The Alpha-Particle Model of the Nucleus

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This paper reviews the theory of nuclei composed of alpha-particles alone, and extends it to cases where in addition to the alpha-particles, a single neutron or proton is present, and also to those light nuclei in which a single neutron or proton is missing from the complete alpha-particle structure. In the latter cases the missing particle is considered as a neutron or proton "hole." In the first section of the paper, the configuration of the alpha-particles is considered together with the motion of the neutron, proton, neutron hole, or proton hole, in the field of the fixed alpha-particles for nuclei lighter than fluorine. Rough values for the binding energies are obtained and compared with the results of the Hartree model. In the second section of the paper, rotations of the same nuclei are discussed, and the symmetry properties of the fundamental states, as well as the symmetry properties and the approximate spacing of some of the excited states are obtained. The symmetry properties of the fundamental states are in agreement with those obtained from the independent-particle model. Furthermore the excited states as obtained from the two models show a marked parallelism, although a greater number of levels is predicted by the alpha-particle model than by the independent-particle model.

INTRODUCTION

THE liquid-droplet model of the nucleus which has been recently discussed by Bohr gives an adequate description of many nuclear processes, and also provides us with some very general statements about nuclear binding energies and the distributions of excited states. More detailed results, however, may be obtained only with the help of models based on more specific assumptions. Two models of this kind have been proposed, namely, the independent-particle model and the alpha-particle model. The main difference between these two models is that, while in the independent-particle model each neutron or proton is assumed to be subject, in the first approximation, to the average force obtained from all other neutrons and protons, in the alpha-particle model the interaction of two neutrons and two protons (constituting an alpha-particle) are first considered, and then the interaction of these groups with each other and with particles outside of the groups are considered later.

It is undoubtedly true that no one is more aware of the weaknesses inherent in the use of the independent-particle model of the nucleus than those theoretical workers who have been making calculations on this basis. Such calculations are based on the model in which each of the nuclear constituents moves in a field obtained by averaging over the positions of the other particles and it neglects, therefore, in the first approximation the phase relation of the nuclear constituents. The success of a similar model in explaining atomic structure is due to the fact that all electrons are subject to the nuclear attraction, which is greater than the interaction between the electrons, and that the latter interaction, too, is a Coulomb force, that is, a long range force. It has been pointed out frequently that the absence of a central body in nuclear structure and the short range interaction between the constituents tend to make the Hartree or independent-particle approximation a very poor one. If the interaction is assumed to be of an exchange type, the independent-particle picture becomes even less justified. The impression that a considerable degree of success has been attained by this model, created perhaps especially among experimentalists, may be due to the fact that it is mainly the successes of the theory, and not its shortcomings, which are published, or at least which are emphasized in publication. It is well to remember in this connection that Breit and Rabi[1] stated at a

[1] G. Breit and I. Rabi, Phys. Rev. **46**, 230–231 (1934).

very early date in the development of nuclear theory that: "According to the usual ideas of nuclear structure the constitution of a nucleus resembles that of a polyatomic molecule or else a liquid. It is questionable whether there is much meaning to a central field which one must necessarily assume in order to assign an orbital quantum number to a particle under these conditions."

An older picture of the nucleus,[2] to which the above quotation implicitly refers, assumes that alpha-particles are present in the nuclei as comparatively stable sub-units. It has been emphasized that the alpha-activity and, to some extent, also the greater stability of nuclei of the $4n$-type may be explained by the independent-particle picture as well,[3] but still it seems impossible to obtain as good absolute values for the binding energy with the help of the Hartree model as is obtained in the zero*th* approximation from the alpha-particle picture. Again, it has been shown in some cases, and will be shown for others in this paper, that the behavior of the nuclei in the vicinity of O^{16}, which seemed to be in particularly good agreement with the closing of the "P-shell" in the Hartree model, can be also accounted for by the alpha-particle model.

Our purpose in this paper is to discuss the alpha-particle model for light nuclei. Such discussions have been given previously[3, 4] for nuclei composed of alpha-particles alone. We shall review the theory of these nuclei and extend it to cases where, in addition to the alpha-particles, a neutron or proton is present, and also to those light nuclei where a single neutron or proton is missing from a complete alpha-particle structure. In the latter cases we may speak of a neutron or proton "hole." Our treatment will be based on the analogy with molecular structure, just as the conventional application of the independent-particle picture is based on an analogy with atomic structure. The alpha-particles will correspond to the nuclei of the molecule, the neutron hole or proton hole to the electron moving in the field of the heavier constituents.

Obviously this picture too is inherently weak. Thus it has been pointed out that, particularly in heavy nuclei, considerations of energy and space-occupation[5] make the stability of alpha-particles as sub-units somewhat doubtful. Furthermore, our calculations, for those nuclei which are not composed of alpha-particles alone, contain the assumption that the mass of the neutron or proton is negligible as compared to that of the alpha-particle.

However, the essential reason that the independent-particle picture has been used so largely in recent theoretical investigations has been the belief that the alpha-particle picture would be more complicated for actual numerical calculations. As will be shown below, many important results can be obtained from the alpha-particle picture in a simple way.

It is a significant fact that, while the Hartree and alpha-particle approximations start from widely different assumptions, the results obtained agree in many of the main features as well as in some of the details. For example, the second approximation of the independent-particle picture, as applied to heavy nuclei,[6] brings about a distribution of neutrons and protons resembling more closely that obtained from the alpha-particle picture. A parallelism has also been found[7] with regard to the properties of the first excited states of heavy nuclei as obtained from the two points of view.

Our treatment consists of two main sections. In the first, the configuration of the alpha-particles is considered together with the motion of the neutron, proton, neutron hole, or proton hole, in the field of the fixed alpha-particles for nuclei lighter than fluorine. Rough values for binding energies are obtained and compared with results of the Hartree model.

