I. General Theory of the Earth's Shadow Effect of Cosmic Radiation

E. J. SCHREMP*
Massachusetts Institute of Technology, Cambridge, Massachusetts
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An inquiry is made into the fundamental mode of origin of the allowed cone of cosmic radiation introduced by Lemaitre and Vallarta. Two kinds of allowed cone are distinguished. One of these, that discussed by Lemaitre and Vallarta, postulates the presence of an impenetrable earth; for it all forbidden directions are due to the earth's shadow effect. The second type of allowed cone, applicable in the first approximation to such problems as that of the sun's magnetic field, does not postulate the presence of an impenetrable earth; for it all forbidden directions are associated with bounded charged particle orbits. The general topological features of each of these kinds of allowed cone, and their relationship to one another, are described.

1

WE shall be concerned with the reduced motions of a charged particle in the meridian plane of a magnetic dipole.[1] These motions may be classified into three types: (1) completely bounded motions which never attain an infinite distance from the dipole; (2) semibounded motions which are bounded in the future but not in the past, or in the past but not in the future; and (3) completely unbounded motions which are bounded neither in the past nor in the future. In this classification it is assumed that an impenetrable earth is absent.

It is convenient to introduce the notion of an orbital section, by which we shall mean any segment of an orbit joining two consecutive points at which the function[2] $P(x, \lambda; \gamma_1)$ attains relative minima. Any orbital section at both ends of which the orbit returns toward the dipole will be termed reentrant. If an entire orbit contains n reentrant sections it will be called an n-reentrant orbit; if it contains no reentrant sections it will be called a nonreentrant orbit. All orbits for which n is finite or zero must be completely unbounded. Orbits for which n is infinite may be either bounded or unbounded.

Two periodic orbits, discovered by Störmer[3] and studied by Lemaitre and Vallarta,[4, 5] are intimately connected with the reentrant orbits defined above. Lemaitre and Vallarta[4] first estimated the range of simultaneous existence of these two periodic orbits to be the interval[2] $0.783 \leqq \gamma_1 \leqq 1$. Lemaitre[5] subsequently refined this interval to $0.78856 \leqq \gamma_1 \leqq 1$. We shall henceforth speak of them as the two principal periodic orbits.

2

In the interval $0.78856 \leqq \gamma_1 \leqq 1$ in which they simultaneously exist, the two principal periodic orbits form left- and right-hand limits (with respect to the coordinate[2] x) of the family of all reentrant sections normal to the equator, as may be seen from certain results of Störmer.[6] Since no orbital section normal to the equator and outside this interval of x may be reentrant, it follows *a fortiori* that no other section outside this interval may be reentrant. Moreover, through every equatorial point within this interval of x, and for a given γ_1 in the above-mentioned range, there passes a singly infinite pencil of reentrant sections, delimited on the left and right by two symmetrical self-reversing reentrant sections making angles[2] η and $-\eta$ with the equator. These facts lead to theorem 1: *For each γ_1 in the interval $0.78856 \leqq \gamma_1 \leqq 1$ the totality of reentrant orbits may be generated by moving the equatorial*

* At present at Washington University, St. Louis, Missouri.

[1] C. Störmer, Zeits. f. Astrophys. 1, 237 (1930), has shown that the actual space trajectories may be reduced to motions in the meridian plane. It is the latter which are dealt with in the Lemaitre-Vallarta theory of the allowed cone. See for instance G. Lemaitre and M. S. Vallarta, Phys. Rev. **49**, 719 (1936).

[2] For the necessary definitions see G. Lemaitre and M. S. Vallarta, Phys. Rev. **49**, 719 (1936).

[3] C. Störmer, Zeits. f. Astrophys. **1**, 237 (1930).

[4] G. Lemaitre and M. S. Vallarta, Phys. Rev. **43**, 87 (1933).

