Coulomb Energy of Light Nuclei

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The Coulomb energy is calculated by determining the nuclear radius so that the energy difference between N^{13} and C^{13} agrees with observation. Two different assumptions are investigated: (a) that the nuclear radius is simply proportional to the cube root of the number of particles, (b) that the wave function of the last neutron or proton extends beyond the surface of the residual nucleus by an amount determined by its binding energy. The latter assumption accounts very well for the irregularities of the positron energies in the series $C^{11}N^{13}O^{15}F^{17}$, and agrees quantitatively with observation within 0.15 milli-mass-units. Theoretically and experimentally, the actual Coulomb energy should lie between the results of (*a*) and (*b*). Application of these considerations to unknown nuclei shows that C^{10} and O^{14} are highly stable, Be^6 certainly and B^9 almost certainly unstable, while the stability of B^8 and N^{12} is doubtful.

IN connection with astrophysical considerations, it became desirable to decide about the stability of unknown nuclei such as Be^6, B^8, B^9, C^{10}, N^{12}, etc. The situation is favorable inasmuch as the "images" of all these nuclei are known, i.e., those nuclei which are obtained by interchanging neutrons and protons (He^6, Li^8, Be^9, Be^{10}, B^{12}). It is therefore only necessary to calculate the Coulomb energy which can be done fairly accurately even though an adequate theory of nuclear forces is still lacking. It is only necessary to estimate the nuclear radius, in particular for nuclei whose last neutron or proton is very loosely bound. In the following, simple formulae will be derived for this purpose which give, with one adjustable parameter, surprisingly good results for the Colomb energy of known isobaric pairs.

We consider a nucleus of charge $Z+1$ and radius R which consists of a "last proton" and a "residual nucleus" of charge Z. If the charges of the proton and the residual nucleus are both distributed uniformly over the sphere of radius R, the Coulomb energy is

$$C=6Ze^2/5R. \tag{1}$$

For a crude approximation, R may be assumed to be proportional to the cube root of the mass number A of the nucleus, *viz.*

$$R=r_0A^{\frac{1}{3}}. \tag{2}$$

In order to get as close an approximation to the empirical values as possible, we use the *observed* Coulomb energy for the isobaric pair $N^{13}-C^{13}$ to determine r_0. The difference between the binding energies of the last neutron in C^{13} and the last proton in N^{13} is equal to the difference between the energy evolutions in the reactions $C^{12}+H^2=C^{13}+H^1$ and $C^{12}+H^2=N^{13}+n^1$ which are 2.71 and -0.28 Mev, respectively.[1] Therefore we have for $N^{13}-C^{13}$:

$$C=2.99 \text{ Mev}=3.21 \text{ mMU}. \tag{2a}$$

This gives

$$C=1.26ZA^{-\frac{1}{3}} \text{ mMU}, \tag{3}$$
$$R=1.47\cdot10^{-13}A^{\frac{1}{3}} \text{ cm}. \tag{4}$$

The energy is given here and in the following in milli-mass-units (mMU, equal to 1/16,000 of the mass of O^{16}, 1 mMU$=0.931$ Mev$=1.83\ mc^2$). The radius is seen to extrapolate, for heavy nuclei, to Gamow's "small" radii.

The result of (3) is given in Table I under the heading "Calc. I." The agreement with the observed Coulomb energies is seen to be fair. However, (3) gives definitely too low values for the pairs $C^{11}-B^{11}$ and $O^{15}-N^{15}$. This is not surprising because (3) varies smoothly with Z whereas the observed values of the Coulomb energy do not. They are "high" for C^{11} and O^{15} and "low" for N^{13} and F^{17}.

In order to obtain such individual fluctuations of C, it appears that we must take into account the larger size of loosely bound nuclei. For a crude approximation, we consider a nucleus as composed of "the last neutron" (or proton) and "the residual nucleus of radius R_0" and assume

[1] Livingston and Bethe, Rev. Mod. Phys. 9, 245 (1937), Table LXXII, p. 371.

that the "last particle" is completely free outside of R_0. This assumption amounts to a one-body model which may be justifiable when the residual nucleus does not possess many low excited levels (light nucleus). We should expect that our procedure gives somewhat too large radii and therefore too small Coulomb energies for nuclei with a loosely bound last particle while formula (3) will give too large Coulomb energies.

