Note on the Theory of the Neutral Particle

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Majorana has recently shown by using a special set of Dirac matrices that the symmetry properties of the Dirac equations make possible the elimination of the negative energy states in the case of a free particle. We present here a further investigation of this possibility, in a treatment based on an arbitrary Hermitian representation of the Dirac matrices instead of Majorana's special representation. The new procedure is compared with Schroedinger's early attempt to eliminate the negative energy states. The question of Lorentz invariance is discussed, and also the possibility of subjecting the particle to forces; it is found that the only sort of force having a classical analogue which is consistent with Majorana's way of eliminating the negative energy states is the nonelectric force of a scalar potential. The theory is worked through for this case, and it is pointed out that, in spite of the fact that the exclusion of negative energy states is accomplished without the introduction of antiparticles, the formalism still shows the stigmata associated with subtraction theories of the positron: the presence of otiose infinite terms which should be removed by subtraction, and the creation and destruction of pairs of particles. The application of Majorana's formalism to the theory of β-radioactivity is discussed at the end of the paper. Here the physical interpretation is quite different from that of the ordinary theory, since only neutrinos appear instead of the neutrinos and antineutrinos of the usual picture. The results predicted for all observed processes are nevertheless identical with those of the ordinary theory. An experimental decision between the formulation using neutrinos and antineutrinos and that using only neutrinos will apparently be even more difficult than the direct demonstration of the existence of the neutrino.

I. INTRODUCTION

IN a recent paper[1] Majorana has presented a derivation of a symmetrical theory of the electron and positron from a new type of variation principle whose use depends essentially on the fact that the quantities involved are q-numbers. In spite of this novel approach, the positron theory he obtains is essentially just a subtraction theory of the simplest type; but Majorana also showed how his ideas can be applied in the theory of the neutral particle to obtain a formalism essentially different from that of the ordinary Dirac theory. Qualitatively the difference appears in the number of states having the same momentum. In the Dirac theory as used at present there are four such states, corresponding to two alternatives for the spin orientation and to the possible existence of both the particles in question and their "antiparticles"—e.g., neutrinos and antineutrinos. In the Majorana theory there are just two states for a given momentum, corresponding to the two possibilities for the spin: there are no "antiparticles" and, in the final formulation, no mention of negative energy states.

As has been remarked by Racah,[2] a theory of Majorana's type cannot very reasonably be supposed to hold for neutrons because of their unsymmetrical relation to the two signs of charge and their possession of a magnetic moment. For the neutrino, however, the Majorana theory is *a priori* just as acceptable as the ordinary Dirac theory. It is interesting to find that it is possible to accomplish all the purposes for which the neutrino theory was devised, including the discussion of both electron emission and positron emission, without the introduction of antineutrinos. This point will be discussed further in the concluding section.

The main formal developments of the present paper are concerned with the generalization of the theory to allow the use of an arbitrary Hermitian representation of the Dirac matrices, instead of the special sort of representation to which Majorana's original treatment is restricted, and with the extension of the theory to include the action on the particles of the nonelectric field of a scalar potential.[3] Although such fields have not had to be postulated in any of the existing applications of the neutrino theory, their introduction into the present dis-

[1] E. Majorana, Nuovo Cimento **14**, 171 (1937).

[2] G. Racah, Nuovo Cimento **14**, 322 (1937).

[3] W. H. Furry, Phys. Rev. **50**, 784 (1936).

cussion is not without interest for two reasons: First, we are enabled to see that the Majorana theory still shows a strong resemblance to the usual Dirac theory, since in spite of the absence of antineutrinos we find that particles are produced in pairs; and second, it helps to bring clearly to light the spin behavior of the particles by showing the existence of a spin-orbit coupling.[4]

The emphasis will here be placed, not on Majorana's interesting new type of variation principle, concerning which we have nothing new to add, but on the application of a symmetry property of the Dirac equations by means of which Majorana accomplishes the elimination of the negative energy states. Before proceeding to the developments of the following sections, we wish to give a general discussion of this idea, comparing it especially with Schroedinger's early suggestion[5] for the elimination of the negative energy states of the electron.

In both cases we may take as the essential problem that of projecting the function space of the Dirac wave functions into one of two equal subspaces. But our use in this connection of the terms "projection" and "subspace" must at once be accompanied by a warning: In the Schroedinger case we are actually concerned with projection into a linear subspace according to the accepted definition[6] of these words, but this is not so in the Majorana case. In Section II we shall discuss in detail the unorthodox and rather artificial sense in which such terms may be used about Majorana's procedure. It is interesting that the procedure which is more artificial is also much more successful.

In the Schroedinger case the two subspaces are those of the positive energy and negative energy wave functions of a free particle, and the projection is to consist in omitting the negative energy components of any given wave function. That is, if

$$\psi=\psi^{(+)}+\psi^{(-)}, \tag{I 1}$$

where $\psi^{(+)}$ consists of positive energy functions of a free particle and $\psi^{(-)}$ of negative energy functions, then the projection is given by

$$\psi\rightarrow\psi^{(+)}. \tag{I 2}$$

This can be formulated in general by using the operator[7] given in the p-representation by

$$\Lambda=(-\boldsymbol{\alpha}\cdot\mathbf{p}-\beta mc)(m^2c^2+p^2)^{-\frac{1}{2}}, \tag{I 3}$$

which has the property

$$\Lambda\psi^{(+)}=\psi^{(+)};\quad \Lambda\psi^{(-)}=-\psi^{(-)}. \tag{I 4}$$

Schroedinger's projection is then simply

$$\psi\rightarrow\tfrac{1}{2}(1+\Lambda)\psi. \tag{I 5}$$

It is well known that an essential difficulty with Schroedinger's suggestion is that it is not relativistically invariant. A Lorentz transformation of ψ is accomplished by an operator $L(a^\nu{}_\mu)$:

$$\begin{aligned}\psi'(x'^\sigma)&=L(a^\nu{}_\mu)\psi(x^\lambda)=S(a^\nu{}_\mu)A(a^\nu{}_\mu)\psi(x^\lambda)\\&=S(a^\nu{}_\mu)\psi(a^\nu{}_\mu x'^\mu).\end{aligned} \tag{I 6}$$

Here $a^\nu{}_\mu$ is the Lorentz transformation of the coordinates, and $S(a^\nu{}_\mu)$ is the corresponding four-rowed matrix operating on the components of the Dirac wave function.[8] The noninvariance of (I 5) then results from the fact that Λ and L do not commute:

$$\Lambda L-L\Lambda\neq 0. \tag{I 7}$$

The projection (I 5) also has the disadvantage that after a wave function has been projected into the positive-energy subspace it will not remain confined to that subspace if the particle is subject to forces of any ordinary sort. This is because Λ does not commute with the Dirac Hamiltonian for a bound particle. Schroedinger accordingly suggested that the terms admitted to the Hamiltonian be restricted to "even" operators, i.e., those which do not connect the two subspaces, or in other words those which commute with Λ. It is well known that this procedure leads to difficulties not only with invariance but also with the physical results of the theory.

