The Paths of Ions in the Cyclotron

I. Orbits in the Magnetic Field

L. H. THOMAS
Mendenhall Laboratory of Physics, Ohio State University, Columbus, Ohio
(Received July 11, 1938)

Bethe and Rose maintain in a recent letter and paper that a maximum energy for the beam from a cyclotron is fixed by the incompatibility of the conditions for resonance and focusing when the relativity increase of mass with velocity is taken into account. It is shown below that, while this result holds for a radially symmetrical magnetic field, it is not necessarily true in general; and that for a field varying with polar angle there is an additional focusing effect. If the relative variation of the field with polar angle is of the order of the ratio of the velocity of the ion to the velocity of light, this focusing effect will compensate the defocusing effect of Bethe and Rose. It is shown further that if this variation has period $\pi/2$, a family of stable periodic orbits exists which are nearly concentric circles. The second order effects due to the simultaneous action of the variations with polar angle of the magnetic field and the accelerating electric field will be considered in a second paper.

INTRODUCTION

In a cyclotron a magnetic field H curves the paths of particles of charge e and mass m so that they describe circuits in time $2\pi/\omega = 2\pi m/eH$. An electric field oscillating in this period continually accelerates those particles that are in phase so that they describe, with increasing velocity v, circuits of radius v/ω. Particles are introduced near the central line and withdrawn near the curved surface of a short cylindrical region of radius A with generators parallel to the magnetic field, when they have attained velocity $A\omega$.

Since the curvature of the path of a particle in a magnetic field is accurately proportional inversely to its momentum $m_0 v/\{1-v^2/c^2\}^{\frac{1}{2}} \approx m_0 v(1+\frac{1}{2}v^2/c^2)$, where m_0 is its rest mass, and c the velocity of light, the time of description of a circuit will increase with increasing velocity, and the particles will get out of phase and never attain great energy unless the magnetic field increases in the same ratio as the momentum. If the magnetic field, which must have zero curl, increases with distance from the axis, the magnetic lines must curve outwards above and below a median plane to which they are all perpendicular, (Fig. 1). An orbit above that plane will therefore be curved away from it as well as around the axis, and the beam will be defocused: this effect, discovered by Bethe and Rose,[1,2] is proportional to the relative change in magnetic field, and so to v^2/c^2.

If the magnetic field varies with polar angle about the central line, the path of a particle must have greatest curvature where the field is greatest, and if the path is a closed orbit it must have greatest radial distance where the field is greatest (X, Fig. 2). Along a portion of the orbit in a region (A) in which the field decreases as the polar angle increases, the magnetic lines must curve backwards from the median plane to which they are all perpendicular, and lie in cylindrical surfaces about the axis. In this region the orbits have decreasing radial distances and an orbit above that plane will therefore be curved towards it as well as around the axis; and the beam will be focused (Fig. 3). The same is true in a region (B) in which the field and radial distance both increase with polar angle. This effect is proportional to the product of the relative changes in radial distance and in magnetic field and so to the square of the relative change in magnetic field with polar angle.

The above argument holds for a variation of the magnetic field with polar angle that does not depend on the distance from the axis: It is shown below in detail that if the variation depends on

[1] H. A. Bethe and M. E. Rose, Phys. Rev. 52, 1254 (1937).
[2] M. E. Rose, Phys. Rev. 53, 392 (1938).

