Symmetry Effects in the Spacing of Nuclear Energy Levels

JOHN BARDEEN,* *Harvard University, Cambridge, Massachusetts*

AND

EUGENE FEENBERG, *New York University, New York City*

(Received June 27, 1938)

The results of Wigner on mass defects and stability relations make possible a calculation of level density for intermediate nuclei ($A<60$) which takes properly into account the dependence of nuclear energies on symmetry character. In general, a configuration of neutrons and protons in single-particle orbits contains many different types of symmetry compatible with the exclusion principle. The various symmetry types arising from one configuration all have the same kinetic energy, but differ in potential energy; decreasing symmetry (increasing number of nodes) is associated with decreasing potential energy. Level densities for the different nuclear types are tabulated. The results indicate a marked increase of density for increasing isotopic number in an isobaric series. Transmutation experiments on isobars should make possible a test of the theory.

INTRODUCTION

TO discuss the problem of nuclear level density Bethe[1] and also Lier and Uhlenbeck[2] utilized the model of noninteracting particles in a potential well. The single-particle energy levels may be ϵ_1, ϵ_2, $\cdots$ arranged in ascending order. The total energy

$$E=\sum\epsilon_i n_i \tag{1}$$

is simply a sum of single-particle energies. Since no more than two neutrons and two protons can occupy a single orbit, the occupation numbers n_i are restricted to the values 0, 1, 2, 3, 4. Let δX designate the number of E's which fall in the energy interval E, $E+\delta E$. If δX is large in comparison with unity, and the average level spacing does not change appreciably over δE, we can define the density of levels $\rho(E)$ by the relation $\rho(E)\sim\delta X/\delta E$; the average spacing between levels is given by $1/\rho(E)\sim\delta E/\delta X$. For excitation energies $Q=E-E_0$ which are much smaller than the normal state energy E_0, Bethe and Lier and Uhlenbeck found the asymptotic formula

$$\rho(Q)\sim\tfrac{1}{4}(\Delta_\nu^2\Delta_\pi^2/54Q^5\Delta^3)^{\frac{1}{4}}\exp 2\pi(2Q/3\Delta)^{\frac{1}{2}},$$
$$2/\Delta=1/\Delta_\nu+1/\Delta_\pi. \tag{2}$$

Here Δ_ν and Δ_π are the average spacings between adjacent single-particle levels at the tops of the neutron and proton distributions for the normal state; these quantities have the values

$$\Delta_\nu\sim 31.6/B^2A^{\frac{2}{3}}(2N)^{\frac{1}{3}}\cdot mc^2,$$
$$\Delta_\pi\sim 31.6/B^2A^{\frac{2}{3}}(2Z)^{\frac{1}{3}}\cdot mc^2, \tag{3}$$
$$\Delta\sim 31.6/B^2A\cdot mc^2$$

for a system of Z protons and $N=A-Z$ neutrons in a spherical box of radius $R=BA^{\frac{1}{3}}e^2/mc^2$. Separate density functions for each value of the orbital angular momentum have also been computed.[1,3]

Certain properties of the density function are readily understood if the general distribution ϵ_1, ϵ_2, $\cdots$ is replaced by a system of uniformly spaced single-particle levels with the spacing Δ. It is clear that the addition or removal of an even number of neutrons or protons does not alter the number of ways in which a given excitation energy can be obtained provided only that the excitation energy is not large enough to raise a particle from the bottom of the single-particle distribution. However, the proportion of excited states in which a single particle obtains a large fraction of the excitation energy is negligible for large values of Q. Consequently the density function will not depend explicitly on Z or N within an even-even,

* Society of Fellows.

[1] H. A. Bethe, Phys. Rev. **50**, 332 (1936); Rev. Mod. Phys. **9**, 69 (1937).

[2] C. van Lier and G. E. Uhlenbeck, Physica **4**, 531 (1937).

[3] J. Bardeen, Phys. Rev. **51**, 799 (1937).

even-odd, or odd-odd series of nuclei; actually the asymptotic expression for $\rho(Q)$ does not involve explicitly even the odd-even character of the nucleus.

To extend the analysis of the independent particle model we have computed separate density functions $\rho_S(Q)$ for each value of the total spin in the nuclear series $4k\pm1$.

If the nuclear Hamiltonian H were known and a complete set of simple approximate wave functions $\psi_1, \psi_2, \cdots$ were available, an approximate evaluation of the density could be obtained by computing the diagonal matrix elements of H and identifying the level spacing with the separation of adjacent diagonal matrix elements. One of the authors[3] has made a calculation in this manner using exchange forces in the Hamiltonian and determinants constructed from products of single-particle wave functions for ψ_i. The problem can be reduced to that of a system of independent particles in a velocity dependent[4] potential well. At the top of the normal state distribution, the spacing between adjacent single-particle levels is about double that given in (3). Thus, for a given nuclear radius and excitation energy, this model yields a much smaller density than the true independent particle model.

The theory could be improved further by a first-order perturbation calculation. Generally a number of determinantal wave functions can be constructed from one configuration of single-particle orbits. In place of the determinants ψ_i, one should use the "correct" zero*th* order normalized and orthogonal linear combinations defined by the two conditions: (a) φ_i is a linear combination of determinants belonging to a single configuration, (b) $\int\cdots\int \varphi_i^* H \varphi_j d\tau = 0$ if $i \neq j$ and the two functions belong to the same configuration. With exchange forces the diagonal matrix elements $\int\cdots\int \varphi_i^* H \varphi_i d\tau$ arising from a single configuration would be spread generally over a wide range in the manner required by the observed dependence of nuclear energies on symmetry character. Because of computational difficulties this calculation has not been attempted. We present, in the following sections, an alternative attack on the problem which avoids the difficulties and some of the inaccuracies of perturbation and variational calculations.

[4] J. H. Van Vleck, Phys. Rev. **48**, 367 (1935).

