The Spacing of Energy Levels in Light Nuclei*

LLOYD MOTZ
Columbia University, New York, New York

AND

EUGENE FEENBERG
New York University, Washington Square College, New York, New York

(Received August 29, 1938)

The single particle model supplemented by a semi-empirical determination of the dependence of potential energy on symmetry character is used to obtain statistical information about the distribution of energy levels in light nuclei ($A \leq 16$). Table II shows the analysis of the low-lying configurations for $A=16$ into states with definite values of the orbital angular momentum and the symmetry character. Similar tables which we have prepared for values of $A<16$ have been omitted because of their length. Tables III to VI give the spin multiplet structure as a function of the symmetry type and the isotopic number $|N-Z|$. Finally, Tables VII and VIII show the dependence of the level spacing on total angular momentum and on $|N-Z|$ in the several isobaric series with $A=10$, 11, 12, and 13.

The most frequently occurring value of the orbital angular momentum is $L=2$ (D states). Levels with total angular momentum $F=2$ (even nuclei) and $F=5/2$ (odd nuclei) are most closely spaced. The level spacing is not a monotonic decreasing function of A; for example, in the series $A=4n$ and $|N-Z|=0$, Be^8 has 87 levels, C^{12} 192 and O^{16} has 60 in the energy range 0–23 Mev. In the series $A=4n$ the level density is considerably greater for $|N-Z|=2$ than for $|N-Z|=0$; the reverse is true in the series $A=4n+2$. For odd nuclei the level density is practically independent of $|N-Z|$ up to an excitation energy of 20 Mev, but for higher excitation energies it increases with $|N-Z|$.

INTRODUCTION

BARDEEN and Feenberg[1] have calculated the level spacing in intermediate nuclei using the statistical model supplemented by Wigner's[2] semi-empirical theory of the dependence of potential energy on symmetry character. The results are not immediately applicable to light nuclei ($A \leq 16$) because the statistical model does not provide a suitable set of single particle levels in this case. In place of the statistical model we utilize here the closely related model of single particle orbits determined by the motion of the individual particles in a spherically symmetrical potential well. The state of a particle is specified by the quantum numbers n, l, m_l, m_s. Protons and neutrons are treated on exactly the same footing.

The arrangement of the energy levels for the single particle model has been worked out by Elsasser[3] for an infinite well and by Margenau[4] for a finite well. In both cases the order of the levels is the same; the effect of the finite potential is to lower and compress all the levels which fall below the top of the well.[5]

The equations which determine the distribution of single particle levels in a finite potential well are given in reference 4. In the present calculations R, the radius, and V, the depth of the well, are chosen so that V is at least as large as the maximum kinetic energy in a completely degenerate neutron gas of 146 particles. This maximum kinetic energy can be computed from the relation

$$N=32\pi^2(RP)^3/9h^3, \tag{1}$$

where N is the number of neutrons and P is the maximum momentum. Placing $R=BA^{\frac{1}{3}}e^2/mc^2$, we obtain

$$\frac{1}{2M}P^2=\frac{18.7}{B^2}(N/A)^{\frac{2}{3}}mc^2. \tag{2}$$

With $A=238$ and $N=146$ (uranium), the maximum kinetic energy becomes $(13.5/B^2)\,mc^2$. If we choose $R=1.7\times10^{-13}A^{\frac{1}{3}}$ cm, B takes on the value 0.61, so that we have

$$\text{Maximum kinetic energy}=36.4\,mc^2 = 18.5 \text{ Mev}. \tag{3}$$

* Publication assisted by the Ernest Kempton Adams Fund for Physical Research of Columbia University.

[1] J. Bardeen and E. Feenberg, Phys. Rev. **54**, 809 (1938).
[2] E. Wigner, Phys. Rev. **51**, 947 (1937).
[3] W. M. Elsasser, J. de phys. et rad. **4**, 549 (1933).
[4] H. Margenau, Phys. Rev. **46**, 613 (1934).
[5] H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 173 (1936), Fig. 8.