[5] G. Lemaitre, Ann. de la Soc. Sci. de Bruxelles **A54**, 194 (1934). Recent work by O. Godart, Ann. de la Soc. Sci. de Bruxelles **A58**, 27 (1938), indicates that the value 0.78856 should be revised to 0.788541.

[6] See reference 3, Fig. 9.

point and its associated pencil of reentrant sections from the inner to the outer principal periodic orbit.

By the same reasoning one is led to theorem 2: *For each γ_1 in the interval $\gamma_1 > 1$ the totality of reentrant orbits may be generated by moving the equatorial point and its associated pencil of reentrant sections from the inner principal periodic orbit to the outer boundary curve $P(x, \lambda; \gamma_1) = 0$ of the bounded region of possible motions.* All such orbits are completely bounded, and of course n is infinite for them.

Together with Lemaitre's result[5] on the range of simultaneous existence of the two principal periodic orbits, theorem 1 leads to theorem 3: *In the interval $\gamma_1 < 0.78856$ no reentrant orbits, and hence no bounded orbits, exist.*

3

By theorem 3, for $\gamma_1 < 0.78856$ all orbits are completely unbounded. By theorem 2, for $\gamma_1 > 1$ all orbits are either completely bounded or completely unbounded according as they lie within the bounded or unbounded regions of possible motion. But by theorem 1, for $0.78856 \leqq \gamma_1 \leqq 1$ bounded and unbounded orbits may coexist.

The problem of ascertaining the relative distributions of bounded and unbounded orbits has been completely solved by theorems 2 and 3 for all values of γ_1 outside the interval $0.78856 \leqq \gamma_1 \leqq 1$. The resolution of this problem in the remaining interval requires an application of point set theory. Using such methods the writer has proved theorem 4: *In the interval $0.78856 \leqq \gamma_1 \leqq 1$ the set of all bounded orbits is nowhere dense on the whole manifold of motions.* After several attempts to extend these methods it has been further conjectured that this set is of zero measure.

By an extension of a theorem due to Poincaré[7] one is led to theorem 5: *All bounded orbits are stable in the sense of Poisson; i.e., they return infinitely often to the neighborhood of their initial state.* The proof of this theorem rests on theorems 1 and 2. We shall give a physical interpretation to this theorem in Section 5.

A detailed mathematical treatment of these matters will be published elsewhere.

4

Having thus determined the relative distributions of bounded and unbounded orbits for all values of γ_1 we find it now possible to give a rather complete description of the structure of the allowed cone in the absence of the earth's shadow effect. In this case the total allowed cone has three distinct parts: (1) the cone for $\gamma_1 < 0.78856$ in which all directions are allowed (all orbits are completely unbounded); (2) the spherical segment $0.78856 \leqq \gamma_1 \leqq 1$ in which all directions are allowed (all orbits are completely unbounded) except for a nondense set of forbidden loci (corresponding to the nondense set of bounded orbits); (3) the cone for $\gamma_1 > 1$ (the Störmer cone[8]) in which all directions are forbidden (all orbits are completely bounded).

These considerations, in which the property of boundedness and not the earth's shadow effect accounts for forbidden directions, may be applied to such problems as that of the allowed cone in the magnetic field of the sun, introduced by Janossy[9] and, independently, by Vallarta.[10] In this problem of the sun's allowed cone at the earth, the foregoing considerations evidently neglect the relatively small shadow effects arising from the presence of the earth and sun in the sun's own magnetic field. The inclusion of these small shadow effects would slightly augment the forbidden directions provided for above by the criterion of boundedness alone.

5

Lemaitre[11] has defined as orbits of the first kind all those orbits which are asymptotic (on the side toward the dipole) to the outer principal periodic orbit. By means of them Lemaitre and Vallarta have shown the existence of a main cone[4, 12] within which all directions are allowed, and of a region of penumbra[12] between the main cone and the Störmer cone within which some directions are allowed and others are forbidden.