In the second section, rotation of the same nuclei is discussed. In this section we obtain the symmetry properties of the fundamental states as well as the symmetry and approximate spacing of some of the excited states. The symmetry properties of the fundamental states are in agreement with those obtained from the

[2] G. Gamow, Proc. Roy. Soc. **A126**, 632–644 (1930).
[3] H. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 168–171 (1936).
[4] W. Wefelmeier, Zeits. f. Physik **107**, 332–346 (1937); J. A. Wheeler, Phys. Rev. **52**, 1083–1106 (1937); C. F. von Weizsäcker, Naturwiss. **26**, 209–217, 225–230 (1938).
[5] W. Elsasser, J. de phys. et le rad. **5**, 389–397, 635–639 (1934).
[6] H. Euler, Zeits. f. Physik **105**, 553–575 (1937).
[7] E. Teller and J. A. Wheeler, Phys. Rev. **53**, 778–789 (1938).

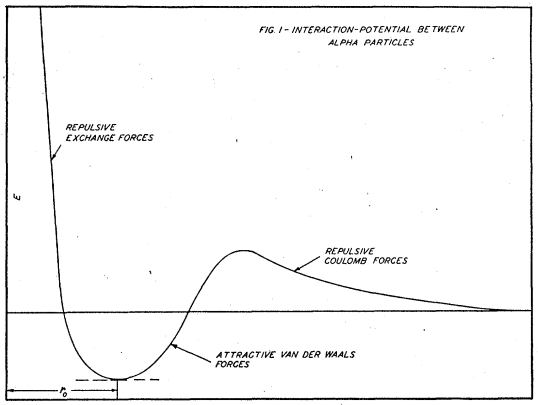

FIG. 1. Interaction potential between alpha-particles.

independent-particle model. Furthermore, the excited states as obtained from the two models show a marked parallelism although a greater number of levels is predicted in the alpha-particle picture than in the independent-particle picture

The agreement of the symmetry properties in the two models seems to us to be the main result of the present paper.

Though we do not think that the model here presented is good enough to give a detailed account of experimental facts, we believe that it is not worse than the Hartree model. Explicitly, we believe that neither of these pictures can give an adequate explanation of the details of nuclear structure. However, it is of some interest to present calculations based on the alpha-particle model which give the same results as the Hartree model. Such results seem to be relatively less sensitive to the approximations which have been made and therefore might be expected with somewhat greater confidence to agree with experimental observations. Such agreement, however, seems to imply only the validity of those considerations which are fundamental to both points of view, and must not be interpreted as supporting the detailed assumptions of either model.

SECTION I

Fundamental states, proper functions, binding energies

Saturated nuclei of 4n-type.—We shall call those nuclei which contain $2n$ neutrons and $2n$ protons, and can therefore be considered as consisting of n alpha-particles, saturated nuclei. The name saturated nucleus refers to the fact that the alpha-particles are the saturated sub-units in nuclear structure. In fact, these nuclei are analogous to the rare gases and other saturated shells in molecular physics. They have, at least for light nuclei, a comparatively high binding energy, a high frequency of occurrence in nature, and no angular or magnetic momentum. These nuclei have been described in accordance with the "saturation character" of nuclear forces.[8] By "saturation character" is meant that the binding energy (and also the volume) of a nucleus is roughly proportional to the number of particles contained in it, and not to the square of the number of particles as might be expected if each particle interacted with all other particles. To obtain this condition, it can be assumed that the alpha-particles repel each other at small distances. In order to build up saturated nuclei containing more than one alpha-particle, an attraction must be introduced at larger distances. Thus an interaction potential between two alpha-particles is obtained of the form shown in Fig. 1.[9] This interaction may be compared to that between two saturated shells in atomic physics. The repulsion at small distances corresponds to the exchange interaction, the attraction at greater distances to the van der Waals, or polarization, forces.[8] The analogy with the closed atomic shells makes it plausible that the interaction forces are of an additive nature, that is, the interaction of two particles is not changed by the presence of a third alpha-particle.

According to these assumptions, alpha-particles will be held in equilibrium-positions within the nucleus, Be^8 corresponding to a dumb-bell model, C^{12} to an equilateral triangle, O^{16} to a tetrahedron, and further saturated nuclei to more complicated structures. If, in analogy to atomic physics, a short range interaction is assumed between the closed shells, the binding energy will be proportional to the number of "bonds" between adjacent particles, that is, to the number of pairs of alpha-particles which for

[8] W. Heisenberg, Zeits. f. Physik **96**, 473–484 (1935).

[9] The effect of the Coulomb interaction at great distances will not be considered in what follows, since it is of relatively small importance for light nuclei.

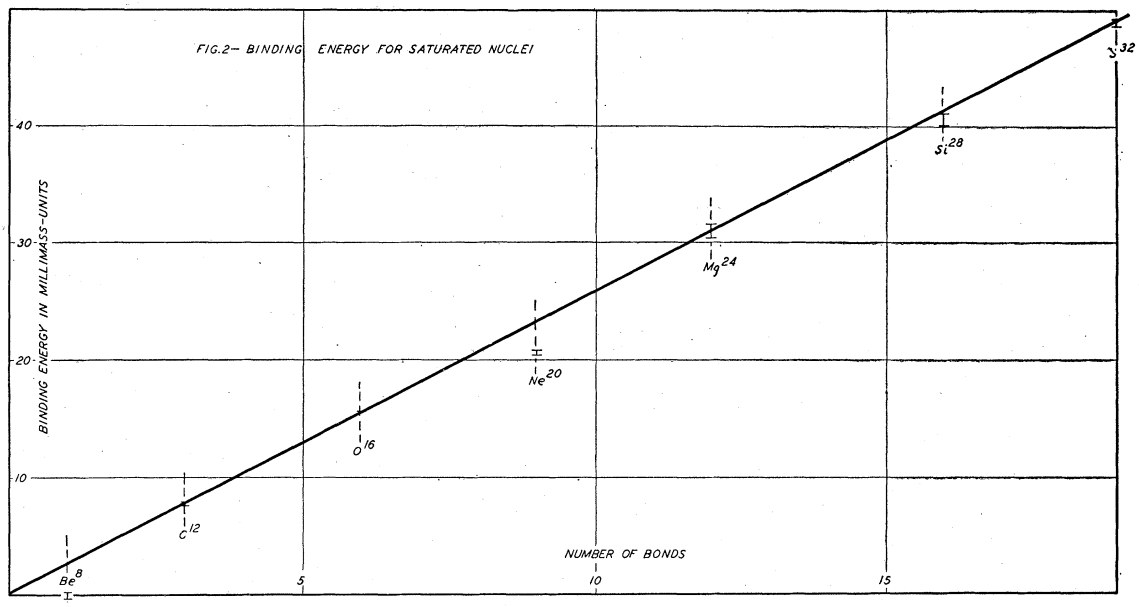

FIG. 2. Binding energy for saturated nuclei.

geometrical reasons can approach to the distance of maximum interaction r_0.