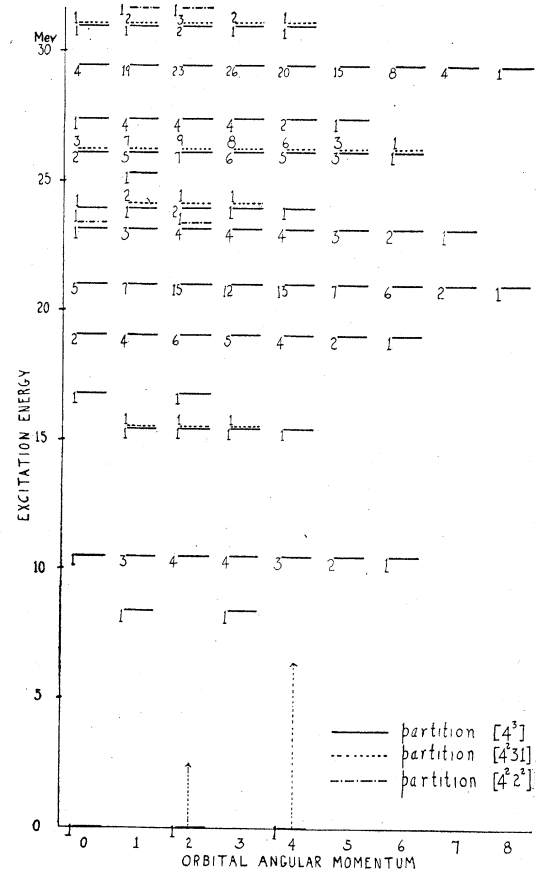

FIG. 1. The energy level diagram for C^{12}. The number to the left of each level is the multiplicity of the level in the configuration of space orbits. The dotted arrows indicate the splitting which occurs within the normal state configuration and the partition $[4^3]$ in the Hartree approximation (references 7, 8).

If we adopt this value for V, we find that the separation between the $1s$ and $2p$ levels in N^{15} is of the order of magnitude found experimentally for the lowest excitation energy of N^{15}, *viz:* six Mev, so that this procedure seems to be reasonable. To find the positions of all the levels for each nucleus, we should have to solve the equations given by Margenau.[4] Since we are not interested in the details of the single particle model, however, but rather in the rough statistical features, an approximate solution will suffice.

The influence of the finite depth of the well on the positions of the levels can be represented fairly well by a change of scale, i.e., by multiplying the levels found by Elsasser for a well of finite depth by a factor $C(A)$. This factor can be computed from the numerical results for the levels $(1s)$ and $(2p)$.

With $V=18.6$ Mev and $A=15$, we find

$$C(15)=0.621 \quad (1s \text{ level})$$
$$C(15)=0.595 \quad (2p \text{ level}).$$

If the variation of $C(A)$ for different levels is neglected, the dependence on A is given with sufficient accuracy in Table I. For the finite well the spacing between single particle levels is proportional to the factor $C(A)/A^{\frac{2}{3}}$ which varies less rapidly with A than $A^{-\frac{2}{3}}$.

THE SYMMETRY PROPERTIES OF NUCLEAR CONFIGURATIONS

The linear manifold in function space generated by the complete set of configurations containing a particles in the $1s$ shell, b particles in the $2p$ shell, etc. is transformed into itself by rotations and permutations of coordinates. Consequently, this manifold can be expressed as a sum of invariant subspaces which are associated with definite values of the total angular momentum and with definite symmetry types. Methods for determining the angular momentum values and the symmetry types contained in a configuration have been described and applied in several recent papers.[6, 7]

We have prepared tables of the multiplet structure (angular momentum values and symmetry types) associated with a number of low lying configurations for all the nuclei from $A=4$ to $A=16$. These tables include all the excited configurations up to about 30 Mev excitation energy for the heavier nuclei and considerably higher excitation energies for the lighter nuclei. Following the convention of atomic

[6] E. Wigner, Phys. Rev. **51**, 106 (1937).
[7] F. Hund, Zeits. f. Physik **105**, 220 (1937).

TABLE I. *Values of $C(A)$ for various values of A.*

A	4	5	6	7	8	9	10	11	12	13	14	15	16
$C(A)$	0.453	0.487	0.514	0.538	0.555	0.568	0.580	0.591	0.601	0.609	0.615	0.621	0.625

spectra, we use the symbols S, P, D, F, G, H, I, J, K for the angular momentum values 0, 1, 2, 3, 4, 5, 6, 7, 8, respectively. The various (irreducible) symmetry types are denoted in the usual manner by the partition symbols $[\lambda_1\lambda_2\cdots\lambda_l]$.

