On The Characteristic Matrices of Covariant Systems

We wish to indicate here how the consideration of certain matrices implicit in a system of tensor equations gives rise to an algebraic formalism both elegant and significant. No attempt to attain generality shall be made at this time, but rather we shall discuss a particular system of current physical interest.

Proca's tensor wave equation[1] is given by the system

$$n_{jklm}p_l\varphi_m=\mu f_{jk},\qquad p_jf_{jk}=\mu\varphi_k, \tag{1}$$

where $n_{jklm}=(\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})$, $p_k=\hbar\partial/i\partial x_k$ and $\mu=m_0c/i$. Let ψ be a single column matrix whose elements are the components of the vector ϕ_j and the distinct components of the tensor f_{jk}. Then Eq. (1) may be written

$$\Sigma\beta_jp_j\psi=\mu\psi. \tag{2}$$

The symbols β_j are ten-row Hermitian matrices which are found to satisfy the following identities:

$$\begin{aligned}&\beta_j^3=\beta_j,\\ &\beta_j\beta_k^2+\beta_k^2\beta_j=\beta_j, \qquad j\neq k,\\ &\beta_j\beta_k\beta_l+\beta_l\beta_k\beta_j=0, \qquad j\neq k, \qquad l\neq k.\end{aligned} \tag{3}$$

It is easy, then, to deduce this theorem. A necessary and sufficient condition for the existence of Eqs. (3) is that

$$\Sigma_{j,\,k,\,l}\beta_j\beta_k\beta_lc_{jkl}=\Sigma_{j,\,k}\beta_jc_{jkk} \tag{4}$$

for every c_{jkl} such that $c_{jkl}=c_{lkj}$. The application of this theorem to Eq. (2) gives the relations $\Sigma p_j^2\psi=\mu^2\psi$ and $\Sigma s_{jk}p_k\psi+p_j\psi=\mu\beta_j\psi$. Here s_{jk} is the matrix $\beta_j\beta_k-\beta_k\beta_j$ and it satisfies the identities:

$$\begin{aligned}&s_{jk}^3=-s_{jk},\\ &s_{jk}\beta_l-\beta_ls_{jk}=\begin{cases}0 & k\neq l\\ \beta_j & k=l.\end{cases}\end{aligned} \tag{5}$$

Another result deducible from Eq. (4) is that, if the symbols β_j are formally transformed as components of a vector, then the relations (3) are covariant under such a transformation.

To attach quantum theoretic significance to these matrices and to the symbol ψ, let us adopt the usual formalism associated with the notation employed here, excepting the postulate that $\psi^*\psi$ is "charge density." Instead we shall suppose that $\psi^*\psi$ is the density of some other mechanical quantity, M, and that if R is an operator, $\int\psi^*R\psi dV$ is the average value of MR. If ψ^* is the matrix adjoint to ψ, then $\psi^*\psi$ when written in terms of ϕ_j and f_{jk} proves to be the expression given by Proca and Kemmer for the relative energy density.

From Eqs. (5) it follows that the operator $(x_1p_2-x_2p_1)-i\hbar s_{12}$ is permutable with the operator of Eq. (2); hence $-i\hbar s_{12}$ is to be interpreted as a component of the spin. The matrix is_{12} has the eigenvalues 1, 0, and -1. The relationship between the spin and the polarization of the vector ϕ_j for plane wave solutions of Eq. (1) is of interest. A circularly polarized wave gives the eigenstates ±1 in the direction of propagation, while a longitudinal wave gives the eigenstate 0 in the same direction. For a wave half-longitudinal and half-plane polarized there are eigenstates ±1 (approximately for low energies) in a direction perpendicular both to the direction of propagation and to the direction of polarization. For all types of low energy waves $-(s_{12}^2+s_{23}^2+s_{31}^2)\psi\cong2\psi$.

The so-called scalar wave equation is given by the system $p_j\phi_j=\mu f$ and $p_jf=\mu\phi_j$ where f is a scalar and ϕ_j is a vector. The matrices here have five rows and again satisfy Eqs. (3) and in addition the relation $\beta_j\beta_k\beta_l=0$. It is again found necessary to add a spin operator to the orbital momentum operator to obtain commutation. However, the spin is "dormant," because for a plane wave solution the average of the spin operator proves to be zero.

Finally we shall give an example of a tensor wave equation with characteristic matrices abstractly isomorphic to the Dirac matrices. This is given by Eqs. (1) if n_{jklm} now signifies the tensor $(\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}+\delta_{jk}\delta_{lm}+\epsilon_{jklm})$. Here ϵ_{jklm} is $+1$ or -1, depending on whether or not $jklm$ is an even or an odd permutation of 1234 and is zero otherwise. Because ϵ_{jklm} is not invariant in sign under certain reflections, there is an equipollent set of equations with ϵ_{jklm} replaced by $-\epsilon_{jklm}$. The matrices have eight rows and columns.

R. J. Duffin

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December 1, 1938.

[1] Proca, J. de phys. et le rad. **7**, 347 (1936); Dirac, Proc. Roy. Soc. **A155**, 477 (1936); Kemmer, Proc. Roy. Soc. **A166**, 127 (1938).

Acknowledgment: Direct Determination of Crystal Structure from X-Ray Data*

I regret that my attention was drawn too late to a paper by H. Ott in the Zeits. f. Krist. **66**, 136 (1927), to include mention of it in the paper on the "Direct Determination of Crystal Structure from X-Ray Data," in the August 15 issue of the *Physical Review*. In this paper Ott gives what is essentially the same method that I have used. Though he confines himself explicitly to the special case of lattices with all atoms alike, and to special reflections, he seems to have indicated the generalization in a subsequent note. Other work which I had not seen, based on Ott's but special in treatment, is that of K. Banerjee, Proc. Roy. Soc. **141**, 188 (1933), and H. Seyfarth, Zeits. f. Krist. **67**, 131, 422, 595 (1927).

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November 17, 1938.

* Publication assisted by the Ernest Kempton Adams Fund for Physical Research of Columbia University.

On the Paths of Ions in the Cyclotron

Thomas[1] has recently shown that a variation of the magnetic field of a cyclotron with polar angle, together with a radial increase in the field, can produce a focusing effect on the beam while maintaining resonance. For polar angle variations of period 2π or π, the orbits are unstable,