Domain Theory of Ferromagnetics Under Stress*

Part III. The Reversible Susceptibility

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It is shown that the statistical theory of Parts I and II can be extended so as to give formulas for the magnetization curve and other properties of an ideal reversible specimen, over the range of field values in which magnetization proceeds by a displacement of the boundaries between domains. The application to actual specimens is made by assuming that the reversible properties of an actual specimen are identical with the properties, at the same magnetization, of an ideal specimen having the same initial susceptibility. This leads to a formula for the reversible susceptibility of nickel and iron that was originally suggested by Gans and verified by him experimentally, but for which it is believed no satisfactory derivation has hitherto been offered. In the case of one nickel specimen and of cobalt, where the data do not follow the Gans curve, formulas agreeing better with experiment are obtained by taking account of the anisotropy of the domains. Reasons are given for believing that the replacement of variable by fixed domain volumes, an arbitrary step taken in order to simplify the problem, is probably not essential to the result.

THE behavior of ferromagnetic crystals at fields of order of magnitude 10^2 oersteds has been satisfactorily explained[1] by supposing that an increase of field causes a "rotation" of the magnetization vector toward the field direction. This vector has a constant magnitude, the saturation value J_s; and except in special cases, it has also a uniform direction throughout each crystal—the direction for which the free energy, consisting of a term due to the field and a term due to the crystalline anisotropy forces, is a minimum. The special cases just referred to are those in which there are several such directions, as is the case when the field direction is [110] or [111] in an iron crystal. In these cases the crystal is saturated not as a whole, but only throughout "domains," equal volumes being magnetized in each of the equivalent directions. The properties of large grained polycrystalline specimens in this same range of field values can be calculated by averaging those of crystals.[2]

At fields of order of magnitude 1 oersted, the process is different and a successful quantitative theory is still lacking. Certain fundamental ideas, however, have proved useful and are probably essentially correct. It is supposed that even in the demagnetized state the specimen is saturated throughout domains; equal volumes are magnetized in each of the possible directions, which are now the directions of minimum free energy determined by the anisotropy forces alone—the "directions of easy magnetization." When a field is applied and gradually increased, the electron spins responsible for the magnetization undergo transitions in the boundary region between domains; the effect is a motion of the boundary, so that the more favorably oriented domains grow at the expense of their less favorably oriented neighbors. This displacement of the boundaries is in part reversible, proceeding an infinitesimal distance dx when the field is changed by an infinitesimal amount dH; and in part irreversible, for when a small change of field enables the boundary to pass a local energy barrier, the motion may continue a considerable distance, with dissipation of energy through the setting up of local eddy currents and in other ways.

The most direct evidence of the correctness of this picture is obtained from measurements of the Barkhausen effect[3] and from the study of magnetic powder and colloid patterns.[4] The regions whose reversal of magnetization is

[1] An outline of the theory and references to the original papers are given in E. C. Stoner, *Magnetism and Matter* (1934), pp. 392 ff., and F. Bitter, *Introduction to Ferromagnetism* (1937), pp. 194 ff.
[2] R. Gans, Ann. d. Physik **15**, 28 (1932).
[3] For references see Stoner, reference 1, p. 436.
[4] F. Bitter, reference 1, pp. 59 ff.; W. C. Elmore, Phys. Rev. **51**, 982 (1937); **53**, 757 (1938).

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observed in the Barkhausen effect, although not necessarily identical with the domains, at least set a lower limit to their order of magnitude. It is consistent with the order of magnitude of the dimensions observed in surface powder patterns, but the latter method of study shows that the actual shapes and arrangements of the differently magnetized regions may be very complicated.

Various types of force are operative in this process. The exchange forces are important only in the boundaries between the domains, where their partial annihilation has the effect of a positive surface energy proportional to the total area of the interdomain surfaces, and therefore plays a role in determining the size of the domains. This size in fact seems to be determined primarily by a balance between these forces and magnetic forces dependent upon the size and shape of the specimen, so that the domain size varies approximately as the square root of the shortest dimension of the specimen.[5] The anisotropy forces are important chiefly at higher fields, but according to one theory[6] they are also instrumental in determining the thickness of the interdomain space and hence indirectly the size of the domains. The theory of the behavior of an ideal crystal in an applied field has been attacked by several authors,[5-7] but even in the simplest cases the problem is difficult because nonlinear partial differential equations occur at the outset.