An example of an energy level diagram constructed from the tables is shown in Fig. 1. All levels associated with a given configuration and symmetry type are plotted with the same excitation energy. In making such diagrams one must take into account the total excitation energy of the configuration, which is equal to the sum of the kinetic energy excitations (denoted in the upper left-hand corners of Table II as Kinetic Excitation Energy), and the potential energy, which depends in a simple manner on the symmetry character. The latter term, which is indicated in Table II as Symmetry Displacement, can be computed from the formula

$$U=\tfrac{1}{2}A(A-1)\left\{\frac{\chi(P)}{\chi(1)}-\left(\frac{\chi(P)}{\chi(1)}\right)_0\right\}\mathfrak{L}(A) \quad (4)$$

found by Wigner[2] for the symmetry dependent part of the potential energy. Here $\chi(P)$ and $\chi(1)$ are the characters associated with an interchange and with the identity, respectively. The subscript 0 indicates the state of greatest symmetry permitted by the exclusion principle. The matrix element $\mathfrak{L}(A)$ must be determined empirically. To obtain numerical values for $\mathfrak{L}(A)$, we use the Li^8-Be^8 and $B^{12}-C^{12}$ mass differences in the relations

$$U(8)=4\mathfrak{L}(8)=(Li^8-Be^8)+(H^1-n^1)+\text{the Coulomb Energy Difference } (Be^8-Li^8), \quad (5)$$

$$U(12)=4\mathfrak{L}(12)=(B^{12}-C^{12})+(H^1-n^1)+\text{the Coulomb Energy Difference } (C^{12}-B^{12}). \quad (6)$$

With estimated Coulomb energy differences[8] of 1.5 Mev in Eq. (5) and 2.5 Mev in Eq. (6), the results are

$$\mathfrak{L}(8)\sim 4.2 \text{ Mev}, \qquad \mathfrak{L}(12)\sim 3.9 \text{ Mev}. \quad (7)$$

The value of $\mathfrak{L}(12)$ is uncertain by ± 0.2 Mev because the B^{12} mass value has not been definitely established. We have used the mean of the upper and lower bounds given by Livingston and Bethe.[9] Although $\mathfrak{L}(A)$ must fall off asymptotically as $1/A$, a much smaller variation for small values of A is not unreasonable.[2]

An estimate of the level density can be obtained from diagrams such as Fig. 1 if one knows the spin multiplicity associated with each symmetry type. Tables III, IV, V, and VI, computed by the methods described in references 1 and 6 give the structure of the spin multiplet as a function of $|N-Z|$ and the symmetry character. All partitions which differ only in the number of four groups have the same spin multiplet structure; also interchange of the numbers of three groups and unit groups leaves the spin multiplet structure unchanged. Thus, for example, the irreducible representations [44331], [331], and [311] are all associated with the same spin values.

[8] E. Feenberg and M. Phillips, Phys. Rev. **51**, 597 (1937).
[9] M. S. Livingston and H. A. Bethe, Rev. Mod. Phys. **9**, 373 (1937).

TABLE II. *The multiplet structure for $A=16$.*

Kinetic Excitation Energy in Mev	Configuration	Symmetry Displacement 0	$4\mathfrak{L}$	$6\mathfrak{L}$	$8\mathfrak{L}$	$10\mathfrak{L}$
		Partition [44]	[4331]	[4322]	[432112] [423312]	[423212]
0	$(1s)^4(2p)^{12}$	S				
9.02	$(1s)^4(2p)^{11}(3d)$	FDP	FDP			
13.32	$(1s)^4(2p)^{11}(2s)$	P	P			
16.24	$(1s)^3(2p)^{12}(3d)$	D	D			
18.04	$(1s)^4(2p)^{10}(3d)^2$	$IHG^4F^3D^6P^2S^3$	$IH^3G^6F^8D^9P^7S^3$	$IHG^3F^2D^4PS^2$	$H^2G^2F^5D^3P^5$	GFD^2PS
19.85	$(1s)^4(2p)^{11}(4f)$	GFD	GFD			
20.54	$(1s)^3(2p)^{12}(2s)$	S	S			
22.34	$(1s)^4(2p)^{10}(3d)(2s)$	$GF^2D^3P^2S$	$G^2F^4D^6P^4S^2$	GFD^2PS	$GF^2D^3P^2S$	FDP
25.26	$(1s)^3(2p)^{11}(3d)^2$	$HG^2F^3D^3P^3S$	$H^2G^4F^6D^6P^6S^2$	HGF^2DP^2	$HG^2F^3D^3P^3S$	GFD^2PS
26.65	$(1s)^4(2p)^{10}(2s)^2$	DPS	$D^2P^2S^2$	DS	DPS	P