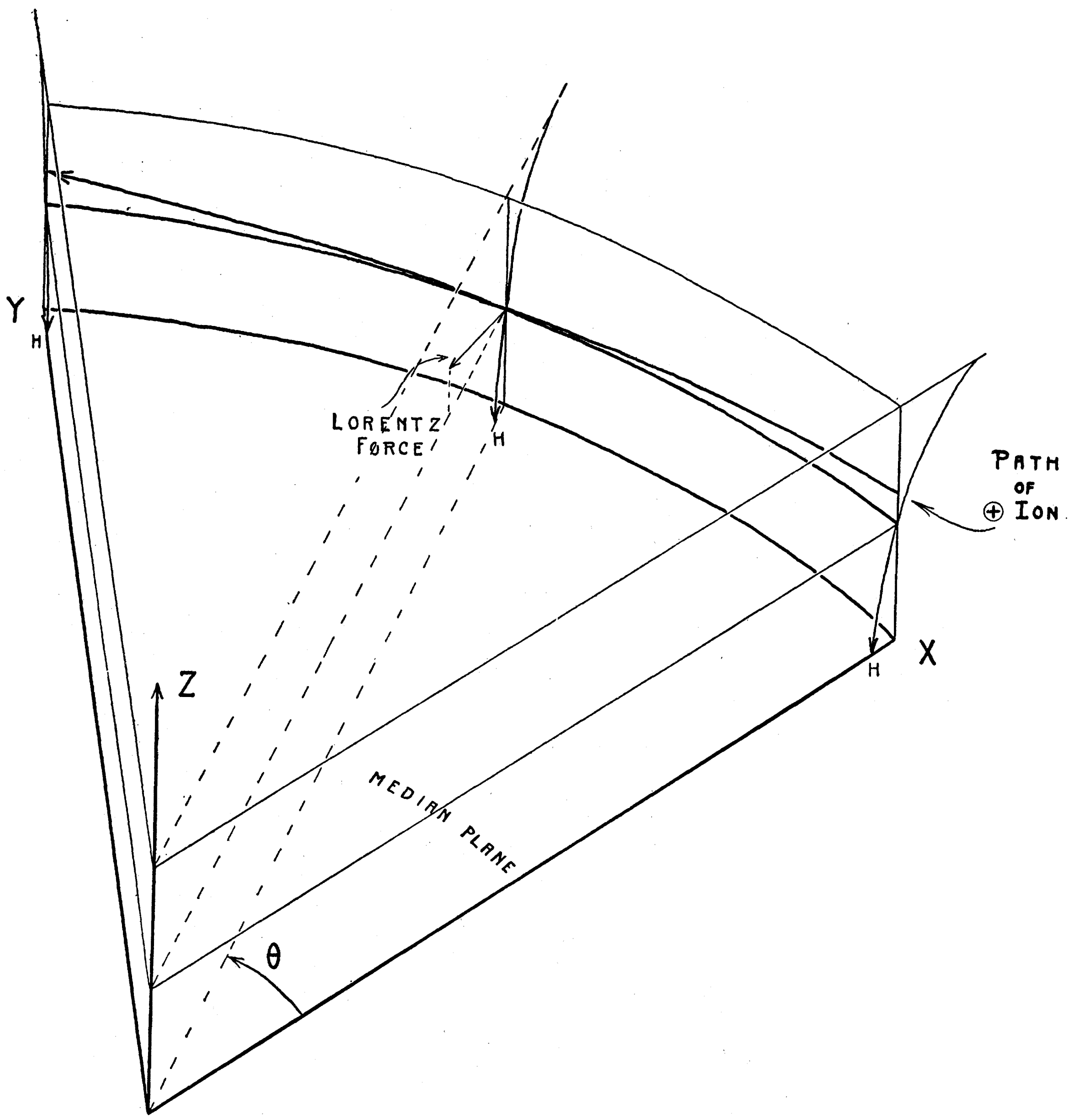

FIG. 1. The defocusing effect of Bethe and Rose.

the distance in such a way as to preserve the time of description of the orbit, the effect is always to focus the beam. A variation of order v/c of the magnetic field with polar angle therefore introduces focusing that will compensate the defocusing effect of Bethe and Rose.

A variation of the magnetic field with position in the median plane will cause a secular change in orbits in that plane regarded as to a first approximation circular. A term in the magnetic field proportional to $\sin\theta$ which we shall suppose is an increase in the y-direction increases the curvature on the side of the circle for which y is positive and decreases it on the side for which y is negative so that the center moves in the positive x-direction (Fig. 4) by an amount per circuit proportional to the product of the radius of the orbit and the change in magnetic field between the two sides.[3] In this case there are no closed orbits.

A term in the magnetic field proportional to $\sin 2\theta$, which we shall suppose is an increase in

[3] I am indebted to a conversation with Professor Bethe for this simple discussion of stability in terms of secular change. See also II §3 below.

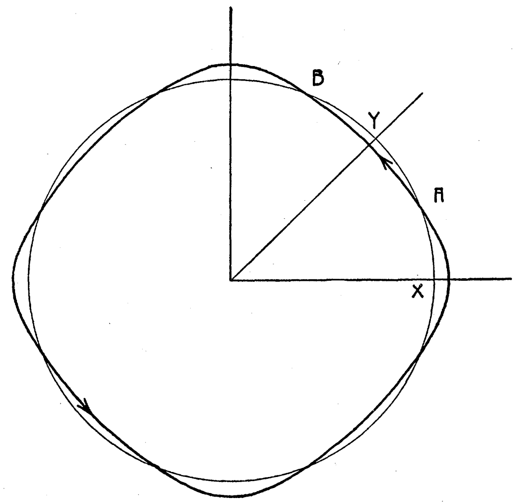

FIG. 2. A periodic orbit in a magnetic field varying with polar angle. (For a positive ion the magnetic field is directed away from the reader.)

both directions for which $x=y$, causes the center to move away from $x=0$ by an amount per circuit proportional to the product of the variation of the magnetic field around the orbit and the distance of the center from $x=0$, as the magnetic field is then greater on the left side by an amount proportional to the product of the ratio of that distance to the radius of the orbit and the variation of the magnetic field around the orbit; and to move towards $y=0$ in a similar way.[3] In this case there is a family of closed orbits but they are unstable.

Thus such terms in the magnetic field will cause particles to approach the outside of the cyclotron along paths of various radii and with various energies; and may, if large enough, so disturb the phase adjustment that the particles will never gain great energy.

It is shown in detail below that a variation of magnetic field proportional to $\cos 4\theta$, or, more generally, any variation periodic with period $\pi/2$ in θ, admits a *stable* family of closed orbits that are approximately circles about the center of the cyclotron.