In an actual crystal, though the order of magnitude of the domain size is probably determined by the factors already mentioned, the exact behavior in a field is dependent very largely on forces that vary irregularly from one point to another. Chief among these are internal stresses due to impurities or to imperfect annealing. When these have been reduced to a minimum, there remain unavoidable stresses resulting from the unequal magnetostriction of differently oriented domains. Becker[8] has shown how the internal stresses limit the initial susceptibility in a somewhat idealized model, and Kersten[9] has shown that the magnitude of the

highest attainable initial susceptibility in iron-nickel alloys of various compositions is adequately accounted for by setting the internal stress equal to the lower limit determined by magnetostriction. In addition to this effect of mild stresses in controlling the reversible boundary displacement, there is the irreversible effect of local high stresses, which set up energy barriers that lead to Barkhausen reversals and hence to hysteresis.[10]

The importance of irregular forces suggests the possibility of handling the problem by a statistical method, in which the exact nature of the process need not be considered in detail. This method of attack was originated by Heisenberg. The quantities that have so far been calculated by it are the magnetostriction as a function of magnetization; the normal component of magnetization as a function of the component parallel to the field; and the elastic constants as functions of magnetization, given the magnetization curve. In parts I and II under the present title,[11] previous work in this field was summarized, more general formulas were developed, and some errors in earlier calculations were pointed out and corrected. No new principles were introduced, except the device of using thermodynamic relations to eliminate certain unknown quantities; by this means it was possible to avoid an incorrect approximation made in some of the earlier work.

It is possible to extend the theory so as actually to calculate the magnetization curve of an ideal reversible specimen, and from this all the reversible properties of an actual specimen, given a single reversible property—the initial susceptibility. This extension is the subject of the present paper; but in order to make the presentation convincing and to insure against misinterpretation of the results, it will be necessary to review the entire theory briefly, from a somewhat different point of view.

A few general remarks will make the aim clearer. The Becker-Kersten rotation theory, applicable only when the internal stresses σ_i are large and fairly uniform over a domain, gives for

[5] J. Frenkel and J. Dorfman, Nature **126**, 274 (1930).
[6] L. Landau and E. Lifshitz, Physik. Zeits. Sowjetunion **8**, 153 (1935).
[7] F. Bitter, reference 1, pp. 185–192.
[8] R. Becker, Physik. Zeits. **33**, 905 (1932).
[9] M. Kersten, Zeits. f. Physik **12**, 665 (1931).

[10] E. Kondorsky, Physik. Zeits. Sowjetunion **11**, 597 (1937).
[11] W. F. Brown, Phys. Rev. **52**, 325 (1937); **53**, 482 (1938): hereafter cited as I and II, respectively.

the initial susceptibility $\chi_0 = (2/9)J_s^2/\lambda_\infty\sigma_i$; λ_∞ is the saturation magnetostriction. Becker's simplified theory of boundary displacement against small, inhomogeneous stresses of amplitude σ_i gives $\chi_0 = (1/\pi)J_s^2/\lambda_\infty\sigma_i$. Although the mechanisms assumed are entirely unlike, the results are practically identical. Again, it is possible to develop a theory similar to that of Becker and Kersten, except that internal stresses are replaced by internal fields; the result of either theory may be written in the form $\chi_0 \cong J_s^2/\epsilon$, where ϵ is an energy density associated with the "random" forces; and in each case the area to the left of the magnetization curve, $\int H dJ$, is approximately equal to the ϵ of the χ_0 formula. In other words, the most important characteristics of the theoretical magnetization curve—its initial slope, and the area between it and the J axis—are determined by energy magnitudes rather than by the exact nature of the internal forces or the exact mechanism through which they make their effect felt. It should therefore be possible to develop at least an approximate theory in which no specific assumptions are made in regard to the nature of the forces or the mechanism. The advantage of such a theory is that it avoids the danger of misinterpreting an agreement between theory and experiment as evidence for the particular forces or mechanism assumed in the theory.

General Theory

The theory will take no account of irreversible phenomena, and therefore the properties predicted by it will be those of an ideal reversible specimen. It is necessary to relate these in some way to the properties of an actual specimen. Now there are certain properties of a ferromagnetic specimen which, to a first approximation, are determined by the magnetization alone and are independent of the previous history and hence of the field strength associated with that magnetization in the particular state under observation. The quantities of which this is true are those measured in a small, approximately reversible change from any initial state: for instance, the elastic constants[12] and the reversible susceptibility[13] (ratio $\Delta J/\Delta H$ when $\Delta H/H$ is small and

negative). This suggests that such quantities should have the same value for the actual specimen and for a reversible specimen at the same magnetization. We assume, therefore, that so far as a specimen's properties are functions of the magnetization alone, they are the same functions of magnetization as are the properties of an ideal reversible specimen having the same initial susceptibility.

