Transition Effects of Cosmic Rays in the Atmosphere

H. SNYDER
University of California, Berkeley, California
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In this paper we give an improved treatment of the multiplication and absorption of cosmic-ray electrons and gamma-rays in the atmosphere. After a brief recapitulation, in I, of the principal processes involved in the cascade theory of showers and the resulting diffusion equations, we give, in II and III, a solution of these diffusion equations, making no further approximations. In IV, an expression for the probable number of charged particles is given; the values of the functions necessary for numerical calculations are given in a tabular form. In V, it is shown how the Bhabha-Heitler cut-off energy method for estimating the effect of ionization losses may be consistently used. In VI, a comparison is made with the Carlson-Oppenheimer calculations. The differences near the top of the atmosphere are small and in a direction to improve the agreement with observations. At sea level the new calculations give an intensity roughly $\frac{2}{3}$ of the Carlson-Oppenheimer results.

RECENT experiments with ionization chambers and vertical coincidence counters have extended the study of cosmic-ray intensities nearly to the top of the atmosphere. In this paper we shall return to the question of a theoretical estimate of the intensity to be expected on the assumption that cosmic rays consist only of electrons, positrons, and gamma-rays. It is known that this assumption fails to give an adequate description of the observations, and that other phenomena, which have to do with the production and absorption of the "penetrating" component, must be essentially involved. We want here to improve the theoretical treatment of the multiplication and absorption of the "soft" radiation, in the hope that this will help, when further experimental results are available, to throw light on the behavior of the penetrating component.

Carlson and Oppenheimer[1] have given an expression for the average number of charged particles that would be expected at a given depth below the top of the atmosphere, provided only that this distance is not too small. In their calculations they made two types of approximations: (a) Analytic forms for the probabilities of pair production and gamma-radiation which closely approximate the quantum theoretical expressions in the high energy region were used; (b) Whereas, in calculating the energy distribution of emitted gamma-rays, the theoretical spectrum with constant intensities was used, a distribution corresponding to constant probability for various quanta was used to estimate the direct effect of radiation on the energy distribution of the charged particles. We have found it possible to calculate the probable number of charged particles at a given depth without making the second approximation, and to extend the calculation to the case of small thicknesses. The approximations (a) are very good, and these have been retained.

I

For convenience, a brief resume of the principal processes will be given. The three essential processes which will be considered are: (1) production of gamma-rays by charged particles, (2) pair production by gamma-rays, (3) ionization losses of the electrons and positrons. The Compton recoil of electrons under impact by photons was not explicitly introduced, because *in air* at the low energies for which pair production gives too few particles, the recoil electrons absorb enough gamma-radiation to keep the gamma-ray absorption coefficient nearly constant. The Compton effect produces one particle per photon, whereas pair production gives two. Either process contributes the same quota to the ionization, since such charged particles have energies so low that they are stopped by ionization losses so quickly that they radiate very little. No consideration will be given to other processes, as production of high energy secondaries or pairs in the nuclear fields by charged particles, because they are relatively much rarer, and because the

[1] Carlson and Oppenheimer, Phys. Rev. **51**, 220 (1937). Referred to as C.O.

resultant particles would be nearly as penetrating as the incident one.

The analytically amenable expressions for the probabilities of pair production, emission of gamma-radiation, and ionization losses, which were chosen by Carlson and Oppenheimer will be used for these calculations.

The probability that an electron or positron of energy greater than E shall radiate a photon of energy between E and $E+\Delta E$, while passing through matter a distance Δx, will be taken to be

$$P\Delta E\Delta x=(K/E)\Delta E\Delta x. \tag{1}$$

If the atomic number of the matter is Z and the density of atoms is N,

$$K=\frac{4Z^2e^6N}{\hbar m^2c^5}\ln(200/Z^{\frac{1}{3}}).$$

The probability that a gamma-ray of energy E shall produce a pair having energies between E' and $E'+\Delta E'$, and $E-E'$ and $E-E'-\Delta E'$, will be taken to be