In Fig. 2 the binding energies of saturated nuclei, that is, the masses of the constituent alpha-particles minus the nuclear mass, are plotted against the number of "bonds." The experimental values are plotted with vertical lines to show their limits of error. The same procedure will be followed in Figs. 5 and 6. In agreement with the model, a very good linear relation is obtained with the exception of Be^8 and Ne^{20}, which both fall below the curve.

According to the model, Be^8 should have a binding energy corresponding to one bond, that is, 2.58 in Mev. Its actual binding energy, however, is close to zero. It has been suggested by Wefelmeier[4] that this discrepancy may be due to the effect of the zero-point energy in analogy to Wigner's explanation[10] of the surprisingly great ratio of the binding energies of the alpha-particle and the deuteron.

In order to check this explanation, we have calculated the zero-point energies for Be^8, C^{12}, and O^{16} for a simplified model, assuming that the zero-point amplitudes are small as compared to the distances between the alpha-particles and that the effects of the anharmonicity of the forces, and the interaction with rotation, may be neglected. If, furthermore, the additivity of the forces is remembered, the ratio of the zero-point energies[11] for the sequence $Be^8 : C^{12} : O^{16}$ is as $1 : [(3/2)^{\frac{1}{2}}+2(3/4)^{\frac{1}{2}}] : (\sqrt{2}+2/\sqrt{2}+3.1)$, or $1 : 2.9 : 5.8$. This ratio is very nearly equal to the ratio $1 : 3 : 6$ of the number of bonds in these nuclei so that the ratios of the total binding energy should remain $1 : 3 : 6$. Thus for the model proposed the zero-point energy does not explain the anomaly for the Be^8-nucleus.

The reason that, in comparing H^2 and H^3, the zero-point energy is of great importance while for Be^8 and C^{12} it is not, seems to us to be as follows. In H^2 and H^3 the equilibrium positions of the neutrons and protons coincide. Therefore, H^2 possesses three "vibrational" degrees of freedom and H^3 six "vibrational" degrees of freedom. Very roughly one may therefore expect the zero-point energy to increase by a factor of two. At the same time the pairs of interacting particles and therefore the number of terms in the potential energy change from one to three, that is, the potential energy increases more rapidly than the zero-point energy. On the other hand, in the models for Be^8 and C^{12} the

[10] E. Wigner, Phys. Rev. **43**, 252–257 (1933).

[11] The normal vibrations may be obtained from Table IV in the paper by Wheeler, reference 4.

number of vibrational degrees of freedom is equal to the number of bonds and it may therefore be expected that the zero-point energy and potential energy will be proportional. This qualitative consideration suggests that a rather great change of the model of saturated nuclei may be necessary to explain the low binding energy of Be^8.

In Fig. 2 it will be noticed that the number of additional bonds per particle increases markedly up to O^{16} but remains constant for the following two-saturated nuclei. This corresponds to the relatively smaller binding energy for the nuclei heavier than O^{16} predicted by the "shell structure" in the independent-particle model. The fact that Ne^{20} lies below the curve shows that the break in binding energy after O^{16} is even more marked than would be expected from the alpha-particle model. It should be remembered, however, that the disagreement for Be^8 and Ne^{20} is small as compared to the disagreement as obtained in any of the direct calculations based on the independent-particle model.

The degree to which alpha-particles may retain their individuality in nuclei will decrease as the total interaction between an alpha-particle and the other constituents of the nucleus increases. It is therefore not improbable that in heavy nuclei where an alpha-particle is completely surrounded by other alpha-particles (and possibly neutrons) the alpha-particle picture is less valid than for light nuclei. This may help to explain the fact that for all heavy nuclei the binding energy of an added neutron or proton is approximately constant, whereas for light nuclei a particularly great binding energy is obtained whenever a new alpha-particle is completed.[12]

One of our reasons for restricting this discussion to light nuclei is that for these the alpha-particle model may be a reasonable approximation even though it may break down for heavy nuclei.

Nuclei of $(4n+1)$ *type.*—For nuclei containing a single neutron or proton in addition to the alpha-particles, the total binding energy will be given by the binding energy of the group of alpha-particles plus an additional energy due to the binding of the single particle to the alpha-particle group.

In the simple case of He^5 the additional energy, which we will call B, will be given directly by the difference between the masses of He^5 and (He^4+n). Thus $He^5-(He^4+n)=B$.

The independent-particle picture predicts that the fundamental state of He^5 should be a P state, consisting of four s particles and one p particle. We take this result over into the alpha-particle picture by assuming that the four s particles correspond to the alpha-particle and that the added neutron in He^5 is in a p state with a node through the center of the alpha-particle. Such a model will satisfy the exclusion principle, and make it possible in the simplest way that the ψ-function of the added neutron should be orthogonal to the ψ-function of the neutrons within the alpha-particle. One consequence of the presence of this node through the alpha-particle is that the added particle will be found with a small probability within the alpha-particle, corresponding to the notion that the density of matter within the alpha-particle closely approaches the maximum density which can be reached within a nucleus.

The case of Li^5 in which a proton is added to the alpha-particle will differ only slightly from that of He^5 due to the effect of the Coulomb forces. This difference, and those for other nuclei in which a proton is added instead of a neutron, can be calculated quite as readily for our present model as for the independent-particle picture. However, since the results are practically identical for the two models (as might be expected, since the effect of the long range Coulomb forces will not depend very greatly on the details of the configuration of nuclear constituents) and, further, since the experimental uncertainties in the available data are too great for definite conclusions to be drawn, the effects of Coulomb interactions will not be considered in this paper.