The correct model for a single crystal is now a group of many domains with fixed orientations of their magnetization vectors, but with variable volumes; the positions of the boundaries are determined by equilibrium between internal and external forces, that is by the minimum of the total free energy due to external fields and stresses and to internal "random" forces. On the supposition that the mechanism by which these internal forces act is unimportant, we proceed to substitute for this model a more manageable one, consisting of a group of N domains of fixed and equal volumes, but with variable orientations of their magnetization vectors: the magnetization of each domain is to be along one of the "easy magnetization" directions, but may change from one of these directions to another when the field or stress is changed. The exact microscopic state of the model under given external conditions is to be that for which the total free energy is a minimum. A macroscopic state of the model, of course, will be described by merely giving the fractional number $n_\sigma = N_\sigma/N$ of domains magnetized in each of the possible directions σ, without regard to which domains these are.

The actual microscopic state, for given field components H_i and stress components X_{ij}, is that corresponding to the minimum value of[14]

$$V = V' + V'', \tag{1}$$

where V' is the ordered free energy per cm^3 due to the applied field and stresses, and V'' is the energy due to the random internal forces. V' is given by

$$V' = -\sum H_i J_i - \sum X_{ii}e_{ii}' - \sum X_{jk}e_{jk}'$$
$$= \sum_\sigma n_\sigma(-\sum H_i J_{i\sigma} - \sum X_{ii}e_{ii\sigma}'$$
$$-\sum X_{jk}e_{jk\sigma}') = \sum_\sigma n_\sigma V_\sigma. \tag{2}$$

[12] E. Giebe and E. Blechschmidt, Ann. d. Physik 11, 905 (1931); O. von Auwers, Ann. d. Physik 17, 83 (1933); W. T. Cooke, Phys. Rev. 50, 1158 (1936).

[13] R. Gans, Physik. Zeits. 12, 1053 (1911); Ann. d. Physik 61, 379 (1920).

[14] I, Eq. (7). The constant term V_0 has been dropped.

In a large number of samples of the same material, microstates of equal V' will occur equally often if the internal forces are random,[15] but microstates of lower V' will occur oftener. On the other hand there are many more microstates of zero ordered energy than of highly negative. Somewhere between 0 and $-\infty$, there is a "normal" value of V' that will occur most of the time; and among samples with about this value of V', certain values of the partition numbers N_σ will be normal. Just what value of V' is normal for given H_i's and X_{ij}'s depends on the average magnitude of the random forces: the larger these are, the larger will be the external field or stress required to produce a specified magnetization or strain.

Assuming that V' has some value, we may proceed to determine the normal values of the N_σ's, and hence the macroscopic properties of the specimen, by methods similar to those used in statistical mechanics. The result is

$$n_\sigma = \exp\,(-L_0 V_\sigma)/\sum_\sigma \exp\,(-L_0 V_\sigma), \qquad (3)$$

where L_0 is determined by

$$\sum_\sigma V_\sigma \exp\,(-L_0 V_\sigma)/\sum_\sigma \exp\,(-L_0 V_\sigma) = V'. \qquad (4)$$

This is in essence the method that was introduced by Heisenberg and used by Bozorth and by Gans and v. Harlem. These writers, however, did not actually assume a given value of V', but rather a given value of the component of magnetization in the field direction, J_p. In the problems treated by them the two quantities differ only by a factor H, and moreover it is really J_p and not V' that is known and must be used to eliminate the Lagrangian multiplier L_0; the two procedures therefore lead to the same result. In the work of Akulov and Kondorsky stresses as well as fields are supposed to be present, and a given value of V' must be assumed. There is now a difficulty, however; for V' is not really known, and so L_0 remains an undetermined quantity, whose dependence on the field and stress are unknown. Akulov and Kondorsky endeavored to overcome this difficulty by supposing the magnetization curve at zero stress to be given, and seeking only

a formula for the N_σ's good to the first order in the stresses. But in doing this they failed to take account of the effect of stresses upon the magnetization curve, and thus obtained an incorrect result. It was shown in I that this error could be corrected, and the result they sought obtained, by using the thermodynamic relations $\partial J_i/\partial X_{jk} = \partial e_{jk}/\partial H_i$. The equations thus obtained are rather complicated, it is still necessary to know the magnetization curve, and the physical significance of the troublesome Lagrangian multiplier never becomes apparent. The theory at this stage is clearly incomplete.