$$P'\Delta E'\Delta x=(K'/E)\Delta E'\Delta x' \tag{2}$$

with $E'<E$, and for which

$$K'/K=\sigma\sim\tfrac{2}{3}.$$

The ionization losses will be assumed to be independent of the energy and will be evaluated at energies for which they are important. If one regauges the measure of distance by the transformation $t=Kx$, the energy loss by ionization per unit distance becomes

$$\beta=(4\pi NZe^4/Kmc^2)\ln(\beta/ZRh). \tag{3}$$

Let $N(t, E)$ be the probable number of charged particles of energy greater than E to be found at a thickness t. Further, take $\gamma(t, E)\Delta E$ to be the probable number of photons having energies between E and $E+\Delta E$ at t. Then, in accordance with (1), (2) and (3), diffusion equations take the form

$$\partial\gamma/\partial t=-\sigma\gamma+N/E, \tag{4}$$

$$\begin{aligned}\frac{\partial N}{\partial t}=\beta\frac{\partial N}{\partial E}+2\sigma\int_E^\infty dE'\int_{E'}^\infty\frac{\gamma(E'')}{E''}dE''&+\int_0^E\frac{N(E+E')-N(E)}{E'}dE'\\&+\int_E^\infty\frac{N(E+E')-N(E')}{E'}dE'.\end{aligned} \tag{5}$$

If one supposes that the incoming radiation is a charged particle of energy E_0, the boundary conditions which the solution to (4) and (5) must satisfy are

$$\gamma(t=0, E)=0;\quad N(t=0, E)=\begin{matrix}1 & E<E_0\\ 0 & E>E_0.\end{matrix} \tag{6}$$

II

An exact solution of Eqs. (4) and (5) with the boundary conditions (6) may be found for $\beta=0$. If one writes

$$\gamma_0=e^{-\sigma t}z_0/E, \tag{7}$$

then it follows that

$$N_0=e^{-\sigma t}\dot{z}_0. \tag{8}$$

In Eq. (8) and succeeding equations a dot will represent differentiation with respect to t. The substitution of (7) and (8) in (5) gives rise to the integral-differential equation

$$\begin{aligned}K(z_0)=\ddot{z}_0-\sigma\dot{z}_0-2\sigma\int_E^\infty dE'\int_{E'}^\infty\frac{z_0(E'')}{(E'')^2}dE''&\\-\int_0^E\frac{\dot{z}_0(E+E')-\dot{z}_0(E)}{E'}dE'&\\-\int_E^\infty\frac{\dot{z}_0(E+E')-\dot{z}_0(E')}{E'}dE'&=0.\end{aligned} \tag{9}$$

Simple solutions of the form $z_0=e^{\mu t}E^{-y}$ exist, provided μ is one of the roots of the quadratic equation

$$\mu^2-(\sigma+L(y))\mu-2\sigma/y(y+1)=0, \tag{10}$$

in which

$$L(y)=\int_0^1\frac{(1+x)^{-y}-1}{x}dx+\int_1^\infty\frac{(1+x)^{-y}-x^{-y}}{x}dx=-[\psi(y)+\gamma] \tag{11}$$

$$\psi(y)=(d/dy)\ln\Gamma(y+1);\quad\gamma=0.577\cdots.$$

The roots of (10) are

$$\begin{matrix}\mu\\ \nu\end{matrix}=\frac{\sigma+L(y)}{2}\pm\left[\left(\frac{\sigma+L(y)}{2}\right)^2+\frac{2\sigma}{y(y+1)}\right]^{\frac{1}{2}};\quad \mu>\nu. \tag{12}$$

The general solution of (9) is now given by

$$z_0=\frac{1}{2\pi i}\int_C \frac{dy}{y}\left(\frac{E}{E_0}\right)^{-y}[A(y)e^{\mu t}+B(y)e^{\nu t}] \quad (13)$$

if the contour C and $A(y)$ and $B(y)$ are determined so that this integral exists.