In cases of more than one alpha-particle, we may reason as follows. The force between the neutron and any one alpha-particle has a short range. This force will therefore contribute to the binding energy only when the neutron is close to some one of the alpha-particles. This fact makes it plausible that the neutron does not

[12] It should be remembered, however, that mass defects for heavy nuclei are relatively less accurately known.

simultaneously interact to a great extent with other alpha-particles which are at the average distance r_0. Thus the neutron will interact with each alpha-particle only part of the time.

For Be^9 we have two alpha-particles each with an associated ψ-function for the neutron which we will designate as ψ_1 and ψ_2, respectively. As in the case of He^5, each neutron ψ-function must have a node through the respective alpha-particle. The total wave function may be any linear combination of ψ_1 and ψ_2, but, since we are seeking the lowest energy state and since long wave-lengths correspond to smallest energies of the particle, we choose a combination $(\psi_1+\psi_2)$ with a single plane node extending through both alpha-particles. The average energy of the neutron will be given by

$$\bar{E}(\psi_1+\psi_2)=\frac{\int(\psi_1+\psi_2)H(\psi_1+\psi_2)}{\int(\psi_1+\psi_2)^2},$$

where $H=(V_1+V_2+T)$ and V_1 refers to the potential energy of the neutron due to alpha-particle number one, V_2 refers to the potential energy of the neutron due to alpha-particle number two, and T is the kinetic energy of the neutron. On expansion we find that the terms can be grouped as binding, interaction, and exchange, terms which we will designate as B, R, and Q, respectively. The B terms will have the form $\int\psi_1(V_1+T)\psi_1$ and refer to the potential and kinetic energy of the neutron due to α_1 (that is, the first alpha-particle) when the neutron is near α_1, with corresponding terms for α_2. The R terms will have the form $\int\psi_1 V_2\psi_1$ and refer to the additional potential energy of the neutron when near α_1 due to the presence of α_2, and conversely. The exchange terms are those having form $\int\psi_1 H\psi_2$ and have the usual meaning. Letting $S=\int\psi_1\psi_2$ we find

$$\bar{E}(\psi_1+\psi_2)=\frac{2(B+R+Q)}{\int\psi_1^2+\int\psi_2^2+2\int\psi_1\psi_2}=\frac{B+R+Q}{1+S}.$$

In keeping with the idea that the added neutron interacts mainly with one alpha-particle at a time and that therefore ψ_1 and ψ_2 do not overlap strongly, we assume that the overlapping of the ψ-function is small and therefore $S\ll 1$,[13] so that

$$E(\psi_1+\psi_2)\cong[B+R+Q].$$

Applying the same type of arguments to C^{13} we will expect that the added neutron proper functions will have a plane node through all three alphas, satisfying the Pauli principle with the smallest amount of added kinetic energy. A similar argument as has been given for Be^8 leads to the energy of the neutron of $[B+2R+2Q]$. The interpretation of this formula is as follows: Since as far as the term B is concerned the neutron will interact with one alpha-particle at a time, this term will be the same as in He^5 and Be^9. On the other hand, in the interaction R, and for the proper functions which we are using also for the exchange term Q, the neutron at one alpha can interact simultaneously with the two neighboring alpha-particles. Therefore, these terms are multiplied by two for C^{13}.

We now wish to build O^{17} by adding a neutron to the tetrahedron constituting O^{16}. The wave function of the added neutron should have a node through each of the alpha-particles. This can no longer be done with one plane node. The added neutron will probably have least kinetic energy while satisfying the Pauli principle if a spherical node passes through the four alpha-particles.

It is not possible to calculate from our picture the energy of the neutron in O^{17} without introducing new constants. It follows, however, that the energy will be greater than $[B+3R+3Q]$, the value which would be obtained by extrapolating the energies of He^5, Be^9, and C^{13}. The different orientation of the nodes on different alpha-particles in O^{16} will make the wave-lengths of the neutron effectively shorter, and therefore the kinetic energy of the added neutron larger and its binding energy smaller.

Summarizing our results for nuclei of the $(4n+1)$ type, we have

$$\begin{aligned}
He^5-(He^4+n)&=B,\\
Be^9-(Be^8+n)&=B+(R+Q),\\
C^{13}-(C^{12}+n)&=B+2(R+Q),\\
O^{17}-(O^{16}+n)&>B+3(R+Q).
\end{aligned}$$

A comparison with experimental observations is given in Fig. 3; the mass-values used are those

[13] One might expect that for similar reasons $Q\ll B$ and also $R\ll B$. This however need not be true since the two terms $\int\psi_1 V_1\psi_1$ and $\int\psi_1 T\psi_1$ in B tend to cancel each other.

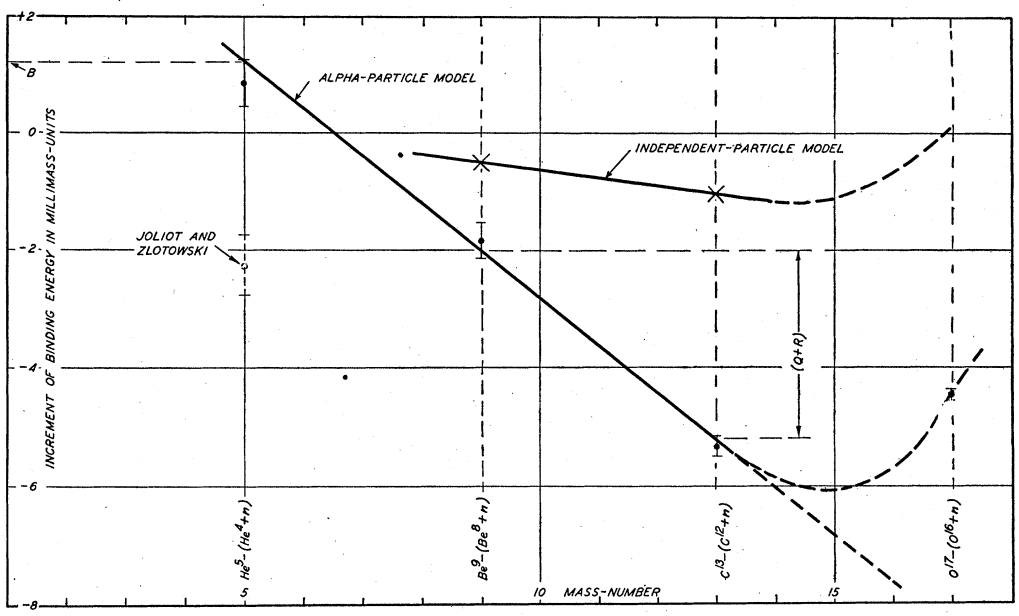

FIG. 3. Increments of binding energy for nuclei of $(4n+1)$ type.