Thus a variation of the magnetic field with angle, periodic with period $\pi/2$ in θ, approximately proportional to $\cos 4\theta$, where θ may be measured from any initial direction, and of order of magnitude v/c; together with nearly the radial increase of relative amount $\frac{1}{2}v^2/c^2$ of Bethe and Rose; gives stable orbits that are in resonance and not defocused.

1. NOTATION

The equations of motion of a particle of charge e, rest mass m_0, and velocity v, in cylindrical polar coordinates r, θ, z, with the time as independent variable, in magnetic field (H_r, H_θ, H_z), are

$$\begin{aligned} &\frac{d}{dt}\left(m\frac{dr}{dt}\right)-mr\left(\frac{d\theta}{dt}\right)^2=\frac{e}{c}\left(r\frac{d\theta}{dt}H_z-\frac{dz}{dt}H_\theta\right),\\ &r\frac{d}{dt}\left(m\frac{d\theta}{dt}\right)+2m\frac{dr}{dt}\frac{d\theta}{dt}=\frac{e}{c}\left(\frac{dz}{dt}H_r-\frac{dr}{dt}H_z\right), \qquad (1.1)\\ &\frac{d}{dt}\left(m\frac{dz}{dt}\right)=\frac{e}{c}\left(\frac{dr}{dt}H_\theta-r\frac{d\theta}{dt}H_r\right), \end{aligned}$$

where $mv=m_0v/(1-v^2/c^2)^{\frac{1}{2}}$ is the momentum, (1.2)

$$v^2=\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2 \qquad (1.3)$$

and c is the velocity of light: e is in electrostatic units and the magnetic field in electromagnetic units: and a right-handed screw rotating in the direction of θ increasing travels along the positive z-direction.

The magnetic field must satisfy

$$\begin{aligned} &\frac{1}{r}\frac{\partial H_z}{\partial\theta}-\frac{\partial H_\theta}{\partial z}=0, \quad \frac{\partial H_\theta}{\partial r}+\frac{1}{r}H_\theta-\frac{1}{r}\frac{\partial H_r}{\partial\theta}=0,\\ &\frac{\partial H_r}{\partial z}-\frac{\partial H_z}{\partial r}=0, \quad \frac{\partial H_r}{\partial r}+\frac{1}{r}H_r+\frac{1}{r}\frac{\partial H_\theta}{\partial\theta}+\frac{\partial H_z}{\partial z}=0. \end{aligned} \qquad (1.4)$$

We shall suppose that in the median plane $z=0$, $H_\theta=0$, $H_r=0$, so that also $\partial H_z/\partial z=0$, and that the field is symmetrical about this plane, so that for small z

$$\begin{aligned} H_z&=-H+0(z^2),\\ H_r&=-z(\partial H/\partial r)+0(z^3), \qquad (1.5)\\ H_\theta&=-(z/r)(\partial H/\partial\theta)+0(z^3), \end{aligned}$$

where H can be adjusted to be any function of r and θ periodic with period 2π in θ. The negative sign is taken so that if H is a positive constant the orbits in the median plane are circles described in the direction of θ increasing.