It is interesting to note that although the operator $\Lambda L-L\Lambda$ is not equal to zero, it gives

[4] Cf. D. R. Inglis, Phys. Rev. **50**, 783 (1936).
[5] E. Schroedinger, Berl. Ber. p. 63, 1931.
[6] Cf. J. von Neumann, *Math. Grundlagen der Quantenmechanick*, p. 40.
[7] Cf. W. Pauli, *Handbuch der Physik*, Vol. 24, pp. 230–231.
[8] Cf. reference 7, p. 221.

the result zero when applied to any *free particle* wave function. This assertion may at first sight seem to be inconsistent with the possibility of analyzing an arbitrary wave function in terms of free particle wave functions, which is postulated in (I 1). But the analysis (I 1) and the projection (I 5) make use only of the instantaneous values of ψ throughout space, and it is only for such three-dimensional purposes that the free particle wave functions form a complete set. It is, however, obviously impossible either to make the Lorentz transformation (I 6) or to tell whether or not ψ is actually just a free particle function unless one makes use of the time dependence of ψ as well as its instantaneous values. If ψ_f is any wave function whose time dependence is that given by the Dirac equation without field, then on using this time dependence in making the Lorentz transformation, one indeed finds that

$$(\Lambda L - L\Lambda)\psi_f = 0. \tag{I 7'}$$

It is this fact which makes it possible to use the wave functions of a free electron as a basis for constructing a positron theory which is at least formally Lorentz invariant, although in practice the invariance is destroyed by divergence difficulties. In using the method of second quantization to set up the formalism of positron theory one of course uses wave functions with the time factors removed; but Lorentz transformations may be made correctly by using the p-representation and regarding p as the space part of a four-vector whose time component is $\pm(m^2c^2+p^2)^{\frac{1}{2}}$. Thus without explicit reference to the time factor one can still verify that $\Lambda L - L\Lambda$ gives zero when applied to such a function.[9]

The projection used by Majorana may be defined for an arbitrary representation of the Dirac matrices by using an operator A which was previously used by the writer to prove a symmetry theorem in the positron theory.[10] The existence of this operator and its unique determination apart from a phase factor have been proved by Pauli.[11] The operator A is a constant unitary four-rowed matrix, and has the property that if ψ is any pure positive energy free particle wave function, then $A\psi$ is the complex conjugate of a pure negative energy function; and if ψ is any pure negative function, $A\psi$ is the complex conjugate of a pure positive function. Majorana's projection is given by

$$\psi \rightarrow \tfrac{1}{2}\{\psi + (A\psi)^*\}, \tag{I 8}$$

where $(A\psi)^*$ is the complex conjugate of $A\psi$.

In the next section we shall show that the operation indicated in (I 8) has the important property of being idempotent, which enables us to regard it in a certain sense[12] as a projection operator; that the projection into the other of the two equal subspaces is given by

$$\psi \rightarrow \tfrac{1}{2}\{\psi - (A\psi)^*\}; \tag{I 9}$$

that the projection (I 8) is relativistically invariant; and that the wave function will remain confined to the subspace if the wave equation is that of a free particle or a particle subject to a nonelectric force,[3] but not if the particle has an electric charge or a magnetic moment introduced as suggested by Pauli.[13]

It is seen from the stated properties of A and the form of (I 8) that instead of discarding the negative energy states Majorana's projection may in a sense be regarded as symmetrizing the wave function with respect to positive and negative states. The fact that the phase of A is not fixed by its definition in terms of its properties shows, however, that one has no right to attach the terms "symmetric" and "antisymmetric" to the functions given by (I 8) and (I 9), respectively.

II. Properties of Majorana's Projection

Notation. In the Dirac theory of a single particle there occur four-row square matrices such as α_i and A, four-row one-column matrices such as the wave function ψ, and one-row four-

[9] It may be remarked that this same circumstance makes Schroedinger's procedure work just as well as Majorana's for the case of a *free particle*, which is the only case Majorana discusses for the neutral particle. The extension to the case of a nonelectric force (Section III) and the application to β-radioactivity (Section IV) are, however, impossible in the Schroedinger case.

[10] W. H. Furry, Phys. Rev. **51**, 125 (1937).

[11] W. Pauli, Ann. Inst. Henri Poincare **6**, 130 (1936). The matrix in question is there denoted by C.

[12] But not at all in the usual sense—cf. Section II and reference 6.

[13] Reference 7, p. 233.

column matrices such as the Hermitian adjoint of ψ. In dealing with such matrices we shall use an asterisk (*) to indicate the complex conjugate matrix, a prime (′) to indicate the transposed matrix, and a dagger (†) to indicate the Hermitian adjoint matrix. Similar notations hold for the two-row and/or two-column matrices which are introduced in the latter part of Section III.

It is important to note carefully how the notation is to be interpreted when we come to apply the method of second quantization, and regard the components of the wave function as q-numbers. Since when the ψ's are c numbers $\psi_i+\psi_i^*$ is a real number, when the ψ's are q-numbers $\psi_i+\psi_i^*$ must be a real q number, or in other words a self-adjoint operator. Thus *when the ψ's are operators, ψ_i^* must be the Hermitian adjoint operator to ψ_i.* By ψ we mean the one-column matrix with the components ψ_1, ψ_2, ψ_3, ψ_4, and by ψ' the one-row matrix with the same components; by ψ^* we mean the one-column matrix with the components ψ_1^*, ψ_2^*, ψ_3^*, ψ_4^*, and by $\psi^\dagger$ the one-row matrix with these components. Such symbols as ψ_i' and $\psi_i^\dagger$ will never be used. The concept of transposition is indeed not a suitable one to be introduced with regard to operators in general, since it is not invariant under canonical transformations and its meaning depends on the representation used. Its use in the case of the matrices of the Dirac theory is, however, free from ambiguity and offers well-known advantages in the way of shortening the formulas.

The matrix A and the projections. We shall develop the theory for an arbitrary Hermitian representation of the Dirac matrices, and we shall denote this set of matrices by α_i, β or by $\boldsymbol{\alpha}$, β. Corresponding to this representation of the Dirac matrices there is a certain matrix A, uniquely determined except for phase, which has the properties described in the introduction. The conditions on this matrix are[14]

$$A^\dagger A=1. \tag{1}$$

$$\begin{aligned}\alpha_i'A&=A\alpha_i,\\ \beta'A&=-A\beta.\end{aligned} \tag{2}$$

[14] Reference 10, Eq. (10) and (14).

Pauli has shown[15] that A also satisfies

$$A^*A=1. \tag{3}$$

From (1) and (3) we see that

$$A'=A. \tag{4}$$

We now represent the projections (I 8) and (I 9) by $\psi\rightarrow P\psi$ and $\psi\rightarrow I\psi$, respectively, with

$$P\psi=\tfrac{1}{2}(\psi+A^*\psi^*), \tag{5}$$

$$I\psi=\tfrac{1}{2}(\psi-A^*\psi^*). \tag{5$'$}$$

The operators P and I obviously satisfy

$$P+I=1 \tag{6}$$

and we can readily show that they also satisfy

$$P^2=P, \qquad I^2=I \tag{7}$$

and

$$PI=IP=0. \tag{8}$$

The meaning of such equations is of course that the application of the left member to an arbitrary ψ gives the same result as the application of the right member. To prove (7) and (8), we have only to substitute from (5) and (5′) and use (3). For instance,

$$\begin{aligned}P^2\psi&=P\cdot\tfrac{1}{2}(\psi+A^*\psi^*)\\ &=\tfrac{1}{2}\{\tfrac{1}{2}(\psi+A^*\psi^*)+A^*\cdot\tfrac{1}{2}(\psi+A^*\psi^*)^*\}\\ &=\tfrac{1}{2}(\psi+A^*\psi^*)=P\psi. \qquad \text{Q.E.D.}\end{aligned}$$

Equation (7) expresses the fact that the operators P and I are idempotent, a property which a projection operator must possess, and (6) and (8) indicate that the "subspaces" into which the two operators project are collectively exhaustive and mutually exclusive.