of Livingston and Bethe.[14] This figure shows clearly the "break" in the curve at O^{17} which has been interpreted[4] as strong evidence for the independent-particle picture but which as we have seen is to be expected on the basis of the alpha-particle picture as well.

The experimental binding energies for He^5, Be^9, and C^{13} can be represented by the theory if we set $B=1.2$ Mev and $(R+Q)=-3.2$ Mev. The fact that $|R+Q|$ is actually greater than B makes our conclusions about the binding energies rather doubtful. If the presence of further alpha-particles modifies the binding of a neutron to one alpha-particle so strongly, it would seem necessary to include higher order approximations, that is, polarization terms in the calculation. However, even these extremely rough considerations account for the *trend* of the binding energies of the light nuclei as well as does the independent-particle picture. Similar statements hold for the nuclei of $(4n-1)$ type which we now proceed to discuss.

Nuclei of (4n−1) type.—With our model we can treat nuclei of the $(4n-1)$ type by considering the missing particle as a "hole" and then proceeding just as in the case of the added particle. However, whereas the added particle was in a p state, with a node through the alpha-particle corresponding to the small probability of the neutron's being within the alpha-particle, the "hole" will be in an s state since it may be expected to be within the alpha-particle. Furthermore, whereas in the previous case we obtained a minimum energy for our model by arranging the wave functions of the added particle so as to have as few nodes as possible, in the present case we will get the lowest energy state for the entire nucleus by giving the "missing particle" as large an energy as possible.

For He^3 we have simply the energy $B=He^3-(He^4-n)$.

For Be^7 we have a case analogous to that of ionized molecular helium; the alpha-particles correspond to the closed shells of the two He atoms and the missing neutron correspond to the missing electron.[15] Guided by the requirement suggested above that the "missing particle" should have high energy, we proceed as in the case of Be^9 but choose $(\psi_1-\psi_2)$ as our wave function, in order to obtain the maximum

[14] M. S. Livingston and H. A. Bethe, Rev. Mod. Phys. **9**, 373 (1937); see however F. Joliot and I. Zlotowski, C. R. Acad. sci. **206**, 1256–1259 (1938) in regard to the mass of He^5.

[15] W. Weitzel and E. Pestel, Zeits. f. Physik **56**, 197–214 (1929).

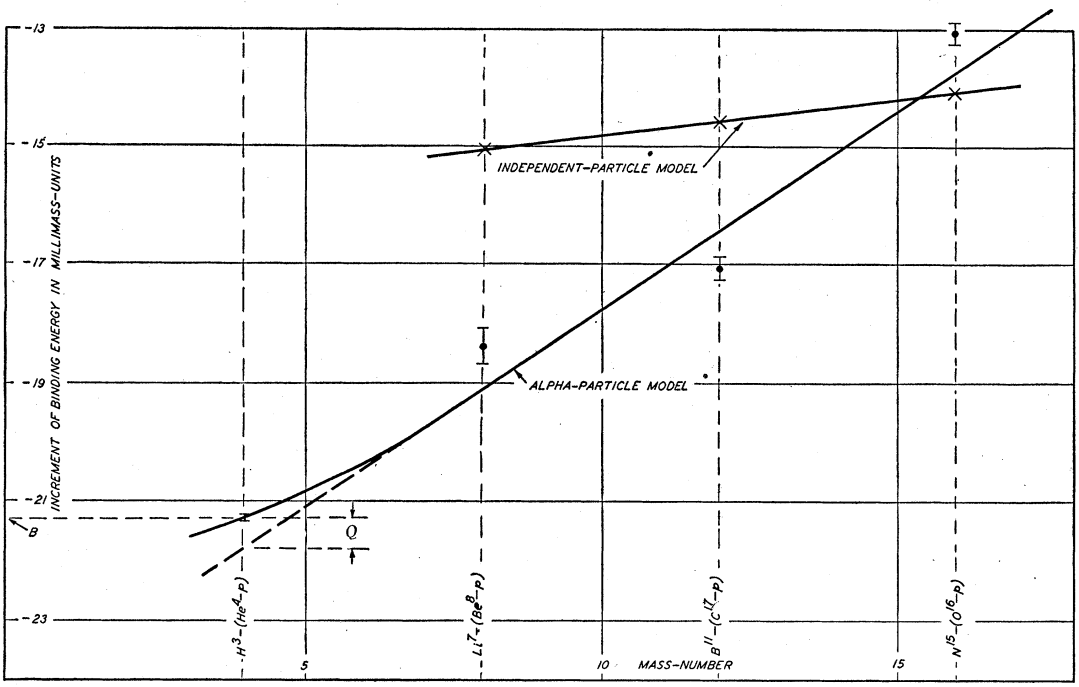

FIG. 4. Increments of binding energies for nuclei of $(4n-1)$ type. (The superscript to the carbon symbol should be 12.)

number of nodes. We evaluate

$$\bar{E}(\psi_1-\psi_2)=\frac{\int(\psi_1-\psi_2)H(\psi_1-\psi_2)}{\int(\psi_1-\psi_2)^2}$$

and find $(B-Q+R)$ as the binding energy for $Be^7-(Be^8-n)$; the negative sign in front of the exchange integral Q is due to the change in the sign of ψ_2. For the missing particle case, however, Q will probably be positive, so that $(B-Q+R)$ rather than $(B+Q+R)$—corresponding to $\psi_1+\psi_2$—will represent the lowest possible energy.