Dependence of Potential Energy on Symmetry Character

The empirical mass defects and stability relations reveal a strong dependence of nuclear binding energy on symmetry character. A semi-empirical theory of this effect has been developed by Wigner[5] under the assumptions (a) the particles interact in pairs, (b) the spin dependent and Coulomb interactions are effectively small in comparison with the part of the Hamiltonian operator which is invariant under permutations of the space coordinates of the particles. The diagonal matrix elements of the potential energy then contain a symmetry dependent term having the form

$$V' = \tfrac{1}{2}A(A-1)\chi(P)/\chi(1)\cdot L(A), \qquad (4)$$

where $\chi(P)$ is the character of a transposition and $\chi(1)$ the character of the identity in the irreducible representation of the symmetric group to which the state in question belongs. The coefficient $L(A)$ is a matrix element involving the density functions for symmetrically and antisymmetrically coupled particles and the magnitude and range of the interaction operator between pairs of particles. The values of $L(A)$ required to fit the experimental material fall on a smooth curve with the correct asymptotic behavior. In this paper we take

$$L(A)\sim\Delta\sim 31.6/B^2A\cdot mc^2, \qquad (5)$$

which is sufficiently close to Wigner's semi-empirical determination and convenient for the discussion of level density. Since energy differences only are involved in the theory of level density (5) may be replaced by a more convenient form

$$V'' = V'(\lambda) - V'(\lambda_0) = \omega(\lambda, \lambda_0)L(A),$$
$$\omega(\lambda, \lambda_0) = \tfrac{1}{2}A(A-1)\{(\chi(P)/\chi(1))_\lambda - (\chi(P)/\chi(1))_{\lambda_0}\} \geqq 0. \qquad (6)$$

Here λ_0 and λ are quantum numbers (partition symbols) describing the symmetry character of the normal and excited states respectively.

[5] E. Wigner, Phys. Rev. **51**, 106 (1937), (A); **51**, 947 (1937), (B).

The unsymmetrical part of the Coulomb interaction,

$$\sum_{1\leq i<j\leq Z} e^2/r_{ij} - Z(Z-1)/A(A-1)\cdot \sum_{1\leq i<j\leq A} e^2/r_{ij}, \quad (7)$$

eventually becomes large enough to destroy the symmetry character as a good quantum number by mixing together states belonging to different irreducible representations of the symmetric group; this effect probably accounts for the failure of Wigner's theory beyond mass 54. However, the usefulness of (5) and (6) in a calculation of level density should extend somewhat beyond $A=54$, because a quantity defined as an average over a number of levels should not be sensitive to a lack of sharp definition in some of the quantum numbers.

Level Densities with Constant and Symmetry Dependent Potential Energies

By using (6) to compute potential energy differences, one avoids the approximations and special assumptions involved in an actual calculation of the potential energy matrix elements. Kinetic energy differences are presumably given with reasonable accuracy by the model of noninteracting particles in a spherical box with a constant spacing Δ of single-particle levels at the top of the normal state distribution.

The symmetry properties of a configuration are completely determined by the distribution of particles in singly, doubly, triply, and quadruply occupied orbits. For the discussion of symmetry properties the configuration may be represented by a partition symbol $\mu=(\mu_l, \mu_{l-1}, \cdots \mu_1)$ with as many μ's equal to m as there are orbits containing m particles. Note that $A=\mu_l+\mu_{l-1}+\cdots+\mu_1$. To avoid ambiguity the μ's are ordered so that $\mu_l\geqslant\mu_{l-1}\geqslant\cdots\geqslant\mu_1\geqslant 1$.

TABLE I. *Constant potential energy. $\rho_S(K\Delta)\Delta$ and $\rho(K\Delta)\Delta$ for the nuclear type $4k\pm1$.*

K	$\rho_S(K\Delta)\Delta$ $S=1/2$	$S=3/2$	$S=5/2$	$S=7/2$	$S=9/2$	$\rho(K\Delta)\Delta$
0	1					2
1	4	1				12
3	37	17	1			148
6	496	316	54	1		2590
9	4290	3220	812	55		26760
12	28500	23780	7510	850	18	204200
15	158200	142200	52300	8000	370	1266800

Let $P(\mu;K)$ denote the number of modes of excitation associated with the partition symbol μ and the kinetic energy $K\Delta$, and $M(S, T_z;\mu)$ the number of times the state with $T_z=\frac{1}{2}(N-Z)$ and total spin S occurs in a configuration belonging to μ.[6] Assuming a constant potential energy, the level densities $\rho_S(K\Delta)$ and $\rho(K\Delta)$ are given by the expressions

$$\begin{aligned}\rho_S(K\Delta)\Delta&=\sum_\mu M(S, T_z;\mu)P(\mu;K),\\ \rho(K\Delta)\Delta&=\sum_\mu\{\sum_S(2S+1)M(S, T_z;\mu)\}P(\mu;K).\end{aligned} \quad (8)$$

The set of functions generated from a single configuration by permutations of coordinates subtends a linear manifold in function space; with respect to the transformations of the symmetric group this manifold, $\Delta(\mu)$, is either irreducible or a linear combination of invariant subspaces which are irreducible. The invariant subspaces are denoted by the symbols $[\lambda]$ $[\lambda_k, \lambda_{k-1}, \cdots, \lambda_1]$, $\lambda_k\geqslant\lambda_{k-1}\geqslant\cdots\geqslant\lambda_1\geqslant 1$, $\lambda_k+\lambda_{k-1}+\cdots+\lambda_1=A$ and the number of times the irreducible representation $[\lambda]$ occurs in $\Delta(\mu)$ by

$$(\lambda/\mu)=\binom{\lambda_k\lambda_{k-1}\cdots\lambda_1}{\mu_l\mu_{l-1}\cdots\mu_1}.$$