A similar problem is encountered in statistical mechanics, in calculating the normal partition numbers for a given energy of an isolated system. The energy is really unknown, and so its Lagrangian multiplier L remains indeterminate until it has been given some thermodynamic interpretation. This may be done in a number of ways, for instance by considering *two* systems capable of exchanging energy. The same equations as before are obtained for each system, but with the same L for both; L therefore represents a quantity which is the same for two systems when they are in thermodynamic equilibrium—namely, the temperature. (A more thorough investigation of course shows that L is proportional to the reciprocal of temperature on the usual absolute scale.)

This and other methods, suitably modified, may be applied here. The phrase "suitably modified" is important, for it must be emphasized that this is *not* a problem in statistical mechanics. Our model is not a group of domains continually jumping from one state to another as a result of exchanges of energy. It is a group of domains in a definite, permanent state of lowest total free energy, under the joint action of ordered and random forces—all constant. The direct effect of temperature agitation on the domains as a whole is completely negligible because of their size; its indirect effect in bringing about the equilibrium of the boundaries is fully taken into account in the temperature-dependent "constants" (such as J_s and λ_∞) appearing in the free energy function. Another way of saying this is that the domain order-disorder term in the free energy is negligible compared with the other terms. This term is $-kT \log W$, where W is the number of ways of

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[15] More accurately, this is our definition of "random" as applied to these forces.

arranging the domains in the given macroscopic state. For unit volume it is of order of magnitude kTN, where N is the number of domains per cm³. For a wire 1 mm in diameter, $N \leq 10^9$ according to Barkhausen effect measurements, so that this term in the free energy density is of order of magnitude 10^{-5} erg/cm³. The energy associated with the internal forces opposing magnetization is measured by the area $\int H dJ$ to the left of the steep part of the magnetization curve, or roughly by J_s^2/χ_0; even in well-annealed specimens, its order of magnitude is between 10^2 and 10^4 ergs/cm³. Thus thermal agitation of the domains as a whole is negligible by a factor of about 10^{-8}.

Consider, then, two adjacent volumes v_1, v_2 of the same material, subject to different values of the fields or stresses. If we now seek the normal N_σ's for a given total ordered free energy $E' = v_1 V_1' + v_2 V_2' = E_1' + E_2'$, we obtain equations of the form (3) for each part, but the same Lagrangian multiplier L_0 occurs in both sets of equations. L_0 is determined by the total E', but also it is equally well determined by the E' for either part. If action at a distance by the random forces is negligible, a variation of the fields or stresses in one part does not affect the state of affairs in the other, and so does not affect L_0, which must therefore be a constant of the material (at the given temperature). Its value depends on the random forces. For cubic crystals or isotropic material, it follows from Eq. (41) of I and the corresponding equation for [100] domains, and from Eqs. (15), (23), and (36) of II, that

$$\chi_0 = \tfrac{1}{3} L_0 J_s^2; \qquad (5)$$

hence $1/L_0$ is an energy density of the same order of magnitude as $\int H dJ$ (if the number of directions of easy magnetization is finite), and measures the energy associated with the internal forces. If we assume that these are internal stresses of mean value σ_i, and set $1/L_0 = \lambda_\infty \sigma_i$, we get $\chi_0 = \tfrac{1}{3} J_s^2/\lambda_\infty \sigma_i$, a formula differing from Kersten's only in having a numerical factor $\tfrac{1}{3}$ instead of 2/9.

Although the discussion has been limited to single crystals for simplicity, the extension to polycrystalline material is easily made by the method of I. Thus we get for the magnetization curve of the ideal reversible specimen of iron or nickel Eq. (12), (20), or (34) of II, according to the type of domain (isotropic, [100], or [111]); in the definition of η preceding each of these equations, L_0 is now to be interpreted as a constant, related to the initial susceptibility by Eq. (5) above. For materials with other types of domain, Eq. (3) or (4) of II must be used.

THE REVERSIBLE SUSCEPTIBILITY

The magnetization curve thus obtained in terms of χ_0 or L_0, for an ideal reversible specimen, is different from any J–H curve of the actual specimen. But the differential susceptibility dJ/dH calculated from it corresponds to a measurable susceptibility for actual specimens, namely the "reversible susceptibility" mentioned before. The value of the reversible susceptibility χ_r at any magnetization should therefore be equal to the differential susceptibility of the ideal specimen at the same magnetization, if L_0 is chosen so as to give the observed initial susceptibility. (The corresponding value of H, however, bears no simple relation to the H of the actual specimen.)