For the cases C^{11} and O^{15}, we assume again that Q is positive. We find then from group theory

$$\psi_1-(\psi_2+\psi_3)/2 \quad \text{and} \quad \psi_1+\psi_2-(\psi_3+\psi_4)$$

for proper functions corresponding to the lowest energies. The first of these functions, corresponding to C^{11}, is one of a twofold degenerate set and leads to the energy $(B-Q+2R)$. The second function corresponding to O^{15} is threefold degenerate and gives $(B-Q+3R)$. It will be seen that the exchange term remains constant regardless of the number of alpha-particles. This is due to the fact that by choosing, the wave function with a node between two alpha-particles, the third alpha-particle in the case of C^{11}, and the third and fourth alpha-particles in the case of O^{15}, lie on this node and therefore do not contribute to the exchange energy. It may be worth while to point out that, since for the case of O^{15} a spherical node such as was needed for O^{17} is not required, the break in the linear relation at O^{15} is not to be expected in the missing particle case.

A comparison with the experimentally observed values of the binding energies of nuclei of this type is given in Fig. 4. In view of the incompleteness of the data for "missing-neutron" elements, the equivalent "missing-proton" elements are used instead. According to theory the points for Li^7, B^{11} and N^{15} should lie on a straight line, whereas the point for H^3 not containing an exchange term should lie above the line by an amount Q. The agreement with experiment for the alpha-particle model obviously is not good but is certainly no worse than that given by the independent-particle model, for which the data were taken from Table V of Feenberg and Wigner.[16]

[16] E. Feenberg and E. Wigner, Phys. Rev. **51**, 95–106 (1937).

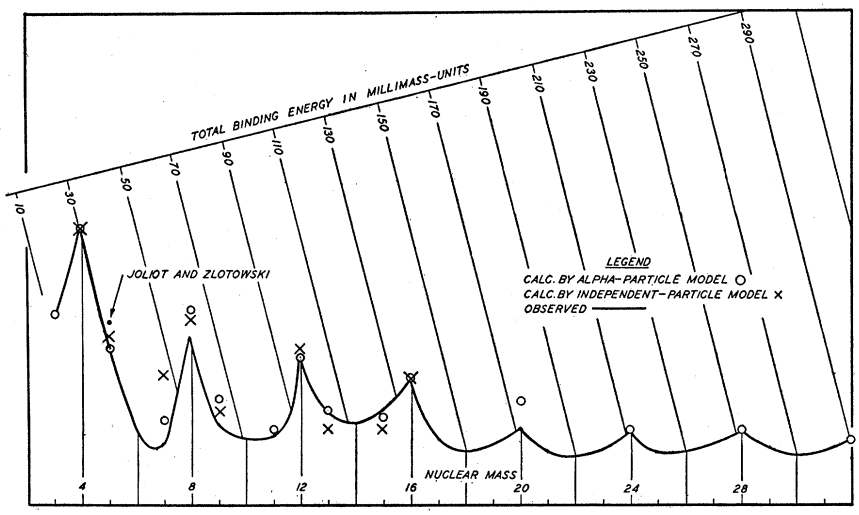

FIG. 5. Comparison of calculated and observed binding energies.

While it is not the purpose of this paper to emphasize detailed agreement with experiment, since we believe that calculations based on either model are too inaccurate to give quantitative results, we include Fig. 5 as a summary of the results so far obtained by each theory for the fundamental states. The data for the independent-particle model were obtained from Fig. 1 in the paper by Feenberg and Phillips.[17] For each model the total number of arbitrary constants used in fitting the experimental curve is large. For the alpha-particle model we determined from the experimental data constants B, R, and Q, both for the added particle and for the hole, and the mass defect of the alpha-particle, as well as the "binding energy per bond" for the saturated nuclei, making a total of eight parameters. For the independent-particle model[16] the parameters are as follows: A, α, G_1, G_3, and two from the linear correction function, making a total of six.

SECTION II

Nuclear rotation, symmetry, excited states

In the previous section nuclear proper functions have been described insofar as they bear upon the *relative* positions of alpha-particles, added neutron or proton, or missing neutron or proton. These proper functions have given us rough values of the binding energies. But if we inquire into the symmetry properties of the states in question, it will be necessary to add the description of the proper functions as depending on the orientation of the nuclear constituents in space, that is, it will be necessary to consider nuclear rotation.

This consideration will yield as a by-product some information on excited nuclear states. Further excited states would be obtained if we would include in our discussion the nuclear vibrations. However, the lowest excited levels will, as a rule, be rotational levels.

The considerable quantitative differences between theory and experiment as found in Section I indicate that no great emphasis can be laid upon the calculated spacing of excited states. Therefore the symmetry properties of both the fundamental and the excited states are of primary interest to us.

Saturated nuclei of 4n-type.—For this simple case, which we include for completeness, the results of Wheeler[4] are represented. As indicated by Wheeler, the rotational levels may be obtained very simply by comparison with the analogous molecules.

For Be^8 we have the case of a diatomic molecule, for which the energy is given by

$$E_J=(\hbar^2/2I_{\mathrm{Be}})J(J+1).$$

However, since the alpha-particles obey Bose

[17] E. Feenberg and M. Phillips, Phys. Rev. **51**, 597–608 (1937).

statistics, and the antisymmetrical ψ-functions for $J=1, 3, 5 \cdots$ are therefore excluded, only values of $J=0, 2, 4 \cdots$ will be allowed. From the fact that reflection at the center of symmetry exchanges the two alpha-particles, we conclude that all states are even.