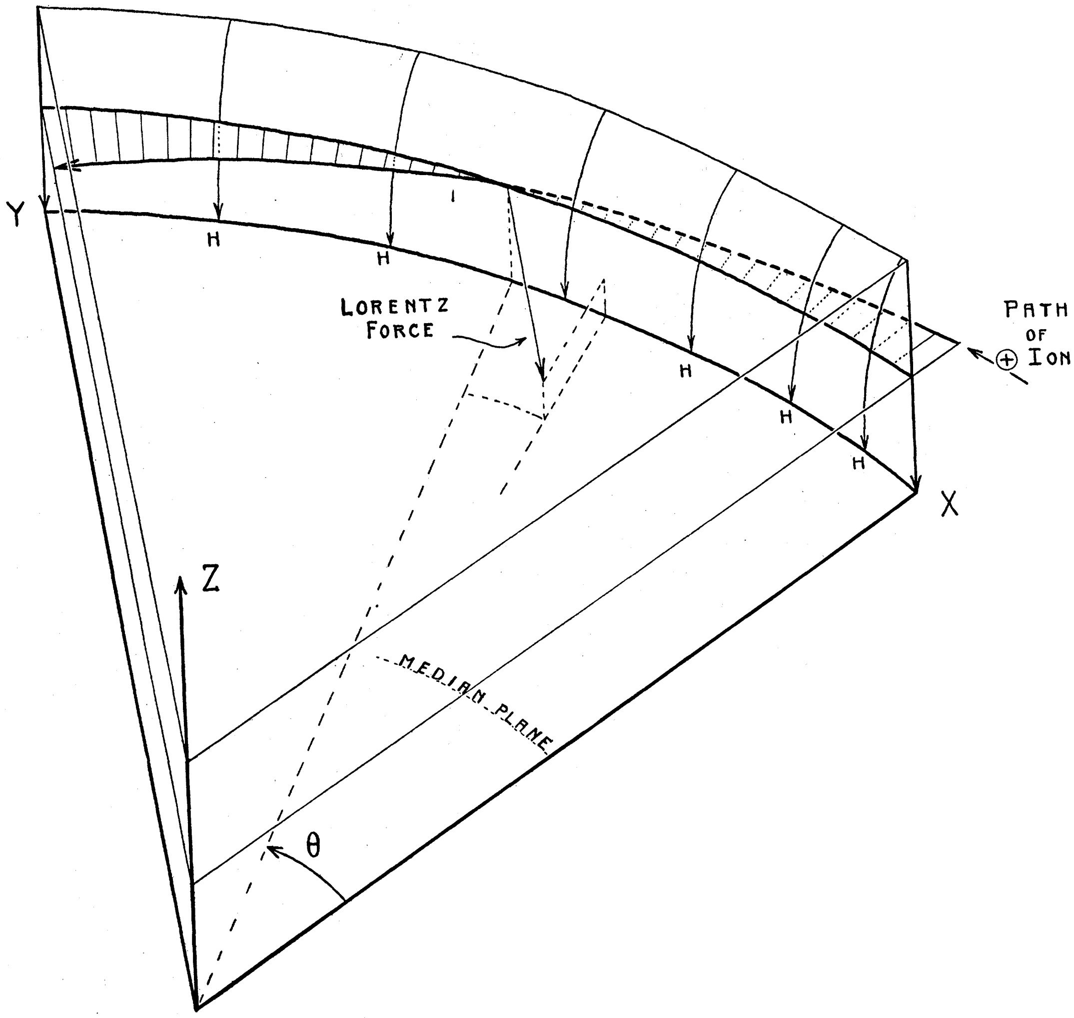

FIG. 3. The focusing effect due to variation of the field with polar angle.

2. THE ORBIT

Eliminating the time, the equation of motion in the plane $z=0$ is

$$\frac{r^2+2(dr/d\theta)^2-r(d^2r/d\theta^2)}{\{r^2+(dr/d\theta)^2\}^{\frac{3}{2}}}=\frac{1}{v}\frac{eH}{mc}, \quad (2.1)$$

the time in the orbit being then given by

$$vdt/d\theta=(r^2+(dr/d\theta)^2)^{\frac{1}{2}}. \quad (2.2)$$

We shall use primes to indicate differentiation with respect to θ. Suppose

$$r=a(1+\epsilon f) \quad (2.3)$$

where f is a function of a and θ, gives a family of orbits depending on the parameter a, which are

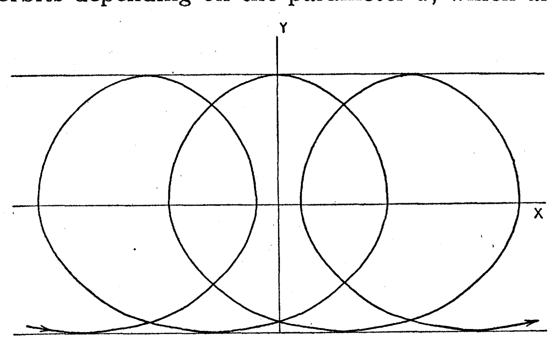

FIG. 4. Instability of a nearly circular orbit due to increase of the field with y. (For a positive ion the magnetic field is directed away from the reader.)

approximately circles about the origin, so that

$$H=\frac{m_0c}{e}\frac{v}{(1-v^2/c^2)^{\frac{1}{2}}}\frac{1}{a}\times\frac{1+\epsilon(2f-f'')+\epsilon^2(f^2+2f'^2-ff'')}{(1+2\epsilon f+\epsilon^2(f^2+f'^2))^{\frac{3}{2}}},\quad(2.4)$$

$$(dt/d\theta)=(a/v)(1+2\epsilon f+\epsilon^2(f^2+f'^2))^{\frac{1}{2}}.\quad(2.5)$$

We shall suppose that our family of orbits is periodic with period 2π in θ so that f is periodic with period 2π in θ, and that a is so chosen that

$$\langle f\rangle_{\text{Av}}\left(=\frac{1}{2\pi}\int_0^{2\pi}fd\theta\right)=0.\quad(2.6)$$

These orbits are exact in the field given by eliminating v and a between (2.3), (2.4), and an arbitrary relation between v and a, which we shall take to be

$$v=a\omega\{1+\epsilon^2g\},\quad(2.7)$$

where g is a function of a only and ω is a constant.

For small ϵ we have approximately, if v/c is small and of the same order, so that

$$(1-v^2/c^2)^{-\frac{1}{2}}\approx1+\tfrac{1}{2}a^2\omega^2/c^2,$$

$$H=(m_0c/e)\omega[1-\epsilon(f+f'')+\tfrac{1}{2}(a^2\omega^2/c^2)+\epsilon^2\{g+2ff''+f^2+\tfrac{1}{2}f'^2\}]\quad(2.8)$$

on $r=a(1+\epsilon f)$, or

$$H=(m_0c/e)\omega[1-\epsilon(f+f'')+\tfrac{1}{2}(a^2\omega^2/c^2)+\epsilon^2\{g+2ff''+f^2+\tfrac{1}{2}f'^2+fa(d/da)(f+f'')\}]\quad(2.9)$$

on $r=a$.