To compute the level density when the dependence of potential energy on symmetry character is given by (6), we introduce two new quantities:

$$P'(\lambda, K)=\sum_\mu(\lambda/\mu)P(\mu;K),$$

the number of times the irreducible representation $[\lambda]$ occurs in the various modes of excitation, and $M'(S, T_z;\lambda)$, the number of ways in

TABLE II. *Symmetry dependent potential energy. $\rho_S'(K\Delta)\Delta$ and $\rho'(K\Delta)\Delta$ for the nuclear types $4k\pm1$, $|N-Z|=1$.*

K	$\rho_S'(K\Delta)\Delta$ $S=1/2$	$S=3/2$	$S=5/2$	$S=7/2$	$\rho'(K\Delta)\Delta$
0	1				2
1	2				4
3	11				22
6	104	8			240
9	695	105			1810
12	3890	833	9		11160
15	18850	5140	133		59000
18	82200	26700	1260	1	278000

[6] Methods used for computing the multiplicities are given in the appendix, part B.

TABLE III. *Symmetry dependent potential energy.* $\rho_S'(K\Delta)\Delta$ *and* $\rho'(K\Delta)\Delta$ *for the nuclear types* $4k\pm1$, $|N-Z|=3$.

K	$\rho_S'(K\Delta)\Delta$ $S=1/2$	$S=3/2$	$S=5/2$	$S=7/2$	$\rho'(K\Delta)\Delta$
0	1				2
1	3				6
3	20	1			44
6	194	24			484
9	1330	258	1		3690
12	7420	1880	34		22550
15	35700	10980	390		117600
18	153800	54700	2970	4	544300

TABLE IV. *Symmetry dependent potential energy.* $\rho_S'(K\Delta)\Delta$ *and* $\rho'(K\Delta)\Delta$ *for the nuclear types* $4k\pm1$, $|N-Z|=5$.

K	$\rho_S'(K\Delta)\Delta$ $S=1/2$	$S=3/2$	$S=5/2$	$S=7/2$	$\rho'(K\Delta)\Delta$
0	1				2
1	3				6
3	23	1			50
6	252	34			640
9	1831	387	3		5230
12	10570	2900	73		33160
15	51900	17080	747		176700
18	226500	85500	5400	14	827000

which the total spin S can be associated with the irreducible representation $[\lambda]$ to yield a completely antisymmetric function in the space, spin and charge spin coordinates of the particles. For the densities $\rho_S'(K\Delta)$ and $\rho'(K\Delta)$, which replace $\rho_S(K\Delta)$ and $\rho(K\Delta)$, we obtain

$$\begin{aligned}\rho_S'(K\Delta)\Delta&=\sum_\lambda M'(S,T_z;\lambda)P'(\lambda;K-\omega(\lambda,\lambda_0))\\&=\sum_\lambda\sum_\mu M'(S,T_z;\lambda)(\lambda/\mu)\\&\qquad\times P(\mu;K-\omega(\lambda,\lambda_0)),\\\rho'(K\Delta)\Delta&=\sum_\lambda\{\sum_S(2S+1)M'(S,T_z;\lambda)\}\\&\qquad\times P'(\lambda;K-\omega(\lambda,\lambda_0))\\&=\sum_\lambda\sum_\mu\{\sum_S(2S+1)M'(S,T_z;\lambda)\}\\&\qquad\times(\lambda/\mu)P(\mu;K-\omega(\lambda,\lambda_0)).\end{aligned}\tag{9}$$

Evidently

$$M(S,T_z;\mu)=\sum_\lambda M'(S,T_z;\lambda)(\lambda/\mu)\tag{10}$$

and the primed and unprimed densities are identical if ω is omitted in the argument of $P'(\lambda;K)$. Since $P'(\lambda;K)$ is a rapidly increasing function of K, the substitution of $K-\omega$ for K results in a considerable decrease in level density.

Numerical results from the evaluation of (8) and (9) are exhibited in Tables I–VIII. The method of calculation is described in the appendix.

The tables reveal a striking *dependence of level density on* $|N-Z|$ *in an isobaric series.* This consequence of the theory can be tested directly by transmutation experiments on isobars. The computed level spacing may very well be considerably in error because of the unavoidable oversimplification in the model and because of uncertainty in the correct value of the energy unit Δ. However the ratios of level densities at a definite excitation energy in a series of isobars should be free from much of the inaccuracy and uncertainty which affects the computed spacing of levels.

The essential reason for the increase in level density with increasing $|N-Z|$ in an isobaric series is that when $|N-Z|$ is small the lowest state belongs to a representation of very high symmetry and states belonging to practically all other representations have much higher energies. Thus most of the low-lying levels belong to a comparatively few representations which have high symmetry. On the other hand, if $|N-Z|$ is large, even the lowest state cannot have high symmetry and states belonging to many other representations have nearly as low an energy.

TABLE V. *Symmetry dependent potential energy.* $\rho_S'(K\Delta)\Delta$ *and* $\rho'(K\Delta)\Delta$ *for the nuclear types* $4k\pm1$, $|N-Z|=7$.

K	$S=1/2$	$\rho'(K\Delta)\Delta$ $S=3/2$	$S=5/2$	$S=7/2$	$\rho'(K\Delta)\Delta$
0	1				2
1	3				6
3	24	1			52
6	278	37			704
9	2120	457	4		6100
12	12700	3570	98		40290
15	64100	21710	1030	1	221200
18	284500	110700	7500	27	1057000

TABLE VI. *Nuclear type* $4k$. $\rho(K\Delta)\Delta$ *and* $\rho'(K\Delta)\Delta$ *as functions of* $|N-Z|$.