In order, however, for an operator to be a projection operator associated with a linear subspace, it must be not only idempotent but linear and Hermitian;[6] and it is obvious from (5) and (5′) that our operators P and I are *not linear.* What we have called the "subspaces" defined by these operators are accordingly not subspaces in the usual sense, i.e., not *linear subspaces;* for from

$$\begin{aligned}P\psi&=\psi, \qquad & I\psi&=0,\\ P\varphi&=\varphi, \qquad & I\varphi&=0\end{aligned}$$

[15] Reference 11, Eq. (33a).

we cannot conclude that

$$P(a\psi+b\varphi)=a\psi+b\varphi, \qquad I(a\psi+b\varphi)=0,$$

unless a and b are real numbers. The subspaces of the Majorana projection are invariant under Lorentz transformation and under time displacement as determined by the Dirac equations of a particle subject to a nonelectric force, as we shall show below, but they are not invariant under simple multiplication by a complex constant. This makes the theory look rather artificial, but does not interfere with its success, because such multiplication plays no part in it.

Lorentz invariance. The projection is Lorentz invariant because it makes no difference whether one makes a Lorentz transformation before or after the projection operator is applied:

$$L(a^{\nu}{}_{\mu})P\psi=PL(a^{\nu}{}_{\mu})\psi. \tag{9}$$

Proof: Since the operations involved in applying P have nothing to do with the dependence of ψ on the x^{μ}, (9) is equivalent to

$$S(a^{\nu}{}_{\mu})P\psi=PS(a^{\nu}{}_{\mu})\psi. \tag{10}$$

By (5), (10) becomes

$$S(a^{\nu}{}_{\mu})\cdot\tfrac{1}{2}(\psi+A^*\psi^*)=\tfrac{1}{2}\{S(a^{\nu}{}_{\mu})\psi+A^*S^*(a^{\nu}{}_{\mu})\psi^*\}$$

and this is true for an arbitrary ψ if

$$S(a^{\nu}{}_{\mu})A^*=A^*S^*(a^{\nu}{}_{\mu}). \tag{11}$$

Now $S(a^{\nu}{}_{\mu})$ can be represented as a product of factors of the following types:

$$\begin{aligned}
&\text{Rotation in } (i,j) \text{ plane:} && S=e^{(\theta/2)\alpha_i\alpha_j}\\
&\text{Velocity along } j \text{ axis:} && S=e^{(\theta/2)\alpha_j}\\
&\text{Reflection of the } i \text{ axis:} && S=i\alpha_j\alpha_k\beta,\ j\neq i\neq k\neq j\\
&\text{Reflection of the time axis:} && S=i\alpha_1\alpha_2\alpha_3\beta.
\end{aligned} \tag{12}$$

On making use of (2) and of the Hermitian character of α and β, one finds that (11) is satisfied by any product of factors of the forms (12), so that the truth of (10) is established.

Because of the noninvariance of the projection P under change of phase, the phase factors in $S(a^{\nu}{}_{\mu})$ must be kept the same as specified in (12), apart from a possible factor -1. The necessity of fixing these phase factors in a definite way in order for (11) to hold has been pointed out by Racah,[2] who calls attention to the fact that these factors have a significance for the choice of the interaction operator in Fermi's original theory of β-decay.

Persistance of the projection in time. The Dirac equations are

$$\partial\psi/\partial t=D\psi, \tag{13}$$

where D is an operator whose various possible forms are listed below. According to (13), if ψ had the value ψ_0 at $t=0$, its value at a later time is

$$\psi=e^{Dt}\psi_0. \tag{14}$$

We now want to investigate whether

$$P\psi_0=\psi_0, \qquad I\psi_0=0,$$

together with (14), implies that

$$P\psi=\psi, \qquad I\psi=0.$$

This will be so if P and I commute with e^{Dt}, or in other words if

$$PD\psi=DP\psi, \qquad ID\psi=DI\psi \tag{15}$$

for arbitrary ψ. Substituting from (5) and (6), we find that (15) will hold for arbitrary ψ if and only if

$$DA^*=A^*D^*. \tag{16}$$

Now D is just $(i\hbar)^{-1}$ times the Dirac Hamiltonian for the particle. Thus it must contain the terms

$$D_0=(i\hbar)^{-1}\{i\hbar c(\boldsymbol{\alpha}\cdot\nabla)-\beta mc^2\} \tag{17}$$

for a free particle, together with terms representing any interactions of the particle with fields of force. The known possibilities for forces having classical analogs are:

$$\begin{aligned}
&D_1=(i\hbar)^{-1}\{e(\boldsymbol{\alpha}\cdot\mathbf{A})+e\varphi\} \quad \text{(charge } e \text{ on particle)},\\
&D_2=(i\hbar)^{-1}\{-\mu(\beta\boldsymbol{\sigma}\cdot\mathbf{H})-i\mu(\beta\boldsymbol{\alpha}\cdot\mathbf{E})\} \quad \text{(magnetic moment } \mu),\\
&D_3=(i\hbar)^{-1}\{-\beta\Phi\} \quad \text{(nonelectric potential energy } \Phi).
\end{aligned} \tag{18}$$

By use of (2), we find that (16) is satisfied for

$$D=D_0+D_3 \tag{19}$$

but not if D_1 or D_2 is included in D. Thus if the wave function is to remain confined to one Majorana subspace the only one of these three kinds of force that can act on the particle is the nonelectric force.

Commutation relations. Hitherto we have been concerned only with certain linear conditions satisfied by the wave function before and after projection. In what follows we shall have to introduce bilinear expressions, and use the technique of second quantization, so that it will be necessary to know the commutation rules obeyed by the components of the projected wave function. Assuming the four components of ψ to satisfy the usual Jordan-Wigner[16] relations

$$\begin{gathered}\psi_i^*(\mathbf{r})\psi_j(\mathbf{r}')+\psi_j(\mathbf{r}')\psi_i^*(\mathbf{r})=\delta_{ij}\delta(\mathbf{r}-\mathbf{r}'),\\ \psi_i(\mathbf{r})\psi_j(\mathbf{r}')+\psi_j(\mathbf{r}')\psi_i(\mathbf{r})=0,\\ \psi_i^*(\mathbf{r})\psi_j^*(\mathbf{r}')+\psi_j^*(\mathbf{r}')\psi_i^*(\mathbf{r})=0.\end{gathered}\tag{20}$$

We find on using (5) that the four components of the function U defined by

$$U=P\psi \tag{21}$$

satisfy the relations

$$\begin{gathered}U_i^*(\mathbf{r})\,U_j(\mathbf{r}')+U_j(\mathbf{r}')\,U_i^*(\mathbf{r})=\tfrac{1}{2}\delta_{ij}\delta(\mathbf{r}-\mathbf{r}'),\\ U_i(\mathbf{r})\,U_j(\mathbf{r}')+U_j(\mathbf{r}')\,U_i(\mathbf{r})=\tfrac{1}{2}A_{ij}^*\delta(\mathbf{r}-\mathbf{r}'),\\ U_i^*(\mathbf{r})\,U_j^*(\mathbf{r}')+U_j^*(\mathbf{r}')\,U_i^*(\mathbf{r})=\tfrac{1}{2}A_{ij}\delta(\mathbf{r}-\mathbf{r}')\end{gathered}\tag{22}$$

III. Transformation of the Hamiltonian Function

The inclusion of a nonelectric potential energy represented, for a single particle, by the scalar function $\Phi(\mathbf{r})$ makes no difference whatever in Majorana's derivation of his Hamiltonian from his special variation principle. With this modification, his result is

$$\begin{aligned}H=\int U_M'(\mathbf{r})\mathcal{H}_M U_M(\mathbf{r})dt=\int U_M'(\mathbf{r})\\ \times\{i\hbar c(\boldsymbol{\alpha}_M\cdot\nabla)-\beta_M(mc^2+\Phi(\mathbf{r}))\}\,U_M(\mathbf{r})d\tau.\end{aligned}\tag{23}$$

Here U_M is a purely real four-component function, or, more strictly, a set of four self-adjoint operators which are functions of $\mathbf{r}$. $\mathcal{H}_M$ is the Dirac Hamiltonian for a single particle, expressed in terms of a specially chosen representation of the Dirac matrices, denoted by α_M, β_M, which satisfy

$$\boldsymbol{\alpha}_M^*=\boldsymbol{\alpha}_M'=\boldsymbol{\alpha}_M;\quad \beta_M^*=\beta_M'=-\beta_M. \tag{24}$$

For this representation, a suitable form of the operator A is

$$A_M=1, \tag{25}$$

as is seen by comparing (24) with (2). Then $P_M\psi_M$ is simply the real part of ψ_M—more strictly the "self-adjoint part"—and this is indeed what Majorana took as U_M.