Suppose the magnetic field has the form

$$H=(m_0c/e)\omega[1+\epsilon h+\epsilon^2k],\quad(2.10)$$

where h is a function of r and θ periodic in θ with period π and such that

$$\langle h\rangle_{\text{Av}}=0\quad(2.11)$$

and k is a function of r only.

That the radial variation of $\langle H\rangle_{\text{Av}}$ is of the second order in (2.10) is not essential; it corresponds to radial variation of v/a being of the second order in (2.7).

The first order equation to be solved is

$$f+f''=-h.\quad(2.12)$$

The solution of this is, when the constants of integration are suitably adjusted, like h, a function f_1 periodic in θ with period π and such that

$$\langle f_1\rangle_{\text{Av}}=0,\quad(2.13)$$

and the condition that it has period π determines the constants of integration.

This solution is substituted into the higher order terms in (2.9); the equation averaged over θ determines, dropping the suffix,

$$g=k-\tfrac{1}{2}(a^2\omega^2/\epsilon^2c^2)-2\langle ff''\rangle_{\text{Av}}-\langle f^2\rangle_{\text{Av}}-\langle\tfrac{1}{2}f'^2\rangle_{\text{Av}}-\langle fa(d/da)(f+f'')\rangle_{\text{Av}},\quad(2.14)$$

and its variation with θ gives an equation of the same form as (2.12) and with its right-hand side satisfying the same conditions, to determine f to the second order; and the process can be continued.

Thus we can solve (2.9) or the exact relation to which it approximates for a function f of a and θ, which will be periodic in θ with period π, and a function g of a only, in power series in ϵ.

If, on the other hand, H is only periodic with period 2π in θ, there appears, at some stage in the process, an equation like (2.12), the right-hand side of which contains a term in $\sin\theta$ or $\cos\theta$, so that it has no periodic solution. In this case a periodic solution of (2.9) expansible in powers of ϵ does not exist, though one expansible in powers of $\epsilon^{\frac{1}{2}}$ may exist.

3. Resonance

These orbits will be exactly in resonance if

$$\left(\frac{dt}{d\theta}\right)_{\text{Av}}=\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{d\theta}d\theta\right)=\frac{1}{\omega},\quad(3.1)$$

where ω is the constant angular velocity for the cyclotron, that is

$$v=a\omega\frac{1}{2\pi}\int_0^{2\pi}\{1+2\epsilon f+\epsilon^2(f^2+f'^2)\}^{\frac{1}{2}}d\theta\quad(3.2)$$

or, approximately, for small ϵ, since $\langle f\rangle_{\text{Av}}=0$,

$$v=a\omega(1+\tfrac{1}{2}\epsilon^2\langle f'^2\rangle_{\text{Av}}).\quad(3.3)$$

Thus the condition for resonance is (from (2.7))

$$g=\tfrac{1}{2}\langle f'^2\rangle_{\text{Av}}.\quad(3.4)$$

4. Focusing

Consider paths in the field given by (2.10) near our family of periodic orbits in the plane $z=0$; on account of the symmetry of the field, the variational equations for motion in the plane and for motion out of it separate; the latter gives the "focusing."

We have

$$\frac{d}{dt}\left(m\frac{dz}{dt}\right)=\frac{e}{c}\left(\frac{dr}{dt}H_\theta-r\frac{d\theta}{dt}H_r\right), \quad (4.1)$$

where H_r and H_θ are given by (1.5) and r, t, are to have their values in terms of θ for the unvaried orbit.

Since the right-hand side is small compared with $m\omega^2 z$, we may integrate over a revolution with z constant and obtain for the change in dz/dt

$$\delta\frac{dz}{dt}=z\frac{e}{mc}\int_0^{2\pi}\left\{r\frac{\partial H}{\partial r}-\frac{1}{r}\frac{dr}{d\theta}\frac{\partial H}{d\theta}\right\}d\theta \quad (4.2)$$

and the condition for focusing is

$$\left\langle r\frac{\partial H}{\partial r}-\frac{1}{r}\frac{dr}{d\theta}\frac{\partial H}{\partial\theta}\right\rangle_{\text{Av}}<0. \quad (4.3)$$

Finding $\partial H/\partial r$ and $\partial H/\partial\theta$ on $r=a$ from 2.9 and changing back to $r=a(1+\epsilon f)$, approximately for small ϵ, gives

$$r\frac{\partial H}{\partial r}-\frac{1}{r}\frac{dr}{d\theta}\frac{\partial H}{\partial\theta}=\frac{m_0c}{e}\omega\left[-\epsilon a\frac{d}{da}(f+f'')+\frac{a^2\omega^2}{c^2}+\epsilon^2\left\{f'(f'+f''')+a\frac{d}{da}(g+2ff''+f^2+\tfrac{1}{2}f'^2)+a\frac{df}{da}a\frac{d}{da}(f+f'')\right\}\right]. \quad (4.4)$$

