mathbf{V}_n$ is

$$(\partial L/\partial N_n)-\nabla\cdot(\partial L/\partial\nabla N_n)-\theta_n' -\mathbf{V}_n\cdot\nabla\theta_n-e_n\phi+e_n\mathbf{V}_n\cdot\mathbf{A}=0, \tag{9}$$

$$(\partial L/\partial\mathbf{V}_n)-N_n\nabla\theta_n+N_n e_n\mathbf{A}_n=0. \tag{10}$$

The Eqs. (9) and (10) can be combined to give

$$\theta_n'=[(\partial L/\partial N_n)-\nabla\cdot(\partial L/\partial\nabla N_n)] -[\mathbf{V}_n\cdot(\partial L/\partial\mathbf{V}_n)]/N_n-e_n\phi, \tag{11}$$

$$\mathbf{V}_n\cdot\nabla\theta_n=[\mathbf{V}_n\cdot(\partial L/\partial\mathbf{V}_n)]/N_n-e_n\mathbf{V}_n\cdot\mathbf{A}, \tag{12}$$

which are frequently convenient.

Any pair of the Eqs. (10) and (11) may be used to eliminate ϕ and $\mathbf{A}$ from any other equa-

[5] P. A. M. Dirac, *Quantum Mechanics* (Oxford Press, 1935), Section 62 (second edition).

tions in which they may occur. This remark is the proof of the gauge invariance of the present theory.

After a somewhat tedious calculation, the energy equation is obtained as

$$\{\mathbf{B}\cdot\mathbf{H}-L+\sum_n\mathbf{V}_n\cdot(\partial L/\partial\mathbf{V}_n)\}'+\nabla\cdot\{\mathbf{E}\times\mathbf{H} +\sum_n[N_n'(\partial L/\partial\nabla N_n)+\mathbf{V}_n\mathbf{V}_n\cdot(\partial L/\partial\mathbf{V}_n) -N_n\mathbf{V}_n\{(\partial L/\partial N_n)-\nabla\cdot(\partial L/\partial\nabla N_n)\}]\},\quad(13)$$

and the momentum equation (x component only) as

$$\{(\mathbf{D}\times\mathbf{B})_x+\sum_nN_n(\partial\theta_n/\partial x)\}'+(\partial/\partial x)\{\mathbf{E}\cdot\mathbf{D} +L-\sum_nN_n[(\partial L/\partial N_n)-\nabla\cdot(\partial L/\partial\nabla N_n)]\} +\nabla\cdot\{-\mathbf{D}E_x-\mathbf{B}H_x-\sum_n[N_n\mathbf{V}_n(\partial\theta_n/\partial x) -(\partial N_n/\partial x)(\partial L/\partial\nabla N_n)]\}=0.\quad(14)$$

The calculation for Eq. (13) is as follows: *a* in the expression for L', the terms $\mathbf{B}\cdot\mathbf{H}'-\mathbf{E}\cdot\mathbf{D}'$ are transformed as in the usual proof of Poynting's theorem; *b* the term in $\nabla N_n'$ is transformed into a divergence and a term proportional to N_n'; *c* the partial derivatives of L are eliminated by Eqs. (7), (9) and (10); *d* on collecting terms, the Eqs. (11) and (12) yield Eq. (13). The calculation for Eq. (14) is similar.

Special Examples

1.

As a particular case of these equations, let there be only one kind of particle, capable of existing in only one state. Then, if

$$L=\tfrac{1}{2}Nm\mathbf{V}^2-(\hbar^2/8mN)(\nabla N)^2+\tfrac{1}{2}(\mathbf{H}^2-\mathbf{D}^2),\quad(15)$$

the Eqs. (7), (10) and (11) become[6]

$$N'+\nabla\cdot(N\mathbf{V})=0,$$
$$m\mathbf{V}=\nabla\theta-e\mathbf{A},$$
$$\theta'=-\tfrac{1}{2}m\mathbf{V}^2+(\hbar^2/4m)[\tfrac{1}{2}(\nabla N/N)^2 +\nabla\cdot(\nabla N/N)]-e\phi.\quad(16)$$

The substitution

$$\psi=N^{\frac{1}{2}}\exp(-i\theta/\hbar)\quad(17)$$

reduces these to Schrödinger's equation

$$-i\hbar\psi'=(1/2m)[-i\hbar\nabla+e\mathbf{A}]^2\psi+e\phi\psi.\quad(18)$$

[6] Since L is independent of θ, the latter may here be considered as a Lagrangian multiplier associated to the auxiliary condition $N'+\nabla\cdot(N\mathbf{V})=0$.

By introducing two and four states, respectively, and a suitable L, the Pauli and Dirac equations can presumably be obtained. In these cases, the Lagrangian must depend explicitly on the variables θ_n.

In considering the significance of this example, there is one point that must not be overlooked. The potentials ϕ and $\mathbf{A}$ that enter into Eq. (18) are not independent of ψ. Eq. (18) is only one consequence of the Eq. (16), and Eqs. (1), (2), (4) and (5) are to be solved simultaneously with Eq. (16). In particular, it can be shown that the assumption

$$\phi=-e/r,\quad \mathbf{A}=0$$

is inconsistent with these other equations. Now, it is only because of this special assumption regarding ϕ and $\mathbf{A}$ that Schrödinger obtained agreement with the empirical data on the spectrum of hydrogen[7]: Consequently it is not correct to say that the present theory contains the Schrödinger theory as a special case. This conclusion applies with redoubled force when polyelectronic atoms are considered.