Their result holds both for the case of the allowed cone in the absence of an impenetrable

[7] H. Poincaré, *Méthodes nouvelles de la mécanique céleste*, Vol. 3, chap. 26. See also G. D. Birkhoff, *Dynamical Systems*, American Mathematical Society Colloquium Publications Vol. IX, p. 189 ff.

[8] C. Störmer, Publ. Univ. Obs. Oslo, No. 11, p. 10 (1933).
[9] L. Janossy, Zeits. f. Physik **104**, 430 (1937).
[10] M. S. Vallarta, Nature **139**, 839 (1937).
[11] G. Lemaitre, Ann. de la Soc. Sci. de Bruxelles **A54**, 162 (1934).
[12] G. Lemaitre and M. S. Vallarta, Phys. Rev. **49**, 720 (1936).

earth and for the case of the allowed cone in the presence of an impenetrable earth, but with considerable differences in the character of the penumbra in the two cases. For in the first case the region of penumbra is densely filled with allowed directions, as we have seen in Section 4; while in the second case the region of penumbra is no longer densely filled with allowed directions but may include alternating allowed and forbidden regions ranging from infinitesimal to finite size, as we shall see in Section 7.

In the case where an impenetrable earth is absent, orbits of the first kind may be associated with at least a part of the nondense set of bounded orbits referred to above. For by an immediate corollary to theorem 1 every orbit of the first kind is a transition orbit between two orbital classes having distinct values of n. In any particular instance where one of these values of n is infinite the orbit of the first kind concerned may actually define one member of the abovementioned nondense set of forbidden loci.

In the case where an impenetrable earth is present, orbits of the first kind may again be generators of the allowed cone. But here they acquire this property as a result of a discontinuous shadow effect. For any asymptotic half-orbit reaching a given point on the earth may be viewed as the limit of an infinite sequence of self-reversing reentrant half-orbits, the reversal points of which form an infinite sequence of points lying on the upper and lower branches of the boundary curve $P(x, \lambda; \gamma_1)=0$, and approaching the two reversal points of the outer principal periodic orbit as limit points.[13] Since every such self-reversing orbit must have emanated from the earth in one sense before it reached the earth in the opposite sense, the direction of incidence for every such orbit must be forbidden in consequence of the earth's shadow effect. Since this argument applies to every orbit in this sequence, including the limiting asymptotic orbit, one is led to the following theorem 6: *If any half-orbit of the first kind, possessing an arbitrary number of reentrant sections, is a limit on one side of half-orbits coming from infinity unobstructed by the earth, then it must be a limit on the other side of self-reversing half-orbits which are blocked by the earth.* It is thus evident that immediately beyond every boundary formed by orbits of the first kind, whether it be a boundary of the main cone or of some allowed region of the penumbra, there must exist an enveloping forbidden region of at least infinitesimal size, associated with orbits emanating from, or in the shadow of, the earth.[14] It is further evident that the criterion for a discontinuous shadow effect, by which we have ascertained if any given orbit of the first kind may be a generator of the allowed cone, includes as a very special case the criterion of boundedness set forth in the preceding paragraph.

It is interesting to note that, while the condition of boundedness is essential for the existence of the forbidden directions discussed in Section 4, the criterion of boundedness may be dispensed with entirely in the present case where an impenetrable earth is postulated, without in any way affecting the resulting form of the allowed cone. For in the latter case theorem 5 leads to the result that every bounded orbit reaching a given point of the earth must have previously emanated from the earth. That is, wherever in the latter case the criterion of boundedness may be applied, the criterion of shadow also applies. However, since the assumption that all cosmic-ray particles originate at infinity is at present the commonly accepted one, we are retaining here the point of view that the Störmer cone and the nondense set of directional loci discussed in Section 4 are forbidden irrespectively of the earth's shadow effect; although there is the alternative possibility of assuming cosmic-ray particles to originate also at finite points, and therefore of abandoning the criterion of boundedness.