Thus, averaging, after some integrations by parts, and since $\langle f\rangle_{\text{Av}}=0$, the condition for focusing is

$$\frac{a^2\omega^2}{c^2}+\epsilon^2 a\frac{d}{da}\{g-\tfrac{1}{2}\langle f'^2\rangle_{\text{Av}}\}+\epsilon^2\{\langle -f^2+2f'^2-f''^2\rangle_{\text{Av}}+\langle(f+a(d/da)f)^2\rangle_{\text{Av}}-\langle(f'+a(d/da)f')^2\rangle_{\text{Av}}\}<0. \quad (4.5)$$

In resonance, by (3.4), the second term of (4.5) vanishes; and we see by expansion of f in a Fourier series, or by the transformations of the calculus of variations, that if f is periodic with period π, and if $\langle f\rangle_{\text{Av}}=0$, the third term is always negative, and vanishes only if f vanishes.

Thus a variation of H with θ, periodic with period π in θ, can always be made so large, of order $a\omega/c$, as to compensate the first term in (4.5), and leave a net focusing effect, while resonance is maintained.

In particular, in the magnetic field,

$$H=\frac{m_0c}{e}\omega\left[1+\left(\frac{30}{19}\right)^{\frac{1}{2}}\frac{\omega}{c}r\cos 4(\theta-\beta)+\frac{15}{38}\frac{r^2\omega^2}{c^2}\right], \quad (4.6)$$

where β is any constant, the orbits, which are approximately given by

$$r=a\left\{1+\left(\frac{2}{285}\right)^{\frac{1}{2}}\frac{a\omega}{c}\cos 4(\theta-\beta)\right\} \quad (4.7)$$

are in resonance and are neither focused nor defocused.

5. Stability

The variational equations for motion in the plane $z=0$ separate into equations giving constant displacements of the parameter a and the origin of time, and the equation obtained by varying (2.1) with fixed v and H.

Suppose, then,

$$r=a(1+\epsilon f)+\varphi. \quad (5.1)$$

Substituting this in (2.1), and using the forms (2.7) for v and (2.9) for H, the terms linear in φ give, after some calculation, to terms in ϵ^2.

$$\varphi''-\varphi'[\epsilon f'+\epsilon^2(3f'f''-ff')]+\varphi\left[1-\epsilon\left(2f''+a\frac{d}{da}(f+f'')\right)+\epsilon^2\left\{3f'^2+2ff''-\left(f-a\frac{df}{da}\right)a\frac{d}{da}(f+f'')+a\frac{d}{da}\left(\tfrac{1}{2}\frac{a^2\omega^2}{\epsilon^2c^2}+g+2ff''+f^2+\tfrac{1}{2}f'^2\right)\right\}\right]=0. \quad (5.2)$$

The substitution

$$\varphi=\psi\{1+\tfrac{1}{2}\epsilon f+\epsilon^2((3/4)f'^2-\tfrac{1}{8}f^2)\} \quad (5.3)$$

reduces (5.2) to its normal form

$$\psi''+\psi[1+\epsilon p+\epsilon^2 q]=0, \tag{5.4}$$

where

$$p=-\tfrac{3}{2}f''-a\frac{d}{da}(f+f'') \tag{5.5}$$

$$q=\tfrac{9}{4}f'^2+\tfrac{3}{2}(ff''+f''^2+f'f''')-\left(f-a\frac{df}{da}\right)a\frac{d}{da}(f+f'')+a\frac{d}{da}\left(\tfrac{1}{2}\frac{a^2\omega^2}{\epsilon^2c^2}+g+2ff''+f^2+\tfrac{1}{2}f'^2\right). \tag{5.6}$$

The condition that the equation

$$\psi''+\psi[1+\epsilon p]=0 \tag{5.7}$$

should give stable oscillations may be found by variation of parameters. Writing

$$\psi=\alpha\cos\theta+\beta\sin\theta,$$
$$\psi'=-\alpha\sin\theta+\beta\cos\theta,$$

we find

$$\alpha'=\epsilon p\sin\theta\cos\theta\,\alpha+\epsilon p\sin^2\theta\,\beta,$$
$$\beta'=-\epsilon p\cos^2\theta\,\alpha-\epsilon p\sin\theta\cos\theta\,\beta.$$

Averaging over θ, we see that the secular charges in α and β contain terms in $\epsilon^{\pm D\theta}$, where the exponents D are given by

$$D^2=\epsilon^2[\langle p\sin\theta\cos\theta\rangle_{\mathrm{Av}}{}^2-\langle p\sin^2\theta\rangle_{\mathrm{Av}}\langle p\cos^2\theta\rangle_{\mathrm{Av}}]$$
$$=\tfrac{1}{4}\epsilon^2[\langle p\cos 2\theta\rangle_{\mathrm{Av}}{}^2+\langle p\sin 2\theta\rangle_{\mathrm{Av}}{}^2-\langle p\rangle_{\mathrm{Av}}{}^2]. \tag{5.8}$$

For p given by (5.5), $\langle p\rangle_{\mathrm{Av}}=0$, (since $\langle f\rangle_{\mathrm{Av}}=0$), so that $D^2>0$ and there is certainly instability, unless $\langle p\cos 2\theta\rangle_{\mathrm{Av}}$ and $\langle p\sin 2\theta\rangle_{\mathrm{Av}}$ vanish, i.e., unless both $\langle f\cos 2\theta\rangle_{\mathrm{Av}}$ and $\langle f\sin 2\theta\rangle_{\mathrm{Av}}$ vary as $a^{-\frac{1}{2}}$ (or vanish); in which case the calculation must be carried to the next order in ϵ.