To satisfy the boundary conditions (6) one must have

$$z_0(t=0,\,E)=0;\quad \dot{z}_0(t=0,\,E)=\begin{matrix}1 & E<E_0\\ 0 & E>E_0.\end{matrix}$$

The contour for the integration with respect to y was chosen to be a straight line parallel to the imaginary axis, at a distance δ to the right, and from $-i\infty+\delta$ to $i\infty+\delta$.

Equation (13) may then be written

$$z_0=\frac{1}{2\pi i}\int_C \frac{dy}{y}e^{\lambda y}[A(y)e^{\mu t}+B(y)e^{\nu t}]$$

$$\text{with}\quad \lambda=\ln(E_0/E).$$

From $z(t=0,\,E)=0$, it must be true that

$$B(y)=-A(y). \quad (14)$$

The second boundary condition leads to

$$\frac{1}{2\pi i}\int_C \frac{dy}{y}e^{\lambda y}A(y)[\mu-\nu]=\begin{matrix}1 & \lambda>0\\ 0 & \lambda<0.\end{matrix} \quad (15)$$

Multiplying (15) by $e^{-y_0\lambda}$ and integrating with respect to λ from 0 to ∞, one gets

$$\frac{1}{2\pi i}\int_C \frac{dy}{y}A(y)\frac{\mu-\nu}{y-y_0}=\frac{1}{y_0}\quad \text{for}\quad R(y_0)>0. \quad (16)$$

The integral in (16) may be evaluated by taking the residue at $y=y_0$, from which one obtains

$$A(y)=1/(\mu-\nu). \quad (17)$$

The solution satisfying the boundary conditions (6) is:

$$z_0=\frac{1}{2\pi i}\int_C \frac{dy}{y}e^{\lambda y}\frac{e^{\mu t}-e^{\nu t}}{\mu-\nu}. \quad (18)$$

Equation (18) is an analytic expression of the solution of the cosmic-ray problem considered by Bhabha and Heitler.[2]

[2] Bhabha and Heitler, Proc. Roy. Soc. **A159**, 432 (1937).

III

If one takes as before

$$\gamma=e^{-\sigma t}z/E;\quad N=e^{-\sigma t}\dot{z}, \quad (19)$$

a solution of (4) and (5) with $\beta\neq 0$ which satisfies the boundary conditions (6) to terms of order β/E_0 may be found. The equation which z must satisfy is:

$$K(z)=\beta\partial\dot{z}/\partial E \quad (20)$$

with the operator K defined in Eq. (9).

In analogy with the work of Carlson and Oppenheimer, we look for simple solutions of (20) which depend upon a parameter, say y, and which may be combined to satisfy the boundary conditions. We set

$$z(y)=\frac{e^{\mu t}}{2\pi i}\int_s ds\left(\frac{E}{E_0}\right)^{-y}\left(\frac{E}{\beta}\right)^{-s} \times C(y,\,s)\frac{\Gamma(-s)\Gamma(y+s)}{\Gamma(y)}. \quad (21)$$

The contour for the integration with respect to s is from $-i\infty$ to $i\infty$. The contour is kept to the left of the poles at 0, 1, 2, etc. and to the right of the poles at $-y$, $-y-1$, $-y-2$, etc. We look for a $C(y,\,s)$ which for $R(y)>0$ is free of singularities in the strip $0\leq R(s)\leq 1$, since one can see that this condition is necessary to make the total number of particles in the shower finite. Substituting (21) into (20), one obtains:

$$\int_s ds\left(\frac{E}{E_0}\right)^{-y}\left(\frac{E}{\beta}\right)^{-s}C(y,\,s)\frac{\Gamma(-s)\Gamma(y+s)}{\Gamma(y)} \times\{\mu^2-(\sigma+L(y+s))\mu-2\sigma/(y+s)(y+s+1)\}$$

$$=-\int_s ds\left(\frac{E}{E_0}\right)^{-y}\left(\frac{E}{\beta}\right)^{-s-1}C(y,\,s)\frac{\Gamma(-s)\Gamma(y+s)}{\Gamma(y)} \times\mu(y+s). \quad (22)$$