Majorana's Hamiltonian (23) differs from the Hamiltonian of the ordinary Dirac theory,

$$\int \psi_M^\dagger(\mathbf{r})\mathcal{H}_M\psi_M(\mathbf{r})d\tau, \tag{23'}$$

only by the omission of the term

$$\int V_M'(\mathbf{r})\mathcal{H}_M V_M(\mathbf{r})d\tau, \tag{23''}$$

where $V_M=-iI_M\psi_M$, $\psi_M=U_M+iV_M$. The components of V_M all anticommute with all the components of U_M, so that any function of V_M commutes with the Hamiltonian (23). If terms of the types $i\hbar D_1$ and $i\hbar D_2$ defined in Eq. (18) were included in $\mathcal{H}_M$, the expression (23′) would contain not only the terms (23) and (23″), but also cross terms in U_M and V_M. Majorana's way of treating the Hamiltonian (23′) by separating ψ_M into the two terms U_M and iV_M then leads directly to a simple subtraction theory of the electron and positron.

We shall now show in detail how the Hamiltonian (23) forms the basis of a theory in which there is just one type of particle, with two states for a given momentum, corresponding to two different spin orientations. First we shall free ourselves from the restriction to the special representation $\boldsymbol{\alpha}_M$, β_M for the Dirac matrices.

By a well-known general theorem,[17] the representation $\boldsymbol{\alpha}_M$, β_M is connected with our represen-

[16] P. Jordan and E. Wigner, Zeits. f. Physik **47**, 631 (1928).

[17] Cf. reference 11, p. 109. The representations of the Dirac matrices used there are not subject to the restriction that they be Hermitian, so that the transformation matrix need not be unitary.

tation $\boldsymbol{\alpha}$, β by a unitary transformation:

$$\boldsymbol{\alpha}_M = T^\dagger \boldsymbol{\alpha} T, \quad \beta_M = T^\dagger \beta T, \quad T^\dagger T = 1. \tag{26}$$

These relations suggest that we should subject (23) to the substitution

$$U_M = T^\dagger U, \quad U_M' = U^\dagger T \tag{27}$$

and regard the result as the Hamiltonian expressed in terms of our function U which satisfies the commutation relations (22).

To justify this procedure we shall show that (27) establishes a one-to-one correspondence between the set of all possible quantities satisfying

$$PU = U, \qquad IU = 0 \tag{21'}$$

and all possible real four-component quanties U_M. This is most readily proved by noting that according to (24) and (26) all of the conditions (1)–(4) on A are satisfied by

$$A = T^* T^\dagger. \tag{28}$$

Then $IU=0$ means $U = A^* U^*$, and $T^\dagger U = T^\dagger A^* U^* = T' U^* = (T^\dagger U)^*$, so that if U satisfies (21′), $T^\dagger U$ is real. Conversely, if U_M is real, $A^*(TU_M)^* = TT'T^* U_M = TU_M$, so that TU_M satisfies (21′). The fact that the phase of A is arbitrary is exactly taken care of by the arbitrariness of the phase of T, which makes no difference in the final result of substituting (27) in (23).

Majorana establishes the Lorentz invariance of his formalism by relating it to the general results of applying the Jordan-Wigner method to the Dirac theory. That the invariance is not damaged by the substitution (27) follows from the fact that corresponding Lorentz transformation matrices in the two formalisms are connected by the similarity transformation (26):

$$S_M = T^\dagger S T, \quad \text{so that} \quad U_M = T^\dagger U$$

implies $\qquad S_M U_M = T^\dagger S U.$

On substituting (27) in (23) and using (26), we have

$$H = \int U^\dagger(\mathbf{r}) \{ i\hbar c(\boldsymbol{\alpha}\cdot\nabla) - \beta(mc^2 + \Phi(\mathbf{r})) \} U(\mathbf{r}) d\tau. \tag{29}$$

We now set

$$U(\mathbf{r}) = \Sigma(\mathbf{k})(\mathbf{r}|\mathbf{k})a(\mathbf{k}), \tag{30}$$

where

$$(\mathbf{r}|\mathbf{k}) = (\mathbf{k}|\mathbf{r})^* = V^{-\frac{1}{2}} \exp\{i(mc/\hbar)\mathbf{k}\cdot\mathbf{r}\} \tag{31}$$

and $a(\mathbf{k})$ is a four-component quantity; we suppose the allowed values of $\mathbf{k}$ made denumerable by imposition of the usual periodic boundary condition in a box of volume V, and the integral in (29) extended over this volume. Then

$$H = mc^2 \Sigma(\mathbf{k}) a^\dagger(\mathbf{k}) \{ -(\boldsymbol{\alpha}\cdot\mathbf{k}) - \beta \} a(\mathbf{k}) - \Sigma(\mathbf{k}, \mathbf{k}') a^\dagger(\mathbf{k})(\mathbf{k}|\Phi|\mathbf{k}')\beta a(\mathbf{k}'), \tag{32}$$

$$\text{where} \quad (\mathbf{k}|\Phi|\mathbf{k}') = \int_V (\mathbf{k}|\mathbf{r})\Phi(\mathbf{r})(\mathbf{r}|\mathbf{k}') d\tau. \tag{33}$$

Relations satisfied by $a_i(\mathbf{k})$ and $a_i^*(\mathbf{k})$. From (22) and (31) it follows that the commutation relations for these quantities are:

$$\begin{aligned} &a_i^*(\mathbf{k})a_j(\mathbf{k}') + a_j(\mathbf{k}')a_i^*(\mathbf{k}) = \tfrac{1}{2}\delta_{ij}\delta(\mathbf{k}', \mathbf{k}), \\ &a_i(\mathbf{k})a_j(\mathbf{k}') + a_j(\mathbf{k}')a_i(\mathbf{k}) = \tfrac{1}{2}A_{ij}^*\delta(-\mathbf{k}', \mathbf{k}), \\ &a_i^*(\mathbf{k})a_j^*(\mathbf{k}') + a_j^*(\mathbf{k}')a_i^*(\mathbf{k}) = \tfrac{1}{2}A_{ij}\delta(-\mathbf{k}', \mathbf{k}). \end{aligned} \tag{34}$$

By substituting (5) and (30) into (21′), multiplying by $(\mathbf{k}|\mathbf{r})$ and integrating, one finds the linear relations

$$\begin{aligned} a(\mathbf{k}) &= A^* a^*(-\mathbf{k}), \\ a^*(\mathbf{k}) &= A a(-\mathbf{k}). \end{aligned} \tag{35}$$

It is readily verified that (34) and (35) are consistent with each other.