If in Eq. (5.4) we make the change of independent variable

$$\theta_1=\theta+\epsilon\xi, \tag{5.9}$$

where ξ is a function of θ to be chosen, and write

$$\psi_1=\psi(1+\epsilon\xi')^{\frac{1}{2}}, \tag{5.10}$$

so that the transformed equation will be in its normal form, we obtain

$$d^2\psi_1/d\theta_1^2+\psi_1[1+\epsilon(p-\tfrac{1}{2}\xi'''-2\xi')+\epsilon^2[q-2p\xi'+3\xi'^2+\tfrac{3}{2}\xi'''\xi'+\tfrac{3}{4}\xi''^2]]=0. \tag{5.11}$$

When $\langle p\rangle_{\mathrm{Av}}$, $\langle p\cos 2\theta\rangle_{\mathrm{Av}}$, and $\langle p\sin 2\theta\rangle_{\mathrm{Av}}$, vanish, we can obtain a periodic function ξ such that

$$\xi'''+4\xi'=2p \tag{5.12}$$

and (5.11) becomes

$$d^2\psi_1/d\theta_1^2+\psi_1(1+\epsilon^2 n)=0, \tag{5.13}$$

where $n=q-2p\xi'+3\xi'^2+\tfrac{3}{2}\xi'''\xi'+\tfrac{3}{4}\xi''^2$ (5.14)

and the exponents are given by (cf. (5.7) and (5.8))

$$D^2=(\epsilon^4/4)[\langle n\cos 2\theta\rangle_{\mathrm{Av}}{}^2+\langle n\sin 2\theta\rangle_{\mathrm{Av}}{}^2-\langle n\rangle_{\mathrm{Av}}{}^2]. \tag{5.15}$$

The form of (5.8) shows that if the Fourier expansion of f contains terms in $\cos 2\theta$ or $\sin 2\theta$ not varying as $a^{-\frac{1}{2}}$, the orbits will be unstable: and this will follow if H contains such terms not varying as $r^{-\frac{1}{2}}$.

If H is periodic with period $\frac{1}{2}\pi$, the exponent given by (5.8) vanishes: terms in $\cos 2\theta$ or $\sin 2\theta$ cannot arise in any order, so that we conclude from the form of (5.15) that the exponents are either zero to all orders in ϵ, or imaginary in the first order in which they do not vanish; and that the orbit is "stable."

In particular in the magnetic field given by (4.6), (5.15) gives

$$D=\pm\iota\frac{5\times 21}{8\times 19}\frac{a^2\omega^2}{c^2}. \tag{5.16}$$

It should be remarked that the above argument applies only when ϵ is sufficiently small. If we apply it to a field depending only on r,

$$H=\frac{m_0c}{e}\omega(1+\epsilon^2 k), \tag{5.17}$$

so that $f=0$, (5.15) gives

$$D=\pm\iota\frac{\epsilon^2}{2}r\frac{d}{dr}k \tag{5.18}$$

for the variation of α and β,

or

$$D=\pm\iota\left(1+\frac{\epsilon^2}{2}r\frac{d}{dr}k\right) \tag{5.19}$$

for the variation of ψ, agreeing, to terms of order ϵ^2, with the exact result for circular orbits in a field depending only on r,

$$D^2=-\left(1+\frac{r}{H}\frac{dH}{dr}\right), \qquad (5.20)$$

which gives instability if $|rH|$ decreases as r increases.

6. The Nature of the Orbits

The above results about the orbits in the median plane illustrate the general theory of orbits.[4]

An ion moving in a plane under a magnetic field perpendicular to that plane forms a conservative dynamical system with two degrees of freedom. Using rectangular Cartesian coordinates x, y, and the corresponding momentum components X, Y; and the proper time,

$$s=\int(1-v^2/c^2)^{\frac{1}{2}}dt,$$

as independent variable, the Hamiltonian function takes the form

$$\mathcal{H}=\frac{1}{2m_0}\left\{\left(X-\frac{e}{c}F\right)^2+\left(Y-\frac{e}{c}G\right)^2\right\}, \qquad (6.1)$$

where F and G are functions of x and y, the components in the x and y directions of the magnetic vector potential. In fact, the motion depends only on the magnetic field.

$$H_z=-H=(\partial G/\partial x)-(\partial F/\partial y).$$

In the special case of constant magnetic field $-H_0$, when we take $F=\frac{1}{2}H_0y$, $G=-\frac{1}{2}H_0x$, there are three independent integrals uniform in x, y, X and Y, *viz.*

$$\mathcal{H}=\alpha, \quad X+(e/c)F=\beta, \quad Y+(e/c)G=\gamma.$$

In this case all the orbits are periodic, forming the three-parameter family of all circles.