Since we assume the force on the last neutron to be exactly zero outside of R_0, the neutron wave function outside will be

$$\psi=(\alpha/r)e^{-(r-R_0)/2b} \tag{5}$$

with $$b=\hbar/2(2M\epsilon)^{\frac{1}{2}}, \tag{5a}$$

M is the neutron mass, ϵ is the binding energy of the last neutron, and α is a normalizing factor. Inside the nucleus, we assume ψ to be constant, *viz.*

$$\psi=\alpha/R_0. \tag{6}$$

Then the normalization condition is

$$\int\psi^2r^2dr=1=\alpha^2(\tfrac{1}{3}R_0+b). \tag{7}$$

If we assume a uniform charge distribution in the residual nucleus, and the same wave function for a "last proton" as for a neutron, the Coulomb energy becomes

$$\begin{aligned}C&=Ze^2\left[\int_0^{R_0}\left(\frac{3}{2R_0}-\frac{r^2}{2R_0^3}\right)\psi^2r^2dr+\int_{R_0}^{\infty}\frac{1}{r}\psi^2r^2dr\right]\\&=Ze^2\alpha^2\left(\frac{2}{5}+\int_{R_0}^{\infty}e^{-(r-R)/b}\frac{dr}{r}\right)\\&\approx Ze^2\alpha^2\left(\frac{2}{5}+\frac{b}{R_0+b}\right).\end{aligned} \tag{8}$$

By use of (7), this reduces to

$$C=\frac{6}{5}\frac{Ze^2}{R_0+b}\left(1+\frac{b}{2(R_0+3b)}\right). \tag{9}$$

The parenthesis gives only a few percent correction.

We now assume that R_0 is proportional to $A_0^{\frac{1}{3}}$, where $A_0=A-1$ is the mass number of the *residual* nucleus. To fix the coefficient, we use again the Coulomb energy of $N^{13}-C^{13}$ (3.21 mMU); we have in this case

$\epsilon=5.3$ mMU (reference 1, p. 378),
$b=1.03\cdot10^{-13}$ cm (Eq. (5a)),
$R_0=2.74\cdot10^{-13}$ cm (Eq. (9), with $Z=6$),
$r_0=1.20\cdot10^{-13}$ cm (Eq. (2), with $A_0=12$).

Inserting this value of r_0 into (9), we find

$$C=\frac{1.54Z}{A_0^{\frac{1}{3}}+(3.9/\epsilon)^{\frac{1}{2}}}\left[1+\tfrac{1}{2}\cdot\frac{(3.9/\epsilon)^{\frac{1}{2}}}{A_0^{\frac{1}{3}}+(3.9/\epsilon)^{\frac{1}{2}}}\right], \tag{10}$$

TABLE I. *Coulomb energy of light nuclei.*

Nucl. (Neutr.)	ϵ_N mMU	$A_0^{1/3}$	$(3.9/\epsilon)^{1/2}$	Z	C (mMU) Calc. I	Calc. II	Obs.	Nucl. (Prot.)	ϵ_P mMU
H^3	6.6_5	1.26	0.77	1	0.87	0.84	0.74	He^3	5.9
He^6	0.5*	1.71	2.79	2+3	3.5	1.9_5	—	Be^6	−1.3 to −0.5*
Li^7	7.6_5	1.82	0.71	3	2.0	1.99	2.02**	Be^7	5.6
Li^8	2	1.91	1.40	3+4	4.4	3.6	—	B^8	−0.3 to +0.5
Be^9	1.8_5	2.00	1.45	4	2.4	2.0	—	B^9	−0.7 to −0.3
Be^{10}	7.3	2.08	0.73	4+5	5.2	5.4	—	C^{10}	1.9*
B^{11}	12.3_5	2.15	0.56	5	2.7_5	3.0_5	3.1	C^{11}	9.3
B^{12}	>2	2.22	1.40	5+6	6.0_5	5.2	—	N^{12}	−0.9 to −0.1
	<3.3	2.22	1.09	5+6	6.0_5	5.6	—	N^{12}	0.4 to 0.8
C^{13}	5.3	2.29	0.86	6	3.21	3.21	3.21	N^{13}	2.1
C^{14}	8.9	2.35	0.66	6+7	6.8	7.1_5	—	O^{14}	5.1
N^{15}	11.6	2.41	0.58	7	3.5_5	3.8_5	3.7	O^{15}	7.9
O^{17}	4.5	2.52	0.93	8	3.9	3.9_5	3.9	F^{17}	0.6