We now make a linear substitution on the $a(\mathbf{k})$ and $a^*(\mathbf{k})$, in order to diagonalize the free-particle terms in (32). Set

$$\begin{aligned} a(\mathbf{k}) &= S(\mathbf{k})b(\mathbf{k}), \\ a^\dagger(\mathbf{k}) &= b^\dagger(\mathbf{k})S^\dagger(\mathbf{k}). \end{aligned} \tag{36}$$

Here $S(\mathbf{k})$ is required to satisfy the conditions

$$S^\dagger(\mathbf{k})S(\mathbf{k}) = 1, \tag{37}$$

$$S^\dagger(\mathbf{k})\{-(\boldsymbol{\alpha}\cdot\mathbf{k}) - \beta\}S(\mathbf{k}) = \epsilon\rho_3 = (1+k^2)^{\frac{1}{2}}\rho_3. \tag{38}$$

ρ_3 is one of the matrices originally defined by Dirac:[18]

$$\rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{39}$$

[18] P. A. M. Dirac, Proc. Roy. Soc. **A117**, 614 (1928).

the $\mathbf{0}$'s and $\mathbf{1}$'s being two-rowed matrices. Thus by (38) the free-particle terms are diagonalized, the four eigenvalues for a given $\mathbf{k}$ being ϵ, ϵ, $-\epsilon$, $-\epsilon$.

The existence of a matrix which satisfies (37) and (38) is most easily shown by exhibiting one, as we shall do later. Here we shall show that if $S_1(\mathbf{k})$ and $S_2(\mathbf{k})$ both satisfy (37) and (38), then

$$S_2(\mathbf{k})=S_1(\mathbf{k})\begin{pmatrix}\mathbf{u}_1(\mathbf{k}) & \mathbf{0}\\ \mathbf{0} & \mathbf{u}_2(\mathbf{k})\end{pmatrix}, \tag{40}$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are two-rowed unitary matrices. Proof: By (37) and (38),

$$S_1\rho_3S_1{}^\dagger=S_2\rho_3S_2{}^\dagger=\epsilon^{-1}\{-(\boldsymbol{\alpha}\cdot\mathbf{k})-\beta\}.$$

Then $$\rho_3S_1{}^\dagger S_2=S_1{}^\dagger S_2\rho_3$$

and by an elementary result in matrix theory[19] it follows from (39) that

$$S_1{}^\dagger S_2=\begin{pmatrix}\mathbf{u}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{u}_2\end{pmatrix}.$$

The unitary character of $\mathbf{u}_1$ and $\mathbf{u}_2$, and (40) itself, then follow from (37).

The result of substituting (36) in (32) is

$$H=\Sigma(\mathbf{k})\epsilon mc^2b^\dagger(\mathbf{k})\rho_3b(\mathbf{k})$$
$$-\Sigma(\mathbf{k},\mathbf{k}')b^\dagger(\mathbf{k})S^\dagger(\mathbf{k})(\mathbf{k}|\Phi|\mathbf{k}')\beta S(\mathbf{k}')b(\mathbf{k}'). \tag{41}$$

Relations satisfied by $b_i(\mathbf{k})$ and $b_i{}^*(\mathbf{k})$. From (34) and (36) we obtain the commutation relations

$$b_i{}^*(\mathbf{k})b_j(\mathbf{k}')+b_j(\mathbf{k}')b_i{}^*(\mathbf{k})=\tfrac{1}{2}\delta_{ij}\delta(\mathbf{k}',\mathbf{k}),$$
$$b_i(\mathbf{k})b_j(\mathbf{k}')+b_j(\mathbf{k}')b_i(\mathbf{k})=\tfrac{1}{2}T_{ij}{}^*(\mathbf{k})\delta(-\mathbf{k}',\mathbf{k}), \tag{42}$$
$$b_i{}^*(\mathbf{k})b_j{}^*(\mathbf{k}')+b_j{}^*(\mathbf{k}')b_i{}^*(\mathbf{k})=\tfrac{1}{2}T_{ij}(\mathbf{k})\delta(-\mathbf{k}',\mathbf{k}),$$

where $$T(\mathbf{k})=S'(\mathbf{k})AS(-\mathbf{k}). \tag{43}$$

From (35) and (36) we get the linear relations

$$\begin{aligned}b(\mathbf{k})&=T^*(\mathbf{k})b^*(-\mathbf{k}),\\ b^*(\mathbf{k})&=T(\mathbf{k})b(-\mathbf{k}),\end{aligned} \tag{44}$$

(42) and (44) are of course consistent.

The elimination from (41) of any reference to the negative energy states is made possible by the nature of the relations (44). We shall show that

$$T(\mathbf{k})=\begin{pmatrix}\mathbf{0} & \mathbf{u}(\mathbf{k})\\ \mathbf{v}(\mathbf{k}) & \mathbf{0}\end{pmatrix}, \tag{45}$$

where $\mathbf{u}(\mathbf{k})$ and $\mathbf{v}(\mathbf{k})$ are two-rowed unitary matrices. The proof is similar to that of (40). By (38) we have, since $\boldsymbol{\alpha}$, β are Hermitian,

$$S(-\mathbf{k})\rho_3S^\dagger(-\mathbf{k})=\epsilon^{-1}\{(\boldsymbol{\alpha}\cdot\mathbf{k})-\beta\} \tag{46}$$

and $$S^*(\mathbf{k})\rho_3S'(\mathbf{k})=\epsilon^{-1}\{-(\boldsymbol{\alpha}'\cdot\mathbf{k})-\beta'\}. \tag{47}$$

Now by (2), if we multiply (46) by A from the left and (47) by A from the right, the right-hand members become equal and opposite. Thus we have

$$AS(-\mathbf{k})\rho_3S^\dagger(-\mathbf{k})=-S^*(\mathbf{k})\rho_3S'(\mathbf{k})A.$$

On multiplying this from the left by $S'(\mathbf{k})$ and from the right by $S(-\mathbf{k})$, we have

$$T(\mathbf{k})\rho_3=-\rho_3T(\mathbf{k}) \tag{48}$$

and it is readily seen from (39) that a unitary matrix which anticommutes with ρ_3 must be of the form (45).

Accordingly we have

$$b(\mathbf{k})=\begin{pmatrix}\mathbf{0} & \mathbf{u}^*(\mathbf{k})\\ \mathbf{v}^*(\mathbf{k}) & \mathbf{0}\end{pmatrix}b^*(-\mathbf{k}). \tag{49}$$

In order to eliminate $b_3(\mathbf{k})$ and $b_4(\mathbf{k})$ from H we want to use for the one-column matrix $b(\mathbf{k})$ the expression

$$b(\mathbf{k})=\begin{bmatrix}b_1(\mathbf{k})\\ b_2(\mathbf{k})\\ v_{11}{}^*b_1{}^*(-\mathbf{k})+v_{12}{}^*b_2{}^*(-\mathbf{k})\\ v_{21}{}^*b_1{}^*(-\mathbf{k})+v_{22}{}^*b_2{}^*(-\mathbf{k})\end{bmatrix}. \tag{50}$$

The actual use of the expression (50) and of an analogous expression for the one-row matrix $b^\dagger(\mathbf{k})$ is not at all difficult, but involves a considerable amount of writing. A simple way to get the same result is to make the substitutions

$$\begin{aligned}b(\mathbf{k})&\rightarrow b(\mathbf{k})+T^*(\mathbf{k})b^*(-\mathbf{k}),\\ b^\dagger(\mathbf{k})&\rightarrow b^\dagger(\mathbf{k})+b'(-\mathbf{k})T'(\mathbf{k})\end{aligned} \tag{51}$$

[19] Cf. E. Wigner, *Gruppentheorie*, p. 10.