TABLE III. *The spin multiplet structure for* $|N-Z|=0$.

SYMMETRY TYPE [PARTITION]	SPIN 0	1	2	3	4
[2]	1	1			
[1^2]	1	1			
[31]	1	2			
[2^2]	2	1	1		
[21^2]	1	3	1		
[1^4]	1	1	1		
[321]	2	4	2		
[31^3]	1	3	2		
[2^3]	2	2	1	1	
[2^21^2]	2	4	3	1	
[21^4]	1	3	3	1	
[3^21^2]	2	3	3		
[32^21]	3	6	4	2	
[321^3]	2	6	6	2	
[2^4]	3	2	2	1	1
[2^31^2]	2	6	4	3	1
[2^21^4]	2	4	6	3	1

TABLE IV. *The spin multiplet structure for* $|N-Z|=2$.

SYMMETRY TYPE [PARTITION]	SPIN 0	1	2	3	4
[2]	1				
[1^2]		1			
[31]	1	1			
[2^2]	1	1			
[21^2]	1	2	1		
[1^4]		1	1		
[321]	2	3	1		
[31^3]	1	2	2		
[2^3]	2	1	1		
[2^21^2]	1	4	2	1	
[21^4]	1	2	3	1	
[3^21^2]	1	3	2		
[32^21]	3	5	3	1	
[321^3]	2	5	5	2	
[2^4]	2	2	1	1	
[2^31^2]	2	5	4	2	1
[2^21^4]	1	4	5	3	1

DISCUSSION OF THE TABLES

The multiplet structure for $A=16$ is shown in Table II. Similar tables which we prepared for $A<16$ are omitted because of their great length. An examination of these tables reveals that the number of levels with given orbital angular momentum attains a maximum at $L=2$ (D states) for almost all the configuration and partitions. A closely related dependence on the total angular momentum is found when the spin multiplicities (Tables III–VI) are taken into account. For even values of A the number of levels with given total angular momentum ($\mathbf{F}=\mathbf{S}+\mathbf{L}$) has a maximum at $F=2$; for odd values of A the maximum occurs at $F=5/2$. This is shown in Tables VII and VIII.

The tables indicate a general increase of level density with n in each nuclear series $A=4n$, $4n\pm1$, $4n+2$, up to about $A=12$ and then a sharp decrease. An estimate of this effect in the $4n$ series can be obtained by counting all the levels up to 23 Mev excitation energy for Be^8, C^{12}, and O^{16}. In making the count, the multiplicity of levels with definite values of the total angular momentum is determined from the orbital angular momentum and spin multiplicities in the usual manner. The numbers found are 87, 192 and 60, respectively, standing in the ratios 5 : 11 : 3.4. In the $4n+2$ series the maximum occurs at $A=10$, and in the $4n\pm1$ series at $A=11$. The reason for the decrease in the number of levels for A larger than 12 is to be found in the operation of the exclusion principle, which becomes very effective in cutting down the number of levels associated with the low lying configurations when the p shell is

TABLE V. *The spin multiplet structure for* $|N-Z|=1$.