6

Lemaitre[11] has defined as orbits of the second kind another class of orbits which may be generators of the allowed cone. These orbits differ from orbits of the first kind in that they are associated with a continuous shadow effect. Their definition, which of course must require the presence of an

[13] G. Lemaitre, reference 11, p. 171, has in another connection shown that certain asymptotic orbits are limits of a sequence of periodic orbits.

[14] G. Lemaitre, reference 11, p. 165, has stated that an asymptotic orbit may separate orbits coming from infinity from other orbits coming from the earth, and may therefore be a generator of the allowed cone.

impenetrable earth, will be restated here. Consider any general half-orbit reaching a given point on the earth and having a finite number n of reentrant sections. Such an orbit will have at least n minima and maxima of x. If none of its minima lies within the earth, this half-orbit will have come unobstructed from infinity and its direction of incidence at the given point will be allowed. If any of its minima lies within the earth, the orbit will be in shadow and its direction of incidence at the given point will be forbidden. Between these two cases there may exist a continuous transition, marked by the presence of a half-orbit whose lowest minimum of x is tangent to the earth. All such orbits are termed orbits of the second kind.

We have stated above that an n-reentrant half-orbit reaching a given point of the earth will have at least n minima and maxima of x. The excess minima and maxima, when they exist, arise in much the same way for all half-orbits, independently of the number of reentrant sections possessed by them. One is thus led to distinguish a wide class of orbits of the second kind which are tangent to the earth at one of these excess minima. These orbits will be termed simple orbits of the second kind. Since they may have an arbitrary number of reentrant sections, they may be common generators of the main cone and of all allowed penumbral regions. The continuous locus of their directions of incidence at a given point will be called the simple shadow cone. In the following paper a complete analysis is made of this cone.

7

A few simple considerations of the topology of orbits having various numbers n of reentrant sections readily reveal the existence of distinct families of such orbits, the topological structure of which principally depends on the number n. These considerations further show that all such families having finite values of n form whole continua of similar orbits. The simplest example of this fact is the well-known family of nonreentrant half-orbits associated with the main cone. Such considerations led the writer three years ago to infer the existence, for suitable energies and at suitable points on the earth, of allowed directional regions in the penumbra ranging from infinitesimal to finite size.[15] The existence of the infinite sequence of self-reversing reentrant orbits described in Section 5 enables one in particular to infer the existence of a corresponding infinite sequence of alternating allowed and forbidden regions expanding outward from the main cone, and ranging from infinitesimal to finite size. Not only is it characteristic of the main cone to be bordered by such an infinite sequence of forbidden and allowed regions, but it is characteristic of each allowed member of this sequence, and in fact of any penumbral allowed region, to be similarly bordered. These qualitative inferences, particularly with regard to the existence of finite allowed penumbral regions, have recently been confirmed by Dr. R. Albagli, whose quantitative results on the structure of the penumbra will be published shortly.

It is possible to give a more definitive form to the necessary condition mentioned above for the existence of allowed directional regions in the penumbra. Clearly this necessary condition applies only in the presence of an impenetrable earth, and then it may be expressed in the statement that there must exist reentrant half-orbits coming to a given point unobstructed by the earth. We shall see that not all points (x, λ) in the meridian plane (and hence not all energies and points on the earth) admit this condition for a given value of γ_1. In fact it is possible to establish the following theorem 7: *For any given γ_1 in the range $0.78856 \leqq \gamma_1 \leqq 1$ there exists an upper limiting value x^0 beyond which no point (x, λ) may be reached by reentrant orbits coming from infinity unobstructed by the earth.* Moreover theorem 1 indicates a direct method[16] for ascertaining this x^0 for each γ_1, which has been quantitatively applied by Dr. R. Albagli and will be described in more detail in her forthcoming paper. We shall give one proof[17] of theorem 7 here. Consider any

[15] G. Lemaitre, reference 11, p. 171, speaks of an infinite number of penumbral bands with a point of accumulation, approaching zero size at this point. These bands are again referred to by Lemaitre and Vallarta in their subsequent papers, e.g., Phys. Rev. **47**, 435 (1935); Phys. Rev. **49**, 720 (1936).