In order that the integrals over E obtained by this substitution converge, it is necessary that $R(y+s)>0$, although the expressions thus obtained may be analytically extended into the left half-plane. One may now note that the coefficient of $C(y,\,s)$ in the left-hand side of (22) differs from that of the right side by a first power in E. If one then takes the $C(y, s)$ to satisfy the functional relation

$$\{\mu^2-(\sigma+L(y+s))\mu-2\sigma/(y+s)(y+s+1)\}\times C(y, s)=\frac{s(y+s-1)\mu C(y, s-1)}{(y+s)}. \quad (23)$$

(21) is at least a formal solution of (20). To insure that (21) be a solution of (20), y will be restricted to $R(y)>0$, and μ will be taken to satisfy (10). When these conditions are satisfied, one may deform the contour for the integration with respect to s for the last term in (22) and show that (21) is a solution of (20).

If μ^2 is eliminated from (23) by means of (10), it may be written:

$$\{\mu[\psi(y+s)-\psi(y)]+2\sigma[1/y(y+1)-1/(y+s)(y+s+1)]\}C(y, s)=\frac{s(y+s-1)\mu}{(y+s)}C(y, s-1). \quad (24)$$

Thus for $R(y)>0$, $C(y, s)$ is free of singularities for $-1-R(y)<R(s)$.

As there are two roots to (10), there are two linearly independent solutions of (20) for each value of y. The $C(y, s)$ corresponding to the two roots μ and ν will be denoted by $C_\mu(y, s)$ and $C_\nu(y, s)$. In addition, the C's will be chosen so that $C_\mu(y, 0)=C_\nu(y, 0)=1$. A solution satisfying the boundary conditions (6) to terms of order β/E_0 may now be written:

$$z=-\frac{1}{4\pi^2}\int_C \frac{dy}{y}e^{\lambda y}\int_s ds\left(\frac{E}{\beta}\right)^{-s}\frac{\Gamma(-s)\Gamma(y+s)}{\Gamma(y)}\times\left\{\frac{C_\mu(y, s)e^{\mu t}-C_\nu(y, s)e^{\nu t}}{\mu-\nu}\right\} \quad (25)$$

for evaluating the integral over s in terms of the residues of the poles at 0, 1, 2, etc., we find

$$z=\frac{1}{2\pi i}\int_C \frac{dy}{y}e^{\lambda y}\frac{e^{\mu t}-e^{\nu t}}{\mu-\nu}-\frac{1}{2\pi i}\int_C dy e^{\lambda y}\left(\frac{\beta}{E}\right)\times\left\{\frac{C_\mu(y, 1)e^{\mu t}-C_\nu(y, 1)e^{\nu t}}{\mu-\nu}\right\}+0(\beta/E)^2+\cdots. \quad (26)$$

From (26) it follows that (25) satisfies the boundary conditions (6) for $E\gg\beta$, at least to order β/E; this suggests that the total energy of the shower is E_0 to terms of order β. This is shown to be true in Eq. (41). Further, the total number of particles of energy greater than zero at $t=0$ determined from (25) is $1+0(\beta/E_0)$, as $\beta/E_0\rightarrow 0$, which follows from Eqs. (28) and (34). It is only for large E_0/β that (25) satisfies the boundary conditions, and it is only for this case that we shall calculate the total number of particles.