in (41), and then discard all terms involving $b_3(\mathbf{k})$, $b_4(\mathbf{k})$, $b_3^*(\mathbf{k})$ and $b_4^*(\mathbf{k})$. The result is

$$\begin{aligned} H=&\tfrac{1}{2}\Sigma(\mathbf{k})\epsilon mc^2\{B^\dagger(\mathbf{k})[\rho_3]^0B(\mathbf{k})\\ &+B^\dagger(\mathbf{k})[\rho_3T^*(\mathbf{k})]^0B^*(-\mathbf{k})\\ &+B'(-\mathbf{k})[T'(\mathbf{k})\rho_3]^0B(\mathbf{k})\\ &+B'(-\mathbf{k})[T'(\mathbf{k})\rho_3T^*(\mathbf{k})]^0B^*(-\mathbf{k})\}\\ &-\tfrac{1}{2}\Sigma(\mathbf{k},\mathbf{k}')(\mathbf{k}|\Phi|\mathbf{k}')\{B^\dagger(\mathbf{k})[S^\dagger(\mathbf{k})\beta S(\mathbf{k}')]^0B(\mathbf{k}')\\ &+B^\dagger(\mathbf{k})[S^\dagger(\mathbf{k})\beta S(\mathbf{k}')T^*(\mathbf{k}')]^0B^*(-\mathbf{k}')\\ &+B'(-\mathbf{k})[T'(\mathbf{k})S^\dagger(\mathbf{k})\beta S(\mathbf{k}')]^0B(\mathbf{k}')\\ &+B'(-\mathbf{k})[T'(\mathbf{k})S^\dagger(\mathbf{k})\beta S(\mathbf{k}')T^*(\mathbf{k}')]^0\\ &\times B^*(-\mathbf{k}')\}, \end{aligned} \tag{52}$$

where $[M]^0$ means the two-rowed matrix obtained from the four-rowed matrix M by taking the upper left hand quadrant, and

$$\begin{aligned} B_1(\mathbf{k})&=\sqrt{2}b_1(\mathbf{k}), & B_2(\mathbf{k})&=\sqrt{2}b_2(\mathbf{k}),\\ B_1^*(\mathbf{k})&=\sqrt{2}b_1^*(\mathbf{k}), & B_2^*(\mathbf{k})&=\sqrt{2}b_2^*(\mathbf{k}). \end{aligned} \tag{53}$$

It is important to note that the subscript on B or B^* can take only two values instead of four.

The expression (52) can be simplified considerably by means of the relations (48), (45), (43), (39), (4), (3), and (2). One obtains:

$$\begin{aligned} H=&\tfrac{1}{2}\Sigma(\mathbf{k})\epsilon mc^2\{B^\dagger(\mathbf{k})B(\mathbf{k})-B'(\mathbf{k})B^*(\mathbf{k})\}\\ &-\tfrac{1}{2}\Sigma(\mathbf{k},\mathbf{k}')(\mathbf{k}|\Phi|\mathbf{k}')\{B^\dagger(\mathbf{k})[S^\dagger(\mathbf{k})\beta S(\mathbf{k}')]^0B(\mathbf{k}')\\ &+B^\dagger(\mathbf{k})[S^\dagger(\mathbf{k})\beta A^*S^*(-\mathbf{k}')]^0B^*(-\mathbf{k}')\\ &+B'(-\mathbf{k})[S'(-\mathbf{k})A\beta S(\mathbf{k}')]^0B(\mathbf{k}')\\ &-B'(-\mathbf{k})[S'(-\mathbf{k})\beta'S^*(-\mathbf{k}')]^0B^*(-\mathbf{k}')\}. \end{aligned} \tag{54}$$

Now from (53), (42), and (45) we find that the commutation relations on the B's and B^*'s are

$$\begin{aligned} B_i^*(\mathbf{k})B_j(\mathbf{k}')+B_j(\mathbf{k}')B_i^*(\mathbf{k})&=\delta_{ij}\delta(\mathbf{k}',\mathbf{k}),\\ B_i(\mathbf{k})B_j(\mathbf{k}')+B_j(\mathbf{k}')B_i(\mathbf{k})&=0,\\ B_i^*(\mathbf{k})B_j^*(\mathbf{k}')+B_j^*(\mathbf{k}')B_i^*(\mathbf{k})&=0, \end{aligned} \tag{55}$$

which are just the standard Jordan-Wigner relations. On using (55) in (54) and remembering that by (31) and (33) $(-\mathbf{k}|\Phi|-\mathbf{k}')=(\mathbf{k}'|\Phi|\mathbf{k})$, we get

$$\begin{aligned} H=&\Sigma(\mathbf{k})\epsilon mc^2\{\sum_{r=1}^{2}N_r(\mathbf{k})-1\}\\ &-\Sigma(\mathbf{k},\mathbf{k}')(\mathbf{k}|\Phi|\mathbf{k}')\{B^\dagger(\mathbf{k})[S^\dagger(\mathbf{k})\beta S(\mathbf{k}')]^0B(\mathbf{k}')\\ &-\tfrac{1}{2}\delta(\mathbf{k},\mathbf{k}')\sum_{r=1}^{2}[S^\dagger(\mathbf{k})\beta S(\mathbf{k})]_{rr}{}^0\}\\ &-\tfrac{1}{2}\Sigma(\mathbf{k},\mathbf{k}')\{(\mathbf{k}|\Phi|-\mathbf{k}')B^\dagger(\mathbf{k})\\ &\times[S^\dagger(\mathbf{k})\beta A^*S^*(\mathbf{k}')]^0B^*(\mathbf{k}')+(-\mathbf{k}|\Phi|\mathbf{k}')B'(\mathbf{k})\\ &\times[S'(\mathbf{k})A\beta S(\mathbf{k}')]^0B(\mathbf{k}')\}, \end{aligned} \tag{56}$$

where
$$N_r(\mathbf{k})=B_r^*(\mathbf{k})B_r(\mathbf{k}). \tag{57}$$

We now wish to give (56) a more explicit form by introducing special forms of the matrices β, A, and $S(\mathbf{k})$. Before doing so let us establish conclusively that the results are essentially independent of the choice of these forms. We shall prove that any change in this choice is equivalent to at most a two-rowed unitary transformation

$$\begin{aligned} B(\mathbf{k})&\to t(\mathbf{k})B(\mathbf{k}), & B'(\mathbf{k})&\to B'(\mathbf{k})t'(\mathbf{k}),\\ B^*(\mathbf{k})&\to t^*(\mathbf{k})B^*(\mathbf{k}), & B^\dagger(\mathbf{k})&\to B^\dagger(\mathbf{k})t^\dagger(\mathbf{k}). \end{aligned} \tag{58}$$

In the first place, it is at once evident from (40) that a change in $S(\mathbf{k})$ for fixed $\boldsymbol{\alpha}$, β is equivalent to a transformation of the type (58). Also, for fixed $\boldsymbol{\alpha}$, β the only change we can make in A is a phase factor, which corresponds to a transformation of the type (58) with $t(\mathbf{k})$ a mere phase factor. These results permit us to discuss the case in which the representation of $\boldsymbol{\alpha}$, β is changed by prescribing particular corresponding changes of $S(\mathbf{k})$ and A. From (37), (38), and (1), (2), (3) it is evident that if the new Dirac matrices are

$$\bar{\alpha}_i=R^\dagger\alpha_iR,\quad \bar{\beta}=R^\dagger\beta R,\quad R^\dagger R=1,$$

one may use with them the matrices

$$\bar{S}(\mathbf{k})=R^\dagger S(\mathbf{k}),\quad \bar{A}=R'AR.$$

On substituting in (56) one finds that all the matrices $[M]^0$ which occur in that equation have the same value whether α_i, β, $S(\mathbf{k})$, A or $\bar{\alpha}_i$, $\bar{\beta}$, $\bar{S}(\mathbf{k})$, $\bar{A}$ are used.