* Binding energy of each of the *last two* neutrons or protons.
** Calculated from the observed half-life (43 days), (Roberts, Heydenburg and Locher, Phys. Rev. **53**, 1016 (1938)) on the assumption that a positron is emitted in the transition $Be^7\rightarrow Li^7$, and with the use of the Fermi theory. The experiments of Roberts, Heydenburg and Locher indicate, however, that the transition is due to K electron capture and that there is less than 1 positron in 20 K captures. This would give an upper limit of 1.94 mMU for the Coulomb energy while the actual value cannot be determined at present.

where both ϵ and C are measured in milli-mass-units.

In Table I, column "Calc. II" the Coulomb energies are calculated according to (10) for a number of light isobaric pairs. The first column of Table I gives that member of the isobaric pair which contains more neutrons; it is, in all cases listed, a known nucleus. The second column gives the binding energy of the last neutron in mMU (observed, cf. reference 1, p. 378). The next two columns give $A_0^{\frac{1}{3}}$ and $(3.9/\epsilon)^{\frac{1}{2}}$; the former quantity is proportional to the radius R_0 of the residual nucleus while the latter measures the extension of the neutron wave function beyond R_0. It is seen that for tightly bound nuclei such as Li^7, Be^{10}, B^{11}, C^{13}, etc., the latter contribution is much smaller than R_0, while for loosely bound ones, such as He^6, the neutron wave function may extend as far out as $2\frac{1}{2}R_0$. The fifth column contains the effective Z; it is simply the charge Z' of the nucleus in the first column when the two nuclei of the isobaric pair differ by one unit of charge (e.g. Li^7-Be^7), while it is $Z'+(Z'+1)$ when the charge difference is two units (e.g. Li^8-B^8). The following three columns contain the values of the Coulomb energy calculated from (3), from (10), and observed, the latter wherever the necessary data are available. Next the symbol of the isobar containing more protons is given, and finally the binding energy ϵ_P of its last proton. When the two isobars differ by one unit of charge, we have simply $\epsilon_P=\epsilon_N-C$ because then the isobars are compared with the same residual nucleus; for a charge difference of two, the relation is slightly more complicated.

The "Calc. II" column reproduces very well both the absolute values and the individual fluctuations of the Coulomb energy. C has almost the same value for the pairs $C^{11}-B^{11}$ and $N^{13}-C^{13}$, then rises sharply to $O^{15}-N^{15}$ and again stays almost the same for $F^{17}-O^{17}$. This experimentally well-known behavior thus finds its explanation in the fact that the last neutron in C^{13} and O^{17} is relatively loosely bound to the very stable residual nucleus (C^{12} and O^{16}) while the binding in B^{11} and N^{15} is much stronger. These latter nuclei are therefore smaller in size and have a larger Coulomb energy.

Actually, our calculation II over-corrects for the effect of loose binding, and gives too high a value for C for $O^{15}-N^{15}$. This might be expected from the "one-body" character of our considerations. That our formula is as good as it is, may be due largely to the fact that in the cases of loosely bound last particles (C^{13}, O^{17}) the residual nucleus (C^{12}, O^{16}) is particularly stable and therefore has particularly few low excited states. This makes the one-body model for nuclei like C^{13}, N^{13}, O^{17}, F^{17} (and also Be^9, B^9) better than usual.

In all cases, the observed value of C falls in between (or very close to) the values predicted by our two formulas, (3) and (10). This was expected from their derivation, since (3) corresponds to an extreme "compound model" and does not take into account the strength of binding of the last particle at all, whereas (10) corresponds to a "one-body model" and overemphasizes the importance of the last particle. Thus we can be confident that generally the actual value of the Coulomb energy will lie between the limits given by the two calculations. In the last column of Table I, we have therefore given the limits of the proton binding energy according to the two calculations; where these limits agreed closely, the mean was given (for C^{10} and O^{14}), where observations were available, the observed value.