2.

Equally important is the possibility of obtaining an analytic dispersion theory. For this purpose, the atoms may be supposed fixed, and the terms in $\mathbf{V}_n$ ignored. The states are to be interpreted as the discrete energy states (energy $=W_n$) of a single kind of atoms. The atoms may be supposed neutral, so that $\rho=0$, $\mathbf{J}=0$, and the polarization of the medium is assumed to be

$$\mathbf{P}=\sum\sum_{(m,\,n)}(N_mN_n)^{\frac{1}{2}}\mathbf{P}_{mn} \times\cos[(\theta_m-\theta_n)/\hbar+\alpha_{mn}],\quad(19)$$

where $\mathbf{P}_{nm}\exp(i\alpha_{nm})$ is the matrix element of the dipole moment. If

$$L=\Sigma_nN_nW_n+\mathbf{D}\cdot\mathbf{P}+\tfrac{1}{2}(\mathbf{H}^2-\mathbf{D}^2),\quad(20)$$

Eqs. (4) become

$$\mathbf{E}=\mathbf{D}-\mathbf{P},\quad \mathbf{B}=\mathbf{H},\quad(21)$$

and Eqs. (7) and (9) become, after neglecting terms in $\mathbf{V}_n$,

$$\begin{aligned}N_n'+\mathbf{D}\cdot(\partial\mathbf{P}/\partial\theta_n)&=0,\\ -\theta_n'+W_n+\mathbf{D}\cdot(\partial\mathbf{P}/\partial N_n)&=0.\end{aligned}\quad(22)$$

[7] E. Schrödinger, Ann. d. Physik (4) **79**, 361, 489 (1926).

A first approximation to the solution of these equations is sufficient to give an adequate account of dispersion phenomena; a second approximation will probably give a correct account of the phenomena of absorption and forced emission,[8] but it will be necessary to add new terms to the Lagrangian if the phenomenon of spontaneous emission is to be correctly included in this approximation. Presumably, with each successive approximation, new terms will be needed in the Lagrangian in order to bring the theory into agreement with the empirical data. If this series of terms converged to some simple function, the situation would be satisfactory; it is not possible to foresee that this will be the case.

It would be interesting to investigate the possibility of replacing Eq. (19) by another; the factors $(N_nN_m)^{\frac{1}{2}}$ are well known from the quantum theory, but seem somewhat strange from the classical point of view, unless the exponent arises from a root-mean-square average based on some statistical theory. It may be that some other dependence on the concentrations will be equally (or even, more) satisfactory.The success of such an investigation seems somewhat doubtful to the author (for reasons similar to those discussed at the end of Example 1), but it should be undertaken in any case; it lies beyond the scope of this outline.

3.

The equations for the irrotational motion of a gas are also a special case of the foregoing. The terms in the electromagnetic fields are unimportant in this connection and may be ignored; the Lagrangian is

$$L=\tfrac{1}{2}Nm\mathbf{V}^2-U(N). \qquad (23)$$

Since it is independent of the variable θ, the total number of atoms remains constant. The Eqs. (9) and (10) become

$$\begin{aligned}\tfrac{1}{2}m\mathbf{V}^2-(\partial U/\partial N)-\theta'-\mathbf{V}\cdot\nabla\theta=0,\\ m\mathbf{V}-\nabla\theta=0,\end{aligned} \qquad (24)$$

so that θ/m is the ordinary velocity potential in this case. The Eqs. (24) are not Euler's equations of hydrodynamics; the latter are essentially Eq. (14), which reduces to

$$\begin{aligned}(NmV_x)'+(\partial/\partial x)[N(\partial U/\partial N)-U]\\ +\nabla\cdot[Nm\mathbf{V}V_x]=0,\end{aligned} \qquad (25)$$

in this case. The pressure is thus related to the function U by the equation

$$p=N(\partial U/\partial N)-U. \qquad (26)$$

If $U=aN^{\gamma+1}$, γ being the ratio of the gas constant to the specific heat at constant volume, then $p=a\gamma N^{\gamma+1}$ and the motion is adiabatic. If $U=NRT\log N$, where R is the gas constant per molecule and $T=\text{constant}$, then $p=NRT$, and the motion is isothermal, with temperature T.

Conclusion

The "analytic" or phenomenologic electrodynamics outlined here cannot pretend to compete with quantum electrodynamics in the domain proper to the latter. However, the former should not be regarded only as a branch of pure mathematics that happens to use the terms of physics. There are undoubtedly many macroscopic phenomena whose laws are special cases of the equations developed above.

Neither is this theory merely the Maxwell-Lorentz theory in a new form. The latter contains Eqs. (1), (2), and (6), together with analogs to Eqs. (4), (13), and (14); it does not contain analogs to Eqs. (9) and (10).

It is my privilege to dedicate this paper to Professor Arnold Sommerfeld, as a greeting on the occasion of his seventieth birthday.

[8] Except for notation, these equations are very similar to those studied by Dirac, Proc. Roy. Soc. **114**, 710 (1927). The statements in the text are based on this similarity.