[16] As will be seen in the following proof of theorem 7, the values of x^0 may be found from a knowledge of the asymptotic family. This family has been generated, as a limiting case, in the course of a general program of generating reentrant orbits in accordance with theorem 1, undertaken by Dr. Albagli.

[17] The writer's original proof was constructed before theorem 1 had been established, and was based on certain

equatorial point lying between the two principal periodic orbits, and the pencil of reentrant sections generated at that point in accordance with theorem 1. This pencil of reentrant sections may be continued until a locus of first minima of x is reached. Now this locus of first minima has the following properties. As the equatorial point is moved continuously from the inner to the outer principal periodic orbit, there arises a continuous sequence of loci of first minima, bounded on the left by the locus of first minima of the asymptotic family and on the right by the inner principal periodic orbit. Let us identify the above-mentioned limiting value x^0 with the greatest value of x on the locus of first minima of the asymptotic family, and let us further consider the orbit tangent to the earth through any point (x, λ') for which $x \geqq x^0$. By the above results every such orbit, if reentrant at all, must be reentrant on both sides of this tangent point (x, λ'). Hence no point (x, λ) for which $x \geqq x^0$ may be reached by reentrant orbits coming from infinity and having all their minima outside the earth. This completes the proof. By use of the above-mentioned relation between x^0 and γ_1 it has been possible to restate theorem 7 in the following equivalent form: *For every point (x, λ) such that $e^x/2\gamma_1 = r \geqq 0.414$ there exists an upper limiting $\gamma_1{}^0 \leqq 1$ beyond which no penumbral allowed directions exist.*[18] From the foregoing discussion it may be seen in particular that all points (x, λ) outside the limiting periodic orbit[5] for $\gamma_1 = 0.78856$ (i.e., for which $r \geqq 0.70389$) have no allowed penumbral directions whatsoever.

Theorem 7 indicates that with decreasing latitude and increasing energy the penumbral region tends toward a limiting condition of complete darkness. Together with a few simple considerations of the force function $P(x, \lambda; \gamma_1)$ it indicates, conversely, that with increasing latitude and decreasing energy the penumbral region tends toward a limiting condition of complete light. Lastly, it indicates that for intermediate latitudes and energies the penumbral region will exhibit its most significant phase, marked especially by the presence of alternating finite allowed and forbidden regions.

Interpreting these results physically, we may draw the following broad conclusions. For low latitudes and high energies the total allowed cone reduces essentially to the main cone of Lemaitre and Vallarta. For intermediate latitudes and energies it exhibits its most complex form, and, as early conjectured by the writer, may be expected to lead to a highly anomalous behavior in the directional distribution of cosmic-ray intensities.[19] For high latitudes and low energies it reduces essentially to the simple shadow cone, to be described in the following paper.

A few general remarks should finally be made concerning the foregoing theorems. Wherever they contain quantitative statements, it should be pointed out that the numerical expressions used are the results of numerical or mechanical integration. Except for one type of argument, on certain continuity properties of the integrals of motion, there is no further dependence upon such calculations. It is conceivable that the whole qualitative discussion of these theorems may be given independently of such calculations, and with complete mathematical rigor. Such a method of approach has been attempted, but has not been sufficiently developed for presentation here.

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properties of the asymptotic family derived from the results of Lemaitre and Vallarta. See reference 2, p. 722, for a description of the asymptotic families and their envelopes.

[18] G. Lemaitre, reference 11, p. 173, has stated that the limit $\gamma_1 = 1$ is thus reached only for values of $r < 0.414$.

[19] E. J. Schremp, Phys. Rev. **53**, 915A (1938).