Thus for material with isotropic domains, χ_r is determined as a function of J by the parametric equations

$$J/J_s = L(\eta), \quad \chi_r/\chi_0 = 3L'(\eta), \qquad (6)$$

where $\qquad L(\eta) = \coth \eta - 1/\eta. \qquad (7)$

These equations were first proposed by Gans in 1911, and were supported by experimental data agreeing very well with the calculated curve.[13] Gans gave no derivation of his formula; he was probably led to it by an attempt to apply the Langevin formula for the susceptibility of a paramagnetic gas. A derivation along that line was very apologetically offered by Debye[16] in 1925. According to this interpretation, η should be equal to $\mu H/kT$, where μ is the magnetic moment of an elementary magnet, in Debye's picture a crystal. Today we must interpret the elementary magnet as a domain. Its magnetic moment is $J_s v_0$, where $v_0 \leq 10^{-9}$. The initial susceptibility thus calculated is 10^8 times too large, as was to be expected from our earlier discussion.

[16] P. Debye, *Handbuch der Radiologie*, Vol. 6 (1925), p. 721.
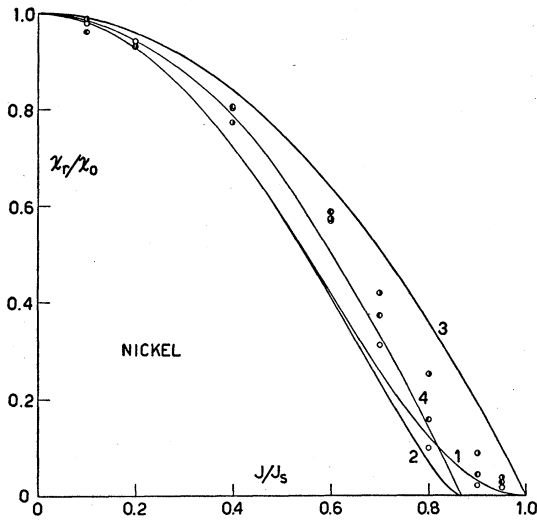
FIG. 1. Reversible susceptibility of polycrystalline nickel. Experimental points (D. Kirkham): open circles, 21.6°C; circles black at left, 99.0°C; black at right, 171.5°C. Theoretical curves: 1, J_s in any direction; 2, J_s along any [111] direction; 3, J_s along $\pm H$; 4, J_s along [111] directions nearest $\pm H$.

If the domains are crystalline, in iron or nickel, the functions are more complicated in form, but the curves differ very little from this one; in fact the series expansions for J are identical to the sixth order in H (see Eqs. (15), (23), and (36) of II). This explains Gans's success in fitting his curve to data on various specimens of iron, steel, and nickel.

Figure 1 shows several attempts to fit Kirkham's[17] results on annealed nickel. These measurements were not taken on the same ellipsoid as the magnetostriction data analyzed in II, but on a thinner ellipsoid, $7\frac{1}{2}''$ long and $\frac{1}{8}''$ in diameter. For this specimen the domains are probably crystalline at room temperature; but according to the value of χ_0 the anisotropy forces are not enormously larger than the random forces, so that the neglected rotation effect may not be entirely negligible, and the specimen may be expected to be intermediate in its properties between one with crystalline domains of high anisotropy and one with isotropic domains.

Curve 1 is calculated for isotropic domains, curve 2 for crystalline domains with [111] as the direction of easy magnetization. There is a difference only at high J, and here 2 is unreliable

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[17] D. Kirkham, Phys. Rev. **52**, 1162 (1937).

because of rotation. The data do not fit at all well.

In the discussion of the magnetostriction of an iron ellipsoid (II, p. 488), it was suggested that when an elongated ellipsoid is demagnetized by a longitudinal field, more domains are left oriented along directions nearly parallel to the ellipsoid axis than along directions nearly perpendicular to it. This applies to the present case. Curve 3 is calculated for the extreme case in which all the domain magnetizations are either parallel or anti-parallel to the field (as might be the case if the specimen were long and thin and the domains magnetically isotropic). Curve 4 is calculated for domain moments oriented along only the [111] directions nearest the field.[18] Most of the data fall between these curves. As the temperature rises, the anisotropy constant decreases, the effect of rotation (or the tendency toward isotropy) increases, and in the range where rotation is important the points should fall nearer to the isotropic curve—as in fact they do.