K	$\rho(K\Delta)\Delta$ N,Z EVEN	N,Z ODD	$\rho'(K\Delta)\Delta$ $\lvert N-Z\rvert=0$	$\lvert N-Z\rvert=2$	$\lvert N-Z\rvert=4$	$\lvert N-Z\rvert=6$	$\lvert N-Z\rvert=8$
0	1	4	1	4	1	4	1
1	8	16	1	12	2	16	2
3	112	192	6	76	21	124	22
5	856	1330	30	352	124	664	141
7	4820	7170	134	1716	589	2890	734
9	22320	31990	535	5040	2380	10880	3210
11	90140	100340	1970	16610	8580	37140	12320
13	327700	446200	6660	50720	28320	116730	42750
15	1095300	1439000	20970	146500	86600	344700	136600

APPENDIX. PART A

Calculation of $P'(\lambda; K)$

We are interested in finding $P'(\lambda; K)$, the number of times the irreducible representation $[\lambda]$ occurs for a given (kinetic) excitation energy K. (To simplify the notation the energy unit is taken equal to Δ). Some of the calculations have been made by using the equation

$$P'(\lambda; K) = \sum_{\mu} (\lambda/\mu) P(\mu; K). \qquad (11)$$

We first find $P(\mu; K)$, the number of configurations which have the energy K, multiply by (λ/μ), the number of times the irreducible representation $[\lambda]$ occurs in the manifold $\Delta(\mu)$, and finally sum over all configurations. For this purpose we have prepared tables of the numbers (λ/μ) for configurations of interest which include as many as twenty particles outside of closed four-groups.[7]

In many cases it is simpler to calculate $P'(\lambda; K)$ by a direct method; in the following sections we indicate how this may be done. In a sense, our problem is a generalization of Fermi-Dirac and Einstein-Bose statistics. In the former we want to know the number of times the antisymmetric representation occurs for a given energy; in the latter the number of times the symmetric representation occurs. In the present case we are interested in representations of intermediate symmetry types.

TABLE VII. *Nuclear type 4k+2. $\rho'(K\Delta)\Delta$ as a function of $|N-Z|$.*

K	$\rho'(K\Delta)\Delta$ $\|N-Z\|=0$	$\|N-Z\|=2$	$\|N-Z\|=4$	$\|N-Z\|=6$	$\|N-Z\|=8$
0	4	1	4	1	4
1	8	2	16	2	16
3	56	18	112	22	128
5	256	88	564	138	704
7	1008	378	2332	696	3150
9	3610	1460	8480	2980	12200
11	11780	5120	28180	11340	42520
13	36160	16770	86500	39370	136000
15	104800	51000	250300	127200	406300

[7] For a discussion of methods which may be used to compute the numbers (λ/μ) see F. D. Murnaghan, A. J. of Math. **59**, 437 (1937). Tables of (λ/μ) for values of n up to and including nine are given in this paper. We have prepared tables of these numbers for configurations of interest with values of n extending to twenty. We will be glad to supply the tables we have computed to any one who has use for them.

TABLE VIII. *Summary of level density dependence on nuclear type and $|N-Z|$. Ratio of $\rho'(K\Delta)$ to the level density for the nuclear type 4k, $|N-Z|=0$.*

K	$\|N-Z\|=0$	$\|N-Z\|=2$	$\|N-Z\|=4$	$\|N-Z\|=6$	$\|N-Z\|=8$
			NUCLEAR TYPE $4k$		
2	1	10.7	2.3	16.0	2.3
5	1	11.7	4.1	22.1	4.7
10	1	8.8	4.4	19.2	6.0
15	1	7.0	4.1	16.4	6.5
			NUCLEAR TYPE $4k+2$		
2	8.0	2.0	14.7	2.3	16.0
5	8.5	2.9	18.8	4.6	23.5
10	6.2	2.6	14.7	5.6	21.7
15	5.0	2.4	11.9	6.1	19.4
			NUCLEAR TYPE $4k\pm1$		
K	$\|N-Z\|=1$	$\|N-Z\|=3$	$\|N-Z\|=5$	$\|N-Z\|=7$	
2	3.3	5.3	6.0	6.0	
5	3.8	7.3	9.7	10.5	
10	3.2	6.5	9.4	11.1	
15	2.8	5.6	8.4	10.5	

A similar problem, which occurs in the case of electrons, is to determine the number of states of a given energy with a given total spin S. The procedure which is generally followed is to determine first the number of states $\rho_M(M_z)$ with a z component of spin equal to M_z. The number of levels, $\rho(S)$, with spin S, is then found from the familiar equation:

$$\rho(S) = \rho_M(S) - \rho_M(S+1). \qquad (12)$$

A similar procedure may be followed in the present case. Instead of a two-valued spin we must now consider a spin with four components $(\eta_1, \eta_2, \eta_3, \eta_4)$ which, according to Wigner,[8] we may define as follows:

η_1 = number of protons with $-$ spin
η_2 = " " neutrons " $-$ "
η_3 = " " protons " $+$ "
η_4 = " " neutrons " $+$ " .

The number of states, $N(\eta_1, \eta_2, \eta_3, \eta_4; K)$, of energy K and with spin components $(\eta_1, \eta_2, \eta_3, \eta_4)$ may be found by some standard method such as that of Sommerfeld.[9] The number of levels, $P'(\lambda; K)$, belonging to a given representation

[8] Reference 5, (A).
[9] A. Sommerfeld, Zeits. f. Physik **47**, 1 (1927). See also H. A. Bethe, Rev. Mod. Phys. **9**, 69 (1937).

$[\lambda]$ of the symmetric group may be obtained from a generalization of (12). While the equation is simple in form, it is rather difficult to use, so that we give only the result.