SYMMETRY TYPE [PARTITION]	SPIN 1/2	3/2	5/2	7/2	9/2
[1]	1				
[21]	2	1			
[1^3]	1	1			
[31^2]	2	2			
[2^21]	3	2	1		
[21^3]	2	3	1		
[321^2]	4	5	2		
[31^4]	2	3	2		
[2^31]	4	3	2	1	
[2^21^3]	3	5	3	1	
[3^21^3]	3	4	3		
[32^21^2]	6	8	5	2	
[321^4]	4	7	6	2	
[2^41]	5	4	3	2	1
[2^31^3]	4	7	5	3	1

TABLE VI. *The spin multiplet structure for* $|N-Z|=3$.

SYMMETRY TYPE [PARTITION]	SPIN 1/2	3/2	5/2	7/2	9/2
[21]	1				
[1^3]		1			
[31^2]	1	1			
[2^21]	2	1			
[21^3]	1	2	1		
[321^2]	3	3	1		
[31^4]	1	2	2		
[2^31]	3	2	1		
[2^21^3]	2	4	2	1	
[3^21^3]	2	3	2		
[32^21^2]	5	6	3	1	
[321^4]	3	5	5	2	
[2^41]	4	3	2	1	
[2^31^3]	3	6	4	2	1

nearly or completely filled. It is clear that beyond O^{16} the level density should again rise rapidly with increasing A. Thus both C^{12} and Ne^{20} should have more levels in a given energy interval than O^{16}.

Tables VII and VIII show the dependence of level density on total angular momentum and $|N-Z|$ in the isobaric series with $A=10$, 11, 12 and 13. The results for the dependence on $|N-Z|$ are very similar to those found in reference 1 for nuclei of intermediate mass. One sees, for example, that the energy levels of B^{10} and B^{12} should be more closely spaced than those of Be^{10} and C^{12}, respectively. For odd nuclei the level density should increase with increasing $|N-Z|$ for sufficiently high excitation energies. This effect does not appear in Table VIII, where the dependence of level density on $|N-Z|$ is shown to be almost constant, because we have stopped at 23 Mev excitation energy. In the range 23–31 Mev the level density for $|N-Z|=3$ is double that for $|N-Z|=1$. In Tables VII and VIII the total number of energy levels obtained by counting all the levels for all values of the total angular momentum is included.

TABLE VII. *The dependence of the level spacing on total angular momentum and on $|N-Z|$ for the $4n$ and $4n+2$ isobaric series. Total number of levels in the range 0–23 Mev excitation energy.*

F	$A=10$ $\|N-Z\|=0$	$\|N-Z\|=2$	$A=12$ $\|N-Z\|=0$	$\|N-Z\|=2$
0	76	34	17	95
1	201	78	32	250
2	261	108	48	318
3	245	89	37	287
4	177	66	30	205
5	101	31	14	132
6	46	14	10	73
7	16	3	3	29
8	4	1	1	8
9	1			1
Total	1128	424	192	1398

TABLE VIII. *The dependence of the level spacing on total angular momentum and on $|N-Z|$ for the $4n\pm1$ isobaric series. Total number of levels in the range 0–23 Mev excitation energy.*

F	$A=11$ $\|N-Z\|=1$	$\|N-Z\|=3$	$A=13$ $\|N-Z\|=1$	$\|N-Z\|=3$
1/2	96	92	84	79
3/2	156	158	136	119
5/2	175	168	140	122
7/2	143	135	113	104
9/2	92	87	73	66
11/2	48	45	41	33
13/2	19	18	18	13
15/2	5	5	6	4
17/2	1	1	1	1
Total	735	709	612	541

A different total number of levels is obtained if one first counts all the levels belonging to the various L values in a partition and then multiplies this number by the spin multiplicity of the partition being considered. The numbers obtained in this way are in all cases slightly larger than those given in the tables.

Recently an alpha-particle molecular model has been used by Wheeler[10] to construct energy level diagrams for C^{12} and O^{16}. The molecular model and the single particle model agree in predicting that the angular momentum value which occurs most frequently is $L=2$. However the former model gives O^{16} a level density double that of C^{12}, whereas with the latter model the same ratio of level densities is less than one-third. The two models also disagree in the magnitude of the level density for C^{12}, the molecular model yielding the greater density by a factor between two and three; agreement can be obtained by a reasonable adjustment of the parameters (nuclear radius, elastic force constants) available in the two methods of calculation.

[10] J. A. Wheeler, Phys. Rev. **52**, 1083 (1937).