In Fig. 2, the points are experimental data of Samuel[19] on soft cobalt. Curve 1 is the Gans formula, which does not fit at all. Cobalt crystals have two opposite directions of easy magnetization; the anisotropy constant is large, so that rotation should be negligible except at high fields, and the calculation for crystalline domains should be a good approximation. This calculation gives

$$J/J_s = E(\eta), \quad \chi_r/\chi_0 = 3E'(\eta), \qquad (8)$$

where
$$E(\eta) = (1/\eta^2) \int_0^\eta u \tanh u \, du. \qquad (9)$$

Series expansions for the function $E(\eta)$ may be obtained from the expansions in II, Eqs. (22)–(27). The function (8) is plotted as curve 2 in Fig. 2.

The deviation at high J is expected. The deviation at low J may be partly due to a poor value of χ_0. The experimental results deviate considerably among themselves; the agreement

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[18] The formulas are: curve 3, $J/J_s = \tanh \eta$, $\chi_r/\chi_0 = \text{sech}^2 \eta$ $= 1 - (J/J_s)^2$; curve 4, $J/J_s = (12/\pi) \int_0^{\pi/3} d\psi \int \alpha^{-1} \alpha \tanh \alpha \eta d\alpha$, $\frac{1}{3}(1 + 4/\pi)\chi_r/\chi_0 = (12/\pi) \int_0^{\pi/3} d\psi \int \alpha^{-1} \alpha^2 \text{sech}^2 \alpha \eta d\alpha$, $\alpha' = \cos \theta'$, $\cot \theta' = \sqrt{2} \cos \psi$; the result of the integration with respect to α can be expressed in terms of the functions E and E' defined by Eq. (9).

[19] M. Samuel, Ann. d. Physik **86**, 798 (1928): specimen H.

with theory is probably as good as can be expected. Gans[2] found a considerably worse deviation between the experimental results at higher fields and the theoretical curve calculated by the rotation theory.

Cobalt is interesting because, with only two directions of easy magnetization, it is easy to work out a more specialized theory, with more definite assumptions about the internal forces. Internal stresses, being impartial with respect to opposite directions, should have no effect on the reversible magnetization, and only internal fields need be considered. Curve 3 is calculated on the assumption that the magnetic moment of each crystalline domain is along that one of the two possible directions favored by the resultant H, external plus internal; it is assumed that the values of the relevant component of internal field are constant and are distributed according to a Gaussian error curve. This gives

$$J/J_s = \left[(2u^2-1) \int_0^u e^{-x^2}dx + ue^{-u^2} \right]/2(\pi)^{\frac{1}{2}}u^2, \quad (10)$$

$$\chi_r/\chi_0 = 3\left[\int_0^u e^{-x^2}dx - ue^{-u^2} \right]/2u^3. \quad (11)$$

The difference between curves 2 and 3 is unimportant at the present stage of development of the theory, and the wider applicability of the general method is strongly in its favor.

MODIFICATION OF EARLIER FORMULAS

The theorem that $L_0(=-L$ of I) is a constant enables us to replace Eq. (16) of I by the simpler relation, which now follows immediately from (14),

$$\partial Y_r/\partial y_s = LP_{rs}. \quad (12)$$

Hence only the first terms remain in the right members of Eqs. (34) when the differentiation is at constant H. From the second of Eqs. (44), $\chi_1 = 0$. This explains why good agreement with experiment was obtained by neglecting χ_1 in Eq. (54).[20] It is now clear, also, that the coefficients χ_1 and χ_2 in Eq. (40) have no relation to the actual magnetization curve of a crystal, but are

[20] It is possible to compute the variation of the elastic constants with magnetization without introducing the approximations of I, but this work is too far from completion to discuss at present.

determined by the curve of the equivalent reversible specimen, which may be obtained from the χ_r vs. J curve by the formula

$$H_r = \int_0^J (1/\chi_r)dJ. \quad (13)$$

Here H_r is the field strength corresponding to the magnetization J for an ideal reversible specimen whose reversible properties coincide with those of the actual specimen.

CRITICISM OF THE THEORY

The fact that the theory gives only reversible properties is a serious limitation, but less serious than might have been feared, provided the concept of the "equivalent reversible specimen" is used as a guide in interpreting the theoretical results. The assumption of complete randomness of the internal forces is an oversimplification; whether it is too drastic an oversimplification can only be decided by comparison of the predictions of the theory with experiment, and it would not be at all surprising if the theory proved useless for pure, well-annealed crystals. The replacement
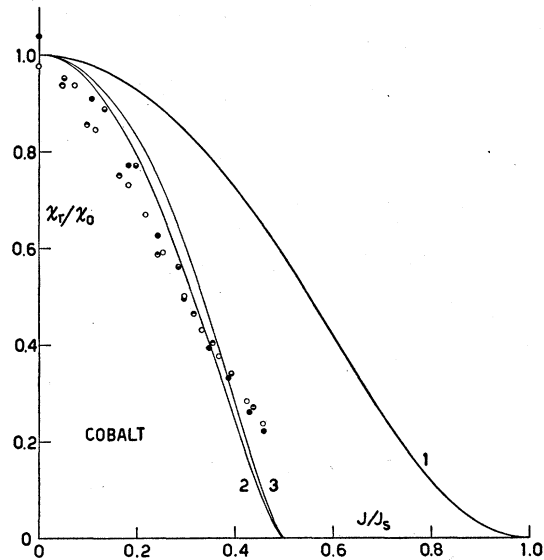