Let us define operators x_1, x_2, x_3, x_4 such that

$$x_1N(\eta_1, \eta_2, \eta_3, \eta_4; K)=N(\eta_1-1, \eta_2, \eta_3, \eta_4; K),$$

$$x_2N(\eta_1, \eta_2, \eta_3, \eta_4; K)=N(\eta_1, \eta_2-1, \eta_3, \eta_4; K), \text{ etc.}$$

The appropriate generalization of (12) is

$$\begin{aligned} P'(\lambda; K)=&(x_4-x_3)(x_4-x_2)(x_4-x_1)\\ &\times(x_3-x_2)(x_3-x_1)(x_2-x_1)\\ &\times N(\Lambda_1, \Lambda_2+1, \Lambda_3+2, \Lambda_4+3), \end{aligned} \quad (13)$$

where $[\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4]$ is the representation associate to $[\lambda_k, \lambda_{k-1}, \cdots \lambda_1]$. If in the representation $[\lambda]$ there are n_1 ones, n_2 twos, n_3 threes, and n_4 fours,

$$\begin{aligned} \Lambda_4&=n_4+n_3+n_2+n_1,\\ \Lambda_3&=n_4+n_3+n_2,\\ \Lambda_2&=n_4+n_3,\\ \Lambda_1&=n_4. \end{aligned} \quad (14)$$

In general, there may be as many as 2^6 or 64 terms to sum in (13), so that the expression is not very suitable for numerical work. We have found that simpler equations are obtained if, instead of considering the spin, we deal directly with the space functions.

Generating function for $P(\mu; K)$

Let us return to Eq. (11). The various quantities in which we are interested can be computed from generating functions. Let us first consider $P(\mu_l, \cdots \mu_1; K)$, the number of modes of excitation of energy K associated with the partition $(\mu_l, \cdots, \mu_1)$. In the partition there are γ_1 singly occupied orbits, γ_2 doubly occupied orbits, etc. The total number of particles $n=\gamma_1+2\gamma_2+\cdots+h\gamma_h$. For the application to nuclei, we are interested in the case where no orbital is occupied by more than four particles, but we will carry through the analysis for the general case. Let us define the quantities*

$$U_i=y^{\epsilon_i}, \qquad (i=1, 2, \cdots m),$$

where y is an indeterminant. In order to avoid dealing with an infinite number of U_i's, we take m finite but so large that $\epsilon_m>K$. It is easily seen that $P(\mu; K)$ is the coefficient of y^K in the symmetric function

$$\begin{aligned} &\sum(U_1\cdots U_{\gamma_1})(U_{\gamma_1+1}\cdots U_{\gamma_1+\gamma_2})^2\\ &\qquad\times(U_{\gamma_1+\gamma_2+1}\cdots U_{\gamma_1+\gamma_2+\gamma_3})^3\cdots. \end{aligned} \quad (15)$$

The sum is over all $m!/(m-\sum\gamma_i)!\gamma_1!\gamma_2!\cdots\gamma_h!$ combinations of the subscripts which lead to different products.

There are no simple general expressions for the numbers (λ/μ) and the generating function (15) for $P(\mu; K)$ is rather complicated. Nevertheless, a rather simple expression can be obtained for the generating function for $P'(\lambda; K)$. In order to derive this expression, we make use of the method of characteristics.

Method of characteristics

Since the method of characteristics is perhaps not very familiar, we will give in outline the results which we will need later. For further details, the reader is referred to a recent article by Murnaghan.[10] The characteristic of an irreducible representation (p) of a finite group is defined by:

$$\phi_p(s)=\frac{1}{N}\sum_q N_q\chi_p{}^{(q)*}s^{(q)}, \quad (16)$$

where N is the total number of elements in the group; N_q is the number of elements in the class (q); $\chi_p{}^{(q)}$ is the character of the irreducible representation p going with the class q; and the $s^{(q)}$ are indeterminants. The sum is over all classes. The asterisk represents the complex conjugate.

The characteristic of a reducible representation which contains the irreducible representation (p) c_p times is simply

$$\begin{aligned} \phi(s)\equiv\sum_q a_q s^{(q)}&=\sum_p c_p\phi_p(s)\\ &=\frac{1}{N}\sum_p\sum_q c_pN_q\chi_p{}^{(q)*}s^{(q)}. \end{aligned} \quad (17)$$

* The ϵ_i are the individual particle energies defined in the introduction.

[10] Reference 7. The complete theory of the representations of the symmetric group is given in this article. We have attempted to follow Murnaghan's notation wherever possible.

From the orthogonality of the characters, we have

$$c_p=\sum_q \chi_p{}^{(q)} a_q. \tag{18}$$

If the characteristic of a reducible representation is known, this equation enables one to obtain the number of times a given irreducible representation is contained in the reducible representation.

Application to the symmetric group

The method of characteristics is particularly suitable for application to the symmetric group. All permutations having the same cyclic structure belong to the same class, which may be denoted by the symbol $(\alpha)=(\alpha_1, \alpha_2, \cdots, \alpha_r)$. The permutations of the class contain α_1 unary cycles, α_2 binary cycles, α_3 ternary cycles, etc. Note that $\alpha_1+2\alpha_2+3\alpha_3+\cdots+r\alpha_r=n$, the total number of particles on which the permutations act. The number of permutations in the class (α) is

$$N_{(\alpha)}=n!/1^{\alpha_1}2^{\alpha_2}\cdots r^{\alpha_r}\alpha_1!\alpha_2!\cdots\alpha_r!. \tag{19}$$

The indeterminant $s^{(q)}$ going with the class (α) may be expressed in the form:

$$s^{(q)}=s_1{}^{\alpha_1}s_2{}^{\alpha_2}\cdots s_r{}^{\alpha_r}. \tag{20}$$

The principal characteristic of the permutation group of order n (i.e., the characteristic of the identity representation) is:

$$q_n(s)=\sum_\alpha 1/\alpha_1!\alpha_2!\cdots\alpha_r! \cdot(s_1/1)^{\alpha_1}(s_2/2)^{\alpha_2}\cdots(s_r/r)^{\alpha_r}. \tag{21}$$