For C^{12} we have the case of a symmetrical top for which

$$E_{JK}=\frac{\hbar^2 J(J+1)}{2I_{\mathrm{Be}}}-\frac{\hbar^2 K^2}{4I_{\mathrm{Be}}}.$$

J is the quantum number for the total angular momentum, while K is the component of J about the figure axis, that is, about the axis perpendicular to the plane of the three alpha-particles. I_{Be} is, as above, the moment of inertia of Be. In our formula it has been implicitly assumed that the separation of the alpha-particles in Be^8 and C^{12} is the same. Since, as above, only symmetrical ψ-functions are allowed by the requirement of Bose statistics, K must be a multiple of 3. J, being the absolute value of the length of a vector, must be zero or positive. K, however, being a component of J, can be either positive or negative, but its absolute value must be smaller than or equal to J. For the case of $K=0$, that is, no rotation about the figure axis, J can have only even values just as in the case of Be^8. When $K\neq 0$, however, J may have any positive value greater than or equal to $|K|$.

The states will be odd or even according to whether K is odd or even.[18]

In the case of O^{16} we have a spherical top for which all moments of inertia are equal. Also from the tetrahedral arrangement we may conclude that $I_{\mathrm{oxygen}}=2I_{\mathrm{Be}}$ so that

$$E_J=\hbar^2 J(J+1)/4I_{\mathrm{Be}}.$$

From group theory one may find that all values of J are allowed except $J=1$, 2, and 5.

The states will be even for $J=0$, 4, and 8, and odd for $J=3$, 7, and 11. For the remaining allowed J values there will be both even and odd states with equal rotational energies.

Nuclei of $(4n+1)$ type

He^5 *or* Li^5.—From our point of view we cannot consider excited states of He^5 or Li^5 for, as in the monatomic molecule, there will be no rotational states. The fundamental state will have $J=1$ and will be odd, that is, it will have the same symmetry properties as in the independent-particle picture.

Be^9 *or* B^9.—The node of the neutron wave function through the two alpha-particles means that there is an angular momentum $\hbar$ of the neutron around the axis connecting the two alpha-particles and therefore we have the lowest state $J=1$, that is, a P state.

The excited states may be obtained by analogy with a diatomic molecule with a π_u electron, for which the nuclei obey Bose statistics. The rotational energies will be roughly given by the formula which held for Be^8. The wave function will be symmetric in the alpha-particles. Two cases arise, because the plane of the neutron wave function may be either (*a*) parallel or (*b*) perpendicular to the axis of rotation. For case (*a*) odd values of J will be permitted; the neutron function changes sign on an interchange of alpha-particles by rotation. For case (*b*) even values of J will be permitted, since now the neutron function does not change sign. All values of $J\geqq 1$ will therefore be allowed. By reflection at the center of symmetry we find no change due to the alpha-particle wave function but a change of sign due to reflection of the neutron wave function which has a node through the center. Therefore all states are odd.

O^{13} *or* N^{13}.—The rotational energies are roughly given by the formula for C^{12}. The former rule that K must be a multiple of three remains valid, because rotation about the figure axis has no effect on the neutron wave function. As in the case of C^{12}, all values of J are allowed for $K\neq 0$.

For $K=0$, the axis of rotation is perpendicular to the figure axis. As in the pure alpha-particle case, an interchange of two alpha-particles must not cause a change in sign. Such interchange of two alpha-particles can be performed by rotating through 180° around an axis passing through one of the alpha-particles and the midpoint between the other two alphas. Such a rotation, however, will also act on the neutron proper function. Since the neutron has a node in the plane of the three alphas it will change its sign. Symmetry with regard to the interchange of two alpha-particles will be obtained, therefore, whenever the rotation through 180° produces a

[18] This may be seen by rotating through π around the figure axis and reflecting in the plane of the alpha-particle.

change of sign, that is, for $K=0$ only odd values of J are allowed.

Because of the influence of the added neutron with its node, there will be even states for odd K values and odd states for even K values.

O^{17} *or* F^{17}.—The added particle will have spherical symmetry and therefore there will be no change in the conditions governing the rotations.

Since an even (spherically symmetrical) proton or neutron is added, the parity of the states will be the same as for O^{16}.

Nuclei of $(4n-1)$ type

Li^7 *or* Be^7.—As in the case of binding energies, we again may choose as our proper function either $(\psi_1+\psi_2)$ or $(\psi_1-\psi_2)$. In our consideration of the fundamental states we were guided in our choice by the fact that we wished to find the lowest possible state. This enabled us to select the $(\psi_1-\psi_2)$ for the "missing particle" case. For excited states both possibilities should be considered. Now the difference between the two cases manifests itself by a change in sign of the exchange integral Q as we have seen above, so that the separation in energy levels calculated for the two cases will be

$$[B+Q+R]-[B-Q+R]=2Q,$$

to which a contribution arising from the rotation must be added. As may be seen from the sequence for the binding energies in the missing particle case, the magnitude of the exchange term Q may be determined, in principle, from the observed binding energies, as shown in Fig. 4. (Q for the added particle case cannot be determined separately, for it appears only in the combination with R.) Actually the errors in the data are too great to permit any conclusion except within wide limits. It is possible that the Li^7 doublet[19] is due to the two states $(\psi_1+\psi_2)$ and $(\psi_1-\psi_2)$. An explanation of this doublet has been proposed by Inglis[20] but this suggestion does not seem to account for a variation in the relative intensity of the two groups with voltage which has been observed.[21] Such a variation, however, is not surprising if the explanation suggested by the alpha-particle model is accepted.

Restricting the discussion of nuclear rotation to the $(\psi_1-\psi_2)$ case, we have a proper function with a node between the two alpha-particles. Except for different values for the moments of inertia this problem is therefore the same as that for C^{13} for $K=0$ which has already been considered. The allowed states will be $J=1, 3, 5, 7\cdots$, with the fundamental state a P state. The rotational energies are roughly given by the Be^8 formula. All states will be odd.

The analogous molecular case for He_2^+ has been investigated by Weizel and Pestel,[15] who found in this case that experimentally only states with odd values of J actually occur.

Considerations regarding the magnetic moment of the Li nucleus, as obtained with the aid of the alpha-particle model, have been given by Bethe.[22]

C^{11} *or* B^{11}.—For this case the proper function of the "hole" in the field of fixed alpha-particles must be considered more closely. The transformation properties under the effect of rotation through $2\pi/3$ and $4\pi/3$ around the figure axis are the same as those of a proper function for a particle with unit angular momentum. Thus unity must be added to, or subtracted from, the angular momentum of the alpha-particles around the figure axis. The latter will be, just as for C^{12}, a multiple of three, so that K for the missing particle case must differ from a multiple of three by ± 1. This means that K can be any integer not divisible by three.