We now fix the Dirac matrices in the form originally used by Dirac:[18]

$$\alpha_i=\begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ \boldsymbol{\sigma}_i & 0 \end{pmatrix}, \quad \beta=\rho_3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{59}$$

where $\mathbf{0}$, $\mathbf{1}$, and $\boldsymbol{\sigma}_i$ are two-rowed matrices, the last being the Pauli matrices. Since σ_x and σ_z are symmetric and σ_y antisymmetric, we can satisfy (1), (2), (3) by setting

$$A=i\alpha_y\beta. \tag{60}$$

It is readily verified that (37) and (38) are satisfied by

$$S(\mathbf{k})=C(k)\{(\epsilon+1)\rho_1-\rho_3(\boldsymbol{\sigma}\cdot\mathbf{k})\}, \tag{61}$$

where

$$C(k)=\{2\epsilon(\epsilon+1)\}^{-\frac{1}{2}},$$
$$\rho_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_i=\begin{pmatrix} \boldsymbol{\sigma}_i & 0 \\ 0 & \boldsymbol{\sigma}_i \end{pmatrix} \tag{62}$$

On substituting (59)–(62) into (56) one readily obtains

$$\begin{aligned} H=\Sigma(\mathbf{k})\epsilon mc^2\{\sum_{r=1}^{2} N_r(\mathbf{k})-1\}-(0|\Phi|0)\Sigma(\mathbf{k})\epsilon^{-1} \\ +\Sigma(\mathbf{k},\mathbf{k}')B^\dagger(\mathbf{k})(\mathbf{k}|V|\mathbf{k}')B(\mathbf{k}') \\ +\Sigma(\mathbf{k},\mathbf{k}')B^\dagger(\mathbf{k})(\mathbf{k},\mathbf{k}'|C|)B^*(\mathbf{k}') \\ +\Sigma(\mathbf{k},\mathbf{k}')B'(\mathbf{k})(|D|\mathbf{k},\mathbf{k}')B(\mathbf{k}'), \end{aligned} \tag{63}$$

$$\left.\begin{aligned} (\mathbf{k}|V|\mathbf{k}')=(\mathbf{k}|\Phi|\mathbf{k}')\{(\epsilon+1)(\epsilon'+1) \\ -(\boldsymbol{\sigma}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\mathbf{k}')\}\{4\epsilon\epsilon'(\epsilon+1)(\epsilon'+1)\}^{-\frac{1}{2}} \\ (\mathbf{k},\mathbf{k}'|C|)=(\mathbf{k}|\Phi|-\mathbf{k}')\{(\epsilon+1)(\boldsymbol{\sigma}\cdot\mathbf{k}') \\ -(\epsilon'+1)(\boldsymbol{\sigma}\cdot\mathbf{k})\}\{16\epsilon\epsilon'(\epsilon+1)(\epsilon'+1)\}^{-\frac{1}{2}}\cdot i\sigma_y \\ (|D|\mathbf{k},\mathbf{k}')=(-\mathbf{k}|\Phi|\mathbf{k}')\cdot i\sigma_y\{(\epsilon+1)(\boldsymbol{\sigma}\cdot\mathbf{k}') \\ -(\epsilon'+1)(\boldsymbol{\sigma}\cdot\mathbf{k})\}\{16\epsilon\epsilon'(\epsilon+1)(\epsilon'+1)\}^{-\frac{1}{2}}. \end{aligned}\right\} \tag{64}$$

The general significance of (63) is clear, since according to (55) and (57) the meanings of $B_r(\mathbf{k})$ and $B_r^*(\mathbf{k})$ can be taken to be those usually given to Jordan-Wigner operators: $B_r(\mathbf{k})$ corresponds to the transition of a particle *out of* the state characterized by the momentum $\mathbf{k}mc$ and the spin index r, and $B_r^*(\mathbf{k})$ corresponds to a transition *into* that state. Thus the terms in $(\mathbf{k}|V|\mathbf{k}')$ correspond to *transitions* between such states, those in $(\mathbf{k},\mathbf{k}'|C|)$ to the *creation* of pairs of particles, and those in $(|D|\mathbf{k},\mathbf{k}')$ to *destruction* of pairs. Further remarks about formulas (63) and (64) are given in the next section.

IV. Discussion

Although the above procedure based on the ideas of Majorana succeeds formally in eliminating the negative energy states without the introduction of corresponding "holes" or anti-particles, the results still show much more resemblance to the positron theory than to a theory based on Schroedinger's early suggestions. This resemblance appears not only in the presence in (63) of terms corresponding to the creation and destruction of pairs of particles, but also in the presence of infinite terms independent of the $B(\mathbf{k})$ and $B^*(\mathbf{k})$. These terms, namely

$$-\Sigma(\mathbf{k})\epsilon mc^2-(0|\Phi|0)\Sigma(\mathbf{k})\epsilon^{-1}$$

can play no part in calculations, and hence may be regarded as typical "subtraction terms," such as appear in the positron theory.

As is the case with much of the formalism of quantum electrodynamics and of positron theory, our final expressions are in a form highly unsymmetrical as regards space and time, so that direct investigation of their Lorentz invariance would be a complicated task. It is scarcely to be doubted that, just as in the positron theory, the Lorentz invariance of the final formulas has become illusory through divergence difficulties associated with the presence of the subtraction terms. It seems, however, not without interest to mention here the question of ordinary rotation invariance, since the unsymmetrical appearance of the expressions (64) might lead one at first glance to suppose that even this invariance is lacking. We can readily show that this is not so, and that the expressions (64) have exactly the sort of form required for the theory to be rotation invariant.

The theory of the spinning electron provides us with the following rule for transforming the two components $B_1(\mathbf{k})$ and $B_2(\mathbf{k})$ on rotation of axes: Corresponding to rotation through the angle θ around an axis in the direction $\mathbf{n}_1$, one subjects the B_r to the unitary transformation

$$B \to u(\mathbf{n}_1;\,\theta)B,$$
$$u(\mathbf{n}_1;\,\theta) = \cos(\theta/2) + i(\boldsymbol{\sigma}\cdot\mathbf{n}_1)\sin(\theta/2) \quad (65)$$

and also

$$B' \to B'u'(\mathbf{n}_1;\,\theta), \quad B^* \to u^*(\mathbf{n}_1;\,\theta)B^*, \quad (65')$$
$$B^\dagger \to B^\dagger u^\dagger(\mathbf{n}_1;\,\theta).$$

Then if the matrix M involves σ only in products $(\boldsymbol{\sigma}\cdot\mathbf{A})$, the product $\bar{B}^\dagger MB$ is invariant under rotation of $\mathbf{A}$ according to ordinary geometry and transformation of B and $\bar{B}^\dagger$ by (65) and (65′). This is the familiar way of establishing the invariance of the "even" terms involving $(\mathbf{k}|V|\mathbf{k}')$. But since $(\mathbf{k},\,\mathbf{k}'|C|)$ and $(|D|\mathbf{k},\,\mathbf{k}')$ are of the forms $M\sigma_y$ and $\sigma_y M$, respectively, we find that $\bar{B}^\dagger CB^*$ and $\bar{B}'DB$ are invariant under these same rules of transformation, because

$$\sigma_y u^*(\mathbf{n}_1;\,\theta) = u(\mathbf{n}_1;\,\theta)\sigma_y;$$
$$u'(\mathbf{n}_1;\,\theta)\sigma_y = \sigma_y u^\dagger(\mathbf{n}_1;\,\theta). \quad (66)$$