The sum is over all values of $\alpha_1, \alpha_2, \cdots, \alpha_r$ subject to the restriction $\alpha_1+2\alpha_2+\cdots+r\alpha_r=n$. The characteristic of the irreducible representation $[\lambda_k, \lambda_{k-1}, \cdots, \lambda_1]$ is:

$$\phi_\lambda(s)=\sum_\alpha \chi_{(\lambda)}{}^{(\alpha)}/\alpha_1!\alpha_2!\cdots\alpha_r! \cdot(s_1/1)^{\alpha_1}(s_2/2)^{\alpha_2}\cdots(s_r/r)^{\alpha_r} \tag{22}$$

$$=\begin{vmatrix} q_{\lambda_k}(s) & q_{\lambda_k+1}(s) & \cdots & q_{\lambda_k+k-1}(s) \\ q_{\lambda_{k-1}-1}(s) & q_{\lambda_{k-1}}(s) & \cdots & q_{\lambda_{k-1}+k-2}(s) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ q_{\lambda_1-k+1}(s) & q_{\lambda_1-k+2}(s) & \cdots & q_{\lambda_1}(s) \end{vmatrix}. \tag{22'}$$

Here $q_\lambda(s)$ is the principle characteristic of the permutation group of order λ. By convention, $q_0(s)=1$ and $q_\lambda(s)$ vanishes if λ is negative. The fact that (22′) gives a characteristic of an irreducible representation was first pointed out by Frobenius, although the above form is due to Schur. It is this expression which gives the justification for the characterization of an irreducible representation of the symmetric group in terms of a partition of the integer n.

It is sometimes convenient to express the indeterminants s_i in terms of other indeterminants $z_1, z_2, \cdots, z_{m'}$, by means of the relations:

$$s_i=z_1{}^i+z_2{}^i+\cdots+z_{m'}{}^i, \tag{23}$$

where m' is some large number ($\geqslant n$). The principle characteristic, $q_n(s)$, when expressed in terms of the z's, is the complete homogeneous symmetric function $p_n(z)$. Thus $q_n(s)=p_n(z)$, and in particular,

$$p_0(z)=1; \quad p_1(z)=\sum z_1; \quad p_2(z)=\sum z_1{}^2+\sum z_1z_2;$$
$$p_3(z)=\sum z_1{}^3+\sum z_1{}^2z_2+\sum z_1z_2z_3; \quad \text{etc.}$$

The $p_n(z)$ may be obtained from the generating function

$$f(z,t)=\{(1-z_1t)(1-z_2t)\cdots(1-z_{m'}t)\}^{-1} =p_0(z)+p_1(z)t+p_2(z)t^2+\cdots. \tag{24}$$

Generating function for $P'(\lambda; K)$

The expression (20) for the indeterminants $s^{(\alpha)}$ going with the different classes of the symmetric group is very suitable for our purpose because of the following theorem[11] which is based on this expression: the characteristic $\phi_{(\mu)}(s)$ of the reducible representation given by the manifold $\Delta(\mu)$ composed of γ_1 singly occupied orbits, γ_2 doubly occupied orbits, etc. is given by the product:

$$\phi_{(\mu)}(s)=(q_1(s))^{\gamma_1}(q_2(s))^{\gamma_2}\cdots(q_h(s))^{\gamma_h}. \tag{25}$$

The characteristic of the (reducible) representation given by the manifold of all functions with a given kinetic energy K is therefore:

$$\phi(s; K)=\sum_\mu P(\mu; K)(q_1(s))^{\gamma_1} \times(q_2(s))^{\gamma_2}\cdots(q_h(s))^{\gamma_h}. \tag{26}$$

[11] For a proof of this theorem, see Murnaghan, reference 7.

Let us now substitute the generating function (15) for $P(\mu; K)$ and replace $q_i(s)$ by $p_i(z)$. Then $\phi(s; K)$ is the coefficient of $y^K t^n$ in the following function:

$$g(U, z, t)=\sum_{\gamma}\{(U_1\cdots U_{\gamma_1}) \times(U_{\gamma_1+1}\cdots U_{\gamma_1+\gamma_2})^2\cdots\} \times p_1^{\gamma_1}p_2^{\gamma_2}\cdots p_h^{\gamma_h}t^{\gamma_1+2\gamma_2+\cdots+h\gamma_h}. \quad (27)$$

The sum is over all values of the γ's. This function may be written in the form:

$$g(U, z, t)=\prod_{i=1}^{m}\prod_{j=1}^{m'}(1-U_iz_jt)^{-1}, \quad (28)$$

as is shown most easily by expanding the product for each value of i into a power series in t and then multiplying the resulting series together. From (28) we have:

$$\log g(U, z, t)=-\sum_i\sum_j \log (1-U_iz_jt) =\sum_i\sum_j(U_iz_jt+U_i^2z_j^2t^2/2 +U_i^3z_j^3t^3/3\cdots). \quad (29)$$

Let us now express the U's in terms of new variables x_β defined by:

$$x_\beta=\sum_i U_i^\beta=\sum_i y^{\beta\epsilon_i}, \quad (\beta=1, 2, 3, \cdots). \quad (30)$$

From (23) and (30) we obtain

$$g(U, z, t)=\exp\{x_1s_1t+x_2s_2t^2/2+x_3s_3t^3/3\cdots\} =\sum_\alpha 1/\alpha_1!\alpha_2!\cdots\alpha_r!\cdot(x_1s_1/1)^{\alpha_1}(x_2s_2/2)^{\alpha_2}\cdots \times(x_rs_r/r)^{\alpha_r}t^{\alpha_1+2\alpha_2+\cdots+r\alpha_r}. \quad (31)$$

The generating function for $P'(\lambda; K)$ is obtained by finding how often the irreducible representation $[\lambda]$ is contained in the representation defined by the above characteristic. From the general equation (18), we find that $P'(\lambda; K)$ is the coefficient of y^K in

$$F(\lambda; y)=\sum_\alpha\chi_{(\lambda)}{}^{(\alpha)}/\alpha_1!\alpha_2!\cdots\alpha_r! \cdot(x_1/1)^{\alpha_1}(x_2/2)^{\alpha_2}\cdots(x_r/r)^{\alpha_r}, \quad (32)$$

where the summation is over all values of $\alpha_1, \alpha_2, \cdots, \alpha_r$ subject to the restriction $\alpha_1+2\alpha_2+\cdots+r\alpha_r=n$. In (32), $\chi_{(\lambda)}{}^{(\alpha)}$ is the character of the irreducible representation $[\lambda]$ going with the class $(\alpha)=(\alpha_1, \alpha_2, \cdots, \alpha_r)$.