We have seen that the proper function of the "hole" transforms as though it had unit angular momentum. Nevertheless, the actual mean value of the angular momentum of the "hole" will be smaller than $\hbar$. If we assume it to be small as compared to $\hbar$, the formula for rotational energies of C^{12} will hold roughly also for C^{11} and B^{11}. The actual values for the rotational energies, shown in Fig. 6, are obtained from that formula. If, however, the mean angular momentum of the "hole" is not small as compared to $\hbar$, interaction between the orbit of the "hole" and the rotation may influence the rotational energy considerably.

All values of $J \geqq |K|$ will be allowed with $J=1$, $K=1$ the fundamental state.

The states will be even if K is even and odd if K is odd.

[19] L. H. Rumbaugh and L. R. Hafstad, Phys. Rev. **50**, 681–689 (1936).

[20] D. R. Inglis, Phys. Rev. **50**, 783–784 (1936).

[21] L. H. Rumbaugh, R. B. Roberts, and L. R. Hafstad, Phys. Rev. **51**, 1013 (1937). J. H. Williams, W. G. Shepherd, and R. O. Haxby, Phys. Rev. **52**, 390–396 (1937). L. H. Rumbaugh, R. B. Roberts, and L. R. Hafstad, Phys. Rev. **54**, 649–672 (1938).

[22] H. Bethe, Phys. Rev. **53**, 842 (1938).

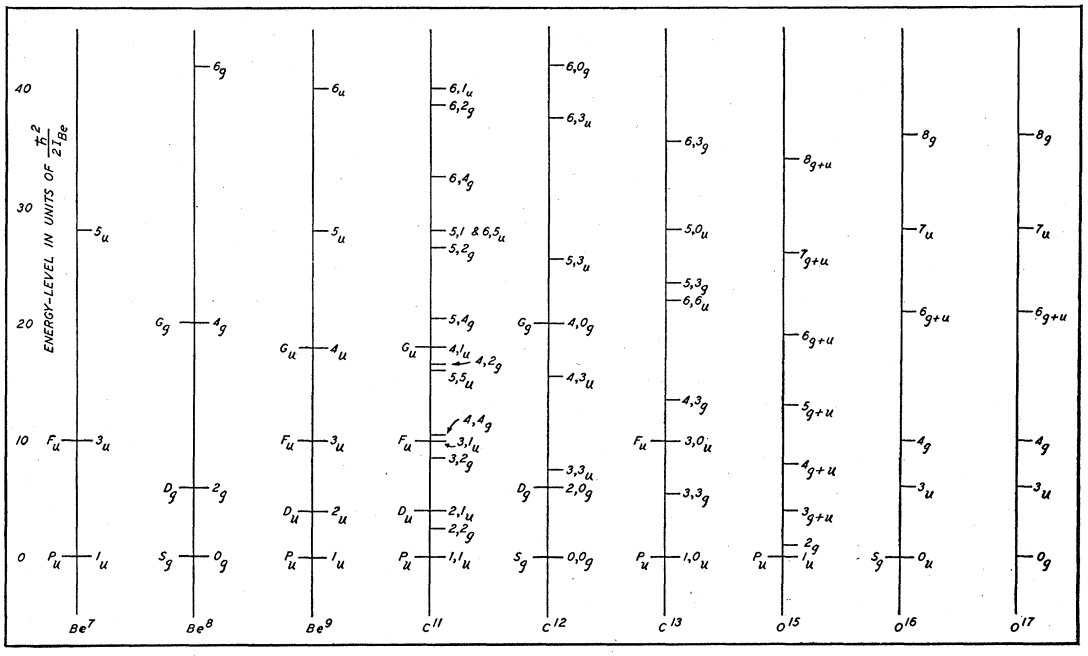

FIG. 6. Excited states of nuclei: Comparison of independent-particle model (left) to alpha-particle model (right) with energy scales adjusted to agree with D level of Be^8 ($_g$=even, $_u$=odd).

O^{15} *or* N^{15}.—Because the proper function of the "hole" transforms as that of a particle with an angular momentum unity, the state $J=0$ cannot occur. All others can.

If the mean angular momentum of the "hole" is small compared to $\hbar$, the formula for the rotation of O^{16} will roughly be applicable. Otherwise the coupling between rotation and the orbit of the "hole" must be taken into account.

The state with $J=1$ will be odd (this is the fundamental state) and that with $J=2$ will be even. States for higher J values will occur with both even and odd parity.

For convenient comparison, the permitted energy states of the various nuclei are shown in Fig. 6. The fundamental states have been assumed to have the energy zero. Actually the rotation will contribute in some cases to the energies of the fundamental states. These contributions are, however, in view of the discrepancies found in Section I, too small to be worth discussion. The excited level system predicted by the alpha-particle model (rotations only) are given to the right of the line for each element of Fig. 6, the numbers indicated referring to J when only one number is given and to the quantum numbers J and K, respectively, when two numbers appear. The levels predicted by Feenberg and Phillips,[17] taken from their Table III, are given on the left. The parity of the levels is indicated for both models.

The ordinate scale for the alpha-particle model is given in units of $\hbar^2/2I_{\mathrm{Be}}$. This scale is fitted to that of Feenberg and Phillips by arbitrarily identifying their D level in Be^8 with our ($J=2$) level. From their Table I it may be seen that their K is exactly equal to two of our units.

It is significant that the symmetry of the fundamental states, as well as that of many of the excited states, are the same for the two models. The agreement for the nuclei Be^7, Be^8, and Be^9 is particularly good. For C^{11}, C^{12}, and C^{13} additional low excited states occur in the alpha-particle picture. They have the opposite parity as the levels obtained from the independent-particle picture. For O^{15} and O^{16} no low excited states are predicted by the independent-particle picture, whereas the alpha-particle model gives some low rotational states. These differences may tend to diminish if in the independent-particle picture excitation of neutrons or protons into higher orbits is taken into account.