FIG. 2. Reversible susceptibility of polycrystalline cobalt. Experimental points (M. Samuel): open circles, virgin curve; solid circles, ideal curve; circles black at bottom, descending magnetization curve; black at top, ascending. Theoretical curves: 1, general theory, isotropic domains; 2, general theory, crystalline domains; 3, special theory of internal fields.

of variable volumes with fixed magnetization directions by fixed volumes with variable magnetization directions is the step that is most difficult to justify. If this step is defensible, then the final results should be capable of derivation without it. Some efforts in that direction are presented in the appendix.

Despite these limitations, the theory seems successful enough to justify its further study; it may help to give a partial understanding of the complex processes that occur at low fields, until a more detailed theory becomes possible.

The writer wishes to express his appreciation to Professor Francis Bitter, of the Massachusetts Institute of Technology, for helpful comments and criticisms.

APPENDIX

(a) Alternative derivations of the distribution formulas.

These parallel familiar arguments of statistical mechanics.

Method 1.—Assume, as before, that in a large number of copies of our simplified model, all subject to the same fields and stresses, microstates of equal ordered energy occur equally often; and that this remains true even if the fields and stresses have different values in two adjacent parts v_1, v_2 into which the model may be divided. Then the relative frequency of occurrence of any microstate, for either part separately or for the model as a whole, is a function only of the corresponding ordered energy $E_1' = v_1 V_1'$, $E_2' = v_2 V_2'$, or $E' = v_1 V_1' + v_2 V_2'$; let this function be $f(E_1')$, $g(E_2')$, or $h(E')$. The probability that a copy selected at random is in a specified microstate, with ordered energies E_1' and E_2', is

$$h(E_1' + E_2') = f(E_1') g(E_2').$$

For two microstates with the same total energy $E_1' + E_2'$, the values of $f(E_1') g(E_2')$ are therefore equal; or, $f(x) g(y) = f(z) g(x + y - z)$, for arbitrary x, y, and z. Differentiation with respect to x and y gives $f'(x) g(y) = f(z) g'(x + y - z) = f(x) g'(y)$, or $f'(x)/f(x) = g'(y)/g(y) = \text{constant} = A$. Hence $f(E_1') = B e^{-A E_1'}$, $g(E_2') = C e^{-A E_2'}$. By varying the fields or stresses in v_2 without varying them in v_1, and assuming action at a distance negligible, we infer as before that A is a constant of the material. Thus for unit volume, the probability of a microstate of ordered energy $V' = \sum n_\sigma V_\sigma$ is $B e^{-A V'}$; B is determined by $\sum_\mu B e^{-A V'} = 1$, where $\sum_\mu$ denotes a summation over all microstates. The mean n_σ's for the whole collection of copies are now given by $(n_\sigma)_{\text{Av}} = B \sum_\mu n_\sigma e^{-A V'}$. When the sum-

mation is carried out,[21] Eq. (3) is obtained, with $L_0 = A/N$.

Method 2.—For given fields and stresses, the actual microstate of any copy of a volume v, containing N domains, is that for which $E' + E'' = v(V' + V'')$ is a minimum. If we knew E'' for each microstate of each copy, we could solve the problem by first finding the normal minimum E'', say E_0'', attainable by varying the microstate, for given N_σ's; and then minimizing $E' + E_0''$ by variation of the N_σ's. The E_0'' obtained in the first step is a function of the N_σ's, but if the internal forces are random it depends on the N_σ's only because the normal minimum V'' to be found in W microstates is lower, the greater the number W of microstates there are to choose from. Thus E_0'' is a function of $W = N!/N_1! N_2! \cdots$, and hence of $\log W$, or of $-N \sum_\sigma n_\sigma \log n_\sigma$. With negligible action at a distance, E_0'' must be proportional to N for large N and for given n_σ's, hence $E_0'' = A N \sum_\sigma n_\sigma \log n_\sigma$, or $V_0'' = \epsilon \sum_\sigma n_\sigma \log n_\sigma$. Upon minimizing $V' + V_0''$ with respect to the n_σ's we obtain the same formulas as before, with L_0 replaced by $1/\epsilon$. In the demagnetized state each $n_\sigma = 1/k$, where k is the number of directions of easy magnetization, and $V_0'' = -\epsilon \log k$; when the field is large one $n_\sigma = 1$ and the others vanish, and $V_0'' = 0$. Hence the work done against internal forces in magnetizing unit volume of the specimen, or $\int H dJ$, is $\epsilon \log k$.