Now the simple characteristic (22) of the irreducible representation $[\lambda]$ of the symmetric group on n particles has exactly the same form as (32). We need merely identify x_i with s_i. Since the equivalence of (22) and (22′) is an algebraic identity, the generating function (32) can also be expressed in the determinantal form (22′). This form is to be preferred because it does not involve the characters.

Generating function for evenly spaced levels

In the special case that the individual particle levels are evenly spaced, the generating function for $P'(\lambda; K)$ is particularly simple. We first obtain an expression for the function $q_n(y)$ defined by substituting x_i for s_i in $q_n(s)$ and then replacing x_i by its value in terms of y. For simplicity, as already remarked, we take the spacing between the individual particle levels as the unit of energy, so that

$$x_i=\sum_{k=0}^{\infty}y^{ki}=1/(1-y^i). \quad (33)$$

From Eqs. (23) and (24)

$$f(s, t)=q_0(s)+q_1(s)t+q_2(s)t^2+\cdots$$

and

$$\log f(s, t)=s_1t+s_2t^2/2+s_3t^3/3+\cdots.$$

The logarithm of the generating function for $q_n(y)$ is therefore

$$\begin{aligned}\log f(y, t)&=t/(1-y)+t^2/2(1-y^2) +t^3/3(1-y^3)+\cdots\\ &=t(1-y)/(1-y)+yt/(1-y) +t^2(1-y^2)/2(1-y^2)\\ &\qquad +y^2t^2/2(1-y^2)+\cdots\\ &=-\log(1-t)+\log f(y, yt),\end{aligned} \quad (34)$$

so that

$$(1-t)f(y, t)=f(y, yt). \quad (35)$$

Written out in full, Eq. (35) becomes

$$q_0+(q_1-q_0)t+(q_2-q_1)t^2+\cdots =q_0+ytq_1+y^2t^2q_2+\cdots.$$

Equating coefficients of t^n, we obtain the recurrence relation

$$q_n(y)=q_{n-1}(y)/(1-y^n). \tag{36}$$

Since $q_0=1$, repeated applications of (36) yield

$$q_n(y)=1/(1-y)(1-y^2)\cdots(1-y^n). \tag{37}$$

There remains to substitute expressions of this form into the determinant corresponding to (22′) and to evaluate this determinant, which can be reduced to the Vandermonde form. We give only the final result. $P'(\lambda;K)$ is the coefficient of y^K in

$$\begin{aligned}F(\lambda;y)&=y^{-k(k-1)(k-2)/6}\cdot q_{\lambda_k+k-1}(y)q_{\lambda_{k-1}+k-2}(y)\\&\quad\cdots q_{\lambda_1}(y)\prod_{i<j}(y^{\lambda_i+i-1}-y^{\lambda_j+j-1})\\&=y^{\lambda_{k-1}+2\lambda_{k-2}+\cdots+(k-1)\lambda_1}q_{\lambda_k+k-1}(y)\\&\quad\times q_{\lambda_{k-1}+k-2}(y)\cdots q_{\lambda_1}(y)\\&\qquad\times\prod_{i<j}(1-y^{\lambda_j+j-\lambda_i-i}).\end{aligned} \tag{38}$$

It may be noted here that, except for a possible factor of a power of y, the generating functions of a given representation and its associate representation are the same.

Application to nuclear levels

In the application to nuclei of heavy or intermediate mass, we are interested in representations for which most of the λ_k's are equal to four; only a few have values less than four. In this case it is simpler to use the notation for the associate representation, which is defined by (14). The generating function for $P'(\lambda;K)$ is then

$$\begin{aligned}F(\lambda,y)&=y^{\frac{1}{2}[\Lambda_1^2+\Lambda_2^2+\Lambda_3^2+\Lambda_4^2]-\frac{1}{2}n}\\&\quad\times G(y)\prod_{i<j}(1-y^{\Lambda_j+j-\Lambda_i-i}),\\&\qquad(i,j=1,2,3,4),\end{aligned} \tag{39}$$

where

$$G(y)=q_{\Lambda_4+3}(y)q_{\Lambda_3+2}(y)q_{\Lambda_2+1}(y)q_{\Lambda_1}(y).$$

In case the number of four-groups is large (Λ_1 large), we have, approximately,

$$G(y)\sim\prod_{i=1}^{\infty}(1-y^i)^{-4}. \tag{40}$$

This form has been used in the calculation of the tables given in the text. Eq. (39) may be used to calculate asymptotic formulae for the $P'(\lambda;K)$.

Appendix. Part B.