For the stability of unknown nuclei we find then:

C^{10} and O^{14} are certainly highly stable, O^{14} with 5.1 mMU against $N^{13}+H$, and C^{10} with 3.8 mMU against disintegration into $2\,He^4+2\,H$. (The stability of Be^{10} is given against Be^9+n^1.)

Be^6 is certainly unstable, being between 1 and 2.6 mMU heavier than $He^4+2\,H$.

B^9 is almost certainly unstable,[2] *viz.* between 0.3 and 0.7 mMU. The lower value seems more likely because the one-body model should be

[2] The existence of B^9 was claimed in a short note by Meitner, Naturwiss. **22**, 420 (1934) who reported a weak positron radioactivity from this nucleus. This result which has never been confirmed is almost certainly incorrect because a positron radioactivity of B^9 would be in definite conflict with its stability. The condition for radioactivity of B^9 is that it be heavier than Be^9 plus two electrons, i.e., (reference 1, p. 373)

$$B^9>9.015\,04+0.001\,10=9.016\,14. \qquad (A)$$

The condition for stability is

$$B^9<2He^4+H=9.015\,85 \qquad (B)$$

in contradiction with (A).

good for this nucleus (Be^8+H). It seems very unlikely that B^9 should be even more stable than is indicated by the one-body model; however, in view of the comparative importance[3] of B^9 for astrophysics, direct experimental evidence would be desirable, e.g. from the reaction $Be^9+H=B^9+n$ which should occur with protons of 2 Mev energy.

B^8 and N^{12} are very doubtful. B^8 is just on the limit of stability, being slightly stable according to "Calc. II," slightly unstable according to I. The stability of N^{12} depends mostly on the exact binding energy of B^{12} which is not known at present.

[3] Bethe, "Energy Production in Stars," to appear shortly in the Phys. Rev.

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Bands of H_2 Ending on the $2p^1\Pi$ Level

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New measurements make possible a much more complete analysis of the band systems ending on $2p^1\Pi$. The constants of this state can be obtained with great accuracy and the irregularities in the Λ-doubling and perturbations traced to the interaction with the higher vibrational levels of $2p^1\Sigma$. New information is obtained about some of the states which have both electrons excited.

THE strongest and most extensive band systems in the visible H_2 spectrum are the transitions from a number of three quantum electronic levels to the $2p^1\Sigma$ state.[1] The data of these band systems can be found in Richardson's book[2] or in the original papers listed there. All the upper levels of these systems should also combine with the $2p^1\Pi$ state. The resulting band systems lie chiefly in the near infra-red around 7600A. Important fragments of these bands have been given by Richardson and Chalk,[2, 3] but the measurements of Gale, Monk, and Lee on which they based their analysis contain only the strongest lines in this region. Therefore it seems desirable to publish the data on these band systems as they were obtained from a more recent analysis based on our own measurements. This re-analysis was undertaken as a preliminary study for the analysis of the corresponding bands of HD and D_2 on which I hope to report in a future paper. I am giving the results of H_2 here separately because they throw some new light on the nature of the levels with both electrons excited about the exact interpretation of which there is still a great deal of uncertainty.

The wave numbers and intensities of the transitions from the singlet $3d$ complex are given in Table I and those from the other initial states in Table II. In general it will be found that the lines which Richardson gives agree with those listed here, but there is a considerable number of discrepancies, which must be ascribed to the scantiness of the data with which Richardson had to work. A few doubtful lines are included in the tables. They are designated by a question mark. In a few bands there are undoubtedly more lines present than those given in the tables. That is indicated by the high intensity of the last listed lines. In such cases there are marked irregularities in either the initial or the final state and a prediction of the position of the line becomes too uncertain. Only when there are good combination relations can such lines be identified with any degree of certainty. In the present instances such combination relations are not always available. However there is no doubt that future reserach will eventually locate these lines.

[1] Whenever only one electron is excited it is sufficient to specify the state of this electron, and it is understood that the other electron is in the normal state $1s\sigma$. The full notation of the $2p^1\Sigma$ state would be $1s\sigma 2p\sigma^1\Sigma$. Only when both electrons are excited is it necessary to specify the state of both.

[2] O. W. Richardson, *Molecular Hydrogen and Its Spectrum* (New Haven, 1934).

[3] O. W. Richardson, Proc. Roy. Soc. **A126**, 487 (1930); M. L. Chalk, Proc. Roy. Soc. **A128**, 579 (1930).