In the special case of magnetic field depending only on the distance r from a fixed point at a finite distance, which we take to be the origin of coordinates, when we take $F=yK$, $G=-xK$, where K is a function of r only, so that

$$H=2K+r(dK/dr),$$

there are just two independent integrals uniform in x, y, X and Y, *viz.*

$$\mathcal{H}=\alpha, \quad Yx-Xy=\beta.$$

In this case the system is of "soluble type." We see from

$$m_0^2\left(\frac{dr}{ds}\right)^2=\frac{1}{r^2}(Xx+Yy)^2=2\alpha m_0-\left(\frac{e}{c}rK+\frac{\beta}{r}\right)^2$$

that in the "stable" case in which $|rH|$ continually increases as r increases, there is a family of orbits which librate between any two values of r. Together these form a three-parameter family of conditionally periodic orbits, from which a denumerable infinity of two-parameter families of "ordinary" periodic orbits with all four of their exponents zero, can be picked out. The circles with center at the origin form a one-parameter family of "singular" periodic orbits with two exponents zero and the other two real or imaginary as $|rH|$ decreases or increases with r.

In the limiting case of the above in which the fixed point is at infinity, in the x-direction say, H is a function of x only, and we take $F=0$, G a function of x only. There are two independent uniform integrals,

$$\mathcal{H}=\alpha, \quad Y=\beta.$$

From
$$m_0^2\left(\frac{dx}{ds}\right)^2=2\alpha m_0-\left(\beta-\frac{e}{c}G\right)^2$$
and
$$m_0\frac{dy}{ds}=\beta-\frac{e}{c}G,$$

we see that in the case which might be stable, x being a periodic function of s, y will generally not remain finite. There will then be no conditionally periodic orbits, and no ordinary periodic orbits; the singular periodic orbits, if they exist, must be unstable.

In general the problem is not of soluble type, and there is only one integral uniform in x, y, X and Y, *viz.*

$$\mathcal{H}=\alpha.$$

There are no two-parameter families of ordinary periodic orbits. There may be, and will be if the field is nearly homogeneous and symmetrical about two perpendicular directions, one-parameter families of singular periodic orbits which

[4] Whittaker, *Analytical Dynamics* (Fourth Edition, Cambridge, 1937), Ch. xv.

will in general have two zero and two nonzero exponents. The stability corresponding to imaginary exponents is really only stability to the first order in the variation of the orbit and it remains open whether second and higher order terms would not lead to instability. However, such second order variations do not increase exponentially with the time, but become large only after a time inversely proportional to the initial displacement; and their effects are not expected to be important in the number of revolutions that take place in the cyclotron: Indeed they may be damped by the effects of the accelerating electric field.

7. Conclusion

It is concluded that in a cyclotron the magnetic field of which varies in such a manner as that given by (4.6), the beam of ions may remain in resonance without being defocused or becoming unstable, in spite of the relativity increase of mass with velocity, at least up to velocities at which terms in v^3/c^3 become important.

Modifications of the above results when the simultaneous action of the variations of the magnetic field with polar angle and of the accelerating electric field with polar angle are taken into account will be considered in a second paper. It appears that these effects become continually less and less important as the relative increase of momentum per revolution decreases with increasing energy.

The author wishes to thank Professor M. L. Pool for his encouragement and interest during the writing of this paper.

OCTOBER 15, 1938 PHYSICAL REVIEW VOLUME 54

The Paths of Ions in the Cyclotron

II. Paths in the Combined Electric and Magnetic Fields

L. H. Thomas
Ohio State University, Columbus, Ohio

(Received July 26, 1938)

It has been pointed out in a recent paper that a variation of the magnetic field of a cyclotron with polar angle can produce a focusing effect on the beam, while preserving resonance and stability. In that paper the effects of the magnetic field alone were considered. It is shown below that the effects of variation with polar angle of the accelerating electric field and of the magnetic field can be considered as almost independent; the second order cross terms between them are without practical effect. Thus the results contained in the above paper (I) may simply be superposed on those obtained by other workers.

Introduction

The argument contained in the former paper[1] is strictly true for orbits in the magnetic field alone; when the effects of the accelerating electric field are taken into account, the closed orbits are replaced by a family of "central paths" which the particles traverse in a phase relationship to the field changing with the radius in a definite way. These central paths can always be found if both magnetic and electric fields are periodic with period π in θ, but will not usually exist if they are only periodic with 2π. If the field is suitably adjusted to resonance, a pencil of the central paths are "spirals" from the center to the outside.

The motion of an ion near a central path separates, on account of the symmetry about the median plane, into motion out of the plane and motion in the plane. The motion out of the plane

[1] L. H. Thomas, Phys. Rev., this issue. (This will be referred to as (I).)