(b) Boundary displacement against random forces.

In the following discussion we abandon the somewhat arbitrary replacement of variable

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[21] Cf. J. Frenkel, *Wave Mechanics: Elementary Theory* (1932), p. 199.

volumes by fixed volumes. We assume that a fraction v_σ of the total volume has J_s in direction σ, and that each of the quantities v_1, v_2, $\cdots$ is a function of the quantities V_1, V_2, $\cdots$.

Consider the ratio v_i/v_j. When $V_i = V_j$, this is 1, whatever the values of the other V_σ's. When V_i is much larger algebraically than V_j, v_i/v_j is practically zero; and when V_i is much smaller algebraically than V_j, v_i/v_j is very large, whatever the values of the other V_σ's. Thus when $V_i - V_j = -\infty$, 0, or $+\infty$, v_i/v_j has a definite value, independent of the other V_σ's and also of the absolute value of V_i or V_j (with respect to any zero). Let us *assume* that in general v_i/v_j is a function only of $V_i - V_j$, say $v_i/v_j = f(V_i - V_j)$. Then for any three directions i, j, k we have

$$1 = (v_i/v_j)(v_j/v_k)(v_k/v_i)$$
$$= f(V_i - V_j)f(V_j - V_k)f(V_k - V_i),$$

or if $\phi = \log f$,

$$\phi(x-y) + \phi(y-z) + \phi(z-x) = 0.$$

Differentiating with respect to z and then setting $z = 0$ gives $\phi'(y) = -\phi'(-x) = \text{constant} = -A$, or $f(y) = Be^{-Ay}$. When $y = 0$, $f = 1$, therefore $B = 1$. Thus $v_2 = v_1 e^{-A(V_2 - V_1)}$, $v_3 = v_1 e^{-A(V_3 - V_1)}$, $\cdots$, or $v_1 e^{AV_1} = v_2 e^{AV_2} = v_3 e^{AV_3} = \cdots = C$, or finally $v_1 = Ce^{-AV_1}$ etc., where C is determined by $v_1 + v_2 + \cdots = 1$.

This derivation fails when there are only two directions of easy magnetization, but this case may be regarded as a limiting case attained by letting the value of V_σ for the other directions become very large.

It is not obvious that the assumption made in this argument is equivalent to the assumption that the internal forces are random. The derivation is therefore incomplete; but it does suggest that the introduction of fixed instead of variable volumes, and the resulting superficial analogy to statistical mechanics, are probably not essential to the theory.

Note added in proof: The following additional results of the theory are interesting because of their relation to some recently published work of Kondorsky.[22] In an iron crystal, when the field is along a specimen axis for which the demagnetizing factor is negligible, it may be assumed that only the [100] directions nearest this axis are occupied. Let the cosine of the acute angle between such a direction and the specimen axis be l, and let $j = J_p/J_s$, $\eta = L_0 J_s H_r$: then the magnetization curve of the equivalent reversible specimen is given by $j = l \tanh \eta l$, so that $\chi_r = L_0 J_s^2 dj/d\eta = L_0 J_s^2 l^2 \operatorname{sech}^2 \eta l = L_0 J_s^2 l^2(1 - j^2/l^2)$. It follows that for three crystals of the same material with their specimen axes along [100], [110], and [111] ($l^2 = 1$, $\frac{1}{2}$, and $\frac{1}{3}$, respectively), and with the field in each case along the specimen axis, the initial susceptibilities should be in the ratios $6 : 3 : 2$, and χ_r/χ_0 should be given by $1 - j^2$, $1 - 2j^2$, and $1 - 3j^2$, respectively. The ratios $6 : 3 : 2$ were observed in Williams'[23] measurements and are predicted by Kondorsky's theory when $\kappa_1 \gg \kappa_2$, and also by earlier theories of Akulov and of Bozorth.[22] The expressions for χ_r/χ_0 are identical with those of Kondorsky's theory.

[22] E. Kondorsky, Phys. Rev. **53**, 319 and 1022 (1938).
[23] H. J. Williams, Phys. Rev. **52**, 1004 (1937).