Multiplicities

The single-particle space orbits u_{j1}, u_{j2}, $\cdots$, with the j's equal in groups containing μ_l, μ_{l-1}, $\cdots$, μ_1 orbits, can be combined with single-particle spin functions g_1, g_2, g_3, g_4 to form a set of determinantal wave functions

$$|u_{j_1}g_{k_1}\,u_{j_2}g_{k_2}\cdots|, \tag{41}$$

which subtends an invariant subspace in the linear manifold $\Delta(\mu)$. Each determinant has definite S_z and T_z quantum numbers. Suitable linear combinations of the determinants with definite S_z, T_z describe states with definite values of S, the total ordinary spin, and T, the total charge spin. We wish first to compute $M(S,T;\mu)$, the number of times the state with given S, T values and $S_z=S$, $T_z=T$ occurs in the manifold $\Delta(\mu)$. One sees immediately that this quantity depends only on the number of two-groups and the total number of one and three-groups. Thus

$$M(S,T;\mu)\equiv M(S,T;\gamma_1+\gamma_3,\gamma_2). \tag{42}$$

Obviously the multiplicity is symmetrical in S and T. If $M(S,T;\gamma_1,\gamma_2)$ is known we compute $M(S,T;\gamma_1+1,\gamma_2)$ and $M(S,T;\gamma_1,\gamma_2+1)$ by means of the recurrence formulae

$$\begin{aligned}M(S,T;\gamma_1+1,\gamma_2)&=M(S-\tfrac{1}{2},T-\tfrac{1}{2};\gamma_1,\gamma_2)\\&\quad+M(S-\tfrac{1}{2},T+\tfrac{1}{2};\gamma_1,\gamma_2)\\&\quad+M(S+\tfrac{1}{2},T-\tfrac{1}{2};\gamma_1,\gamma_2)\\&\qquad+M(S+\tfrac{1}{2},T+\tfrac{1}{2};\gamma_1,\gamma_2),\end{aligned} \tag{43}$$

$$\begin{aligned}M(S,T;\gamma_1,\gamma_2+1)&=M(S-1,T;\gamma_1,\gamma_2)\\&\quad+M(S,T-1;\gamma_1,\gamma_2)+M(S+1,T;\gamma_1,\gamma_2)\\&\quad+M(S,T+1;\gamma_1,\gamma_2)+2M(S,T;\gamma_1,\gamma_2).\end{aligned} \tag{44}$$

Terms with negative S or T must be omitted. These relations follow from the usual rules for combining angular momenta together with the starting values

$$\begin{aligned}M(S,T;1,0)&=1,\quad S=T=\tfrac{1}{2},\\&=0,\quad\text{for all other } S,T.\end{aligned}$$

$$\begin{aligned}M(S,T;0,1)&=1,\quad (S=1,T=0),\\&\qquad\qquad (S=0,T=1),\\&=0,\quad\text{for all other } S,T.\end{aligned} \tag{45}$$

The multiplicity $M'(S, T; \lambda)$ associated with the irreducible manifold $[\lambda]$ is related to $M(S, T; \mu)$ by the equation

$$M(S, T; \mu) = \sum_{\lambda} (\lambda/\mu) M'(S, T; \lambda). \quad (46)$$

Eq. (46) can be used as a recurrence formula for the computation of $M'(S, T; \lambda)$; since $(\lambda/\lambda) = 1$,

$$M'(S, T; \mu) = M(S, T; \mu) - \sum_{\lambda \neq \mu} (\lambda/\mu) M'(S, T; \lambda). \quad (47)$$

One needs also the starting values

$$M'(S, T; 4\cdots4) = 1, \quad S = T = 0,$$
$$= 0, \quad \text{for all other } S, T.$$

$$M'(S, T; 4\cdots41) = M'(S, T; 4\cdots43) = M(S, T; 1, 0). \quad (48)$$

$$M'(S, T; 4\cdots42) = M(S, T; 0, 1).$$

Finally

$$M'(S, T_z; \lambda) = \sum_{T \geqq |T_z|} M'(S, T; \lambda) \quad (49)$$

and

$$M(S, T_z; \mu) = \sum_{T \geqq |T_z|} M(S, T; \mu). \quad (50)$$

NOVEMBER 15, 1938 PHYSICAL REVIEW VOLUME 54

The Focusing of Charged Particles by a Spherical Condenser

EDWARD M. PURCELL
Harvard University, Cambridge, Massachusetts

(Received August 12, 1938)

The paths of charged particles traversing a portion of an ideal spherical condenser are worked out. The section of the condenser considered is bounded by two rays, enclosing an angle Φ, from the common center of curvature, O, of the equipotential surfaces. It is shown that a group of particles, homogeneous in energy, leaving a point P on a normal to one of these boundaries and entering the condenser along this normal as a diverging bundle, will be brought to a focus at a point Q lying on the line PO extended, if the proper potential is applied to the condenser. This permits the whole condenser gap to be used as a focusing energy analyzer, or monochromator, of very large useful aperture. The velocity dispersion and reduced velocity dispersion are calculated for the most general case, and are found to take the same simple form as do the corresponding expressions for the limited homogeneous magnetic field spectrograph. The expressions for the reduced velocity dispersion are identical in the two cases. Compensation for edge effect is discussed. The relativistic modification of the theory required for high speed particles is discussed and results are presented which indicate that the simple theory of the electrostatic spectrograph may be inadequate even for fairly low values of v/c. It is suggested that this difficulty may be avoided by the choice of suitable instrument parameters.

An analyzer is described which has a useful aperture of 0.210 steradians, a theoretical reduced dispersion of 1010, and which requires a total focusing potential of 0.315 E, where E is the particle energy in equivalent volts. The operation of the analyzer in focusing electrons accelerated by a field designed to furnish an equivalent point source is described.

INTRODUCTION

THE possibility of deflecting and focusing a slightly diverging beam of charged particles by means of a cylindrical condenser was first demonstrated by Hughes and Rojansky.[1] In Fig. 1(a), a beam of particles of the same charge and initial energy, diverging from P and traveling between the plates C and D of a cylindrical condenser, will be approximately focused at Q, if the circular arc PBQ, subtending an angle of $\pi/\sqrt{2}$ or 127° 17′, is the trajectory of those particles which leave P in a direction perpendicular to OP. This device is essentially an *energy*-analyzer, for the trajectory of a (nonrelativistic) particle in any given electrostatic field depends only on its initial position and direction and the ratio of its charge to its initial kinetic energy. Such analyzers have been incorporated in successful mass spectrographs.

[1] A. L. Hughes and V. Rojansky, Phys. Rev. **34**, 284 (1929).