These relations follow directly from the commutation rules of the σ's together with the fact that σ_y is imaginary antisymmetric and σ_x and σ_z are real symmetric. The unsymmetrical looking appearance of σ_y in (64) is thus precisely what is required to secure the actual rotation invariance of the formalism. It is evident that this factor should not be regarded as a spin matrix at all, but simply as the antisymmetric unit matrix which plays a part in spinor analysis.[20]

It is evident that the formalism here obtained cannot be regarded as equivalent to a definite wave equation for a single particle, both because of the presence of the "odd" terms, which show that the number of particles is not constant, and because of the highly irrational dependence on $\mathbf{k}$, which would be represented by the operator $(-i\hbar/mc)\nabla$. Apparently the only case in which an approximate treatment by means of a single particle wave equation can readily be given is that of a slowly moving particle in a weak field free of high frequency components. For such a field pairs will not be produced, and the only contributions to the energy from the odd terms will be of the order of $|(\mathbf{k}|\Phi|\mathbf{k}')|^2$, which, since the field is weak, can be neglected compared to the even terms. Also the irrationalities can be removed by expansion in powers of k and k', since for a slow particle these are small of order v/c; one may conveniently work to the order k^2. In this approximation the even matrix elements contained in (63) are, apart from the "subtraction terms" equivalent by partial integration to those of

$$H_1 = mc^2 - (\hbar^2/2m)\nabla^2 + \Phi - (\hbar^2/8m^2c^2)(\nabla^2\Phi) + (\hbar\boldsymbol{\sigma}/4c^2)\cdot[(\nabla\Phi/m)\times(\mathbf{p}/m)]. \quad (67)$$

Here the first three terms are, respectively, proper energy, nonrelativistic kinetic energy, and potential energy, and the last term gives precisely the spin orbit coupling corresponding to the Thomas precession.[21] Just as in the case of the nonrelativistic approximation obtained from the usual Dirac equations for a single particle there appears still another term, which in the present instance is Hermitian.

The matrix elements occurring in (64) are just those which appear in the simple subtraction theory in which both particles and antiparticles occur. The physical interpretation is, however, rather different; the events of creation and annihilation may here involve any two particles which occur at all in the theory, instead of having to involve one particle from each of two distinct kinds.

As was mentioned in the introduction, the neutrino theory of β-radioactivity either with electron emission or with positron emission can be based on this theory of the neutrino quite as well as on the usual formalism in which antineutrinos occur. As Racah[2] has pointed out, the procedure is, schematically, just to replace the neutrino wave function φ by $\frac{1}{2}(\varphi + A^*\varphi^*)$ and replace φ^* by $\frac{1}{2}(\varphi^* + A\varphi)$ throughout whatever formula for the interaction energy it is desired to use. In our present language this means replacing φ by U and φ^* by U^*, after which one may use the formulas of Section III to express the interaction in terms of the operators $B_r^*(\mathbf{k})$ and $B_r(\mathbf{k})(r=1,\,2)$, which refer to the emission and

[20] B. L. van der Waerden, *Gruppentheoretische Methode in der Quantenmechanik*, p. 79; O. Laporte and G. E. Uhlenbeck, Phys. Rev. **37**, 1383 (1931).

[21] Cf. references 4 and 3.

absorption of neutrinos. Using (30), (36), (50) and (53), we have

$$\varphi \rightarrow U = 2^{-\frac{1}{2}} \Sigma(\mathbf{k})(\mathbf{r}|\mathbf{k}) S(\mathbf{k}) \cdot \begin{Bmatrix} B_1(\mathbf{k}) \\ B_2(\mathbf{k}) \\ v_{11}^* B_1^*(-\mathbf{k}) + v_{12}^* B_2^*(-\mathbf{k}) \\ v_{21}^* B_1^*(-\mathbf{k}) + v_{22}^* B_2^*(-\mathbf{k}) \end{Bmatrix}.$$

This differs from the corresponding expression for φ used in the ordinary theory only in the following ways: (a) A factor $2^{-\frac{1}{2}}$ occurs, which can be absorbed into the disposable constant coefficient of the interaction energy; (b) A spin transformation $\mathbf{v}^*$ appears, but this makes no difference in the total transition probabilities summed over both spins for the neutrino. (c) The role which in the ordinary theory is played by operators representing the emission of an antineutrino is here played by the B_r^*, which represent the emission of a neutrino. The actual four-component amplitudes used in calculating matrix elements are the same in both cases, the columns of $S(\mathbf{k})$. Thus in all cases in which the only part taken by the light neutral particles consists in one of them being emitted, there is no difference in the calculated intensities,[22] and the only difference between the two theories consists in our mental picture of what happens: In the ordinary theory one type of β-decay involves the emission of neutrinos and the other the emission of antineutrinos, but in the Majorana theory use is made of neutrinos only.

It should be possible to settle which theory is preferable by considering processes in which neutrinos are absorbed as well as emitted, but actually this does not seem feasible at present. Differences would presumably appear in the results of using the light particle fields to account for the forces between heavy particles, but this part of the subject is in such an unsatisfactory state owing to divergence difficulties that it seems to offer no hope of a decision, and indeed it seems quite doubtful that nuclear forces are to be explained in this way. Another possibility of deciding between the two theories is offered in principle by the phenomenon of β-decay with *absorption of* a light neutral particle instead of its *emission*, the β-ray accordingly having *more* energy than the limit of the spectrum instead of *less*. Here, as Racah has remarked, there is an obvious qualitative difference between the two theories. On the ordinary Dirac theory, a positron emitter can be "stimulated" only by an electron emitter, and vice versa, but on the Majorana theory any emission may "stimulate" any other emission, whether of the same or of opposite type. But since the cross section[23] of a radioactive nucleus for capture of a neutrino is of order of magnitude between 10^{-40} and 10^{-50} cm^2, it seems unlikely that this effect, which would not only serve to decide the question of the existence of antineutrinos but would provide experimental evidence of the best sort for the neutrino hypothesis itself, can ever be observed.

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[22] This assertion may seem to contradict a statement made by Racah (reference 2), that differences can be obtained unless the rest mass of the neutrino is zero. Our present assertion holds, however, for the comparison between the Majorana theory and any neutrino theory in which the interaction energy involves the electron wave function ψ and the neutrino function φ through expressions of the form $\psi^\dagger M\varphi$ and $\varphi^\dagger M^\dagger\psi$. For such theories the phases of the Lorentz transformation matrices (Eq. (12)) are arbitrary, but the way in which they are chosen has no effect on the form of the quantities $\psi^\dagger M\varphi$ and $\varphi^\dagger M^\dagger\psi$ used in expressing the interaction energy; so that the phases may be taken to be those given in Eq. (12), and our arguments about the connection with the Majorana theory are at once applicable. This way of forming the interaction energy was used by Uhlenbeck and Konopinski, and is now the most commonly used. Fermi's original way of forming the interaction energy, however, involved expressions of the form $\psi^\dagger M\varphi^*$ and $\varphi' M^\dagger\psi$, and here the phases of the transformation matrices are not without effect on the way one must choose the form of the expression for the interaction energy in order to obtain Lorentz invariance. As Racah points out, the phases given in (12) are not consistent with Fermi's actual choice of the interaction energy expression, so that his original formalism could not be carried over into the Majorana theory. This lack of exact correspondence between the two points of view is to be regarded as due to objectionably artificial characteristics of the Fermi type of expression, which are avoided in the now generally accepted Konopinski-Uhlenbeck type.

[23] H. Bethe and R. Peierls, Nature **133**, 532 (1934).