IV

For the purpose of calculating the number of charged particles in the shower, deform the contour to the left. There are no singularities between $s=0$ and $-y$. In the limit $E=0$, the only contribution to the integral comes from the simple pole at $s=-y$. One then has

$$z(E=0)=\frac{1}{2\pi i}\int_C \frac{dy}{y}e^{\epsilon y}\left\{\frac{C_\mu(y, -y)e^{\mu t}-C_\nu(y, -y)e^{\nu t}}{\mu-\nu}\right\};\quad \epsilon=\ln(E_0/\beta). \quad (27)$$

From Eqs. (19) and (27), one can easily obtain

$$N=\frac{e^{-\sigma t}}{2\pi i}\int_C \frac{dy}{y}e^{\epsilon y}\times\left\{\frac{C_\mu(y, -y)\mu e^{\mu t}-C_\nu(y, -y)\nu e^{\nu t}}{\mu-\nu}\right\}. \quad (28)$$

It is convenient to divide (28) into two parts,

$$N_\mu=\frac{1}{2\pi i}\int_C \frac{dy}{y}\left[\frac{\mu C_\mu(y, -y)}{\mu-\nu}\right]\cdot e^{\epsilon y+(\mu-\sigma)t} \quad (29)$$

and

$$N_\nu=-\frac{1}{2\pi i}\int_C \frac{dy}{y}\left[\frac{\nu C_\nu(y, -y)}{\mu-\nu}\right]e^{\epsilon y+(\nu-\sigma)t}. \quad (30)$$

The first part, (29), gives the principal contribution to the shower. In the second, ν is less than zero for all y on the contour, so (30) falls off exponentially with increasing t.

For large values of t and ϵ, (29) may be calculated by the saddle point method. The quantity

$$\mu C_\mu(y, -y)/(\mu-\nu) \quad (31)$$

is a slowly varying function of y, and may be taken from under the integral sign and evaluated at the saddle point. One then obtains

$$N_\mu = \frac{C_{\mu_s}(y_s, -y_s)\mu_s e^{\epsilon y_s + (\mu_s - \sigma)t}}{[2\pi(\mu_s'' y_s^2 t + 1)]^{\frac{1}{2}}(\mu_s - \nu_s)} \tag{32}$$

with the saddle point determined by

$$\mu_s' t + \epsilon - 1/y_s = 0, \tag{33}$$

in which $\mu' = d\mu/dy$; $\mu'' = d\mu'/dy$.

In fact, (32) gives satisfactory results even for small t. For small values of t and large values of ϵ the saddle point determined by (33) is at y very near zero, so it is possible to approximate μ by $(2\sigma/y)^{\frac{1}{2}}$. When this is effected, (29) may be evaluated in a rapidly converging series, and one obtains

$$N_\mu = \frac{C_{\mu_s}(y_s, -y_s)\mu_s}{(\mu_s - \nu_s)} e^{-\sigma t} \sum_{n=0}^{\infty} \frac{[(2\sigma\epsilon)^{\frac{1}{2}} t]^n}{\Gamma(\frac{1}{2}n+1)\Gamma(n+1)}. \tag{34}$$

Numerical calculation of (34) shows that even for $t=0$ the corrections to be applied to (32) are small; for $t=0$, they amount to a decrease of eight percent.

The function $C_\mu(y, -y)$ is a very complicated but slowly varying analytic function of y for all y near the saddle points determined from (33). It is, however, easily calculated from (24) for integral values of y. Its values for these points were plotted, and a smooth interpolating curve drawn through them.

If the substitutions

$$H(y) = (2\pi)^{-\frac{1}{2}} C_\mu(y, -y)\mu/(\mu - \nu), \qquad k = \mu - \sigma, \quad a = -\mu' y, \quad b = \mu'' y^2 \tag{35}$$

are made in Eqs. (32) and (33), they may be written in the simple forms

$$N_\mu = (1 + bt)^{-\frac{1}{2}} H e^{kt + \epsilon y} \tag{36}$$

and

$$t = (\epsilon y - 1)/a. \tag{37}$$

In Table I, the values of a, b, k, and H are given over a range of values sufficiently large to cover showers from the top of the atmosphere to sea level. The value of t given by (37) increases uniformly as a function of y.

For air the value of β is about 90 Mev. t is in units of 0.4 m water equivalent.

N_ν is completely negligible for values of t greater than 0.5. For values of t less than this, it is easy to estimate its contribution to the size of the shower: (a) $N_\mu = \frac{1}{2}$, $N_\nu = \frac{1}{2}$ at $t=0$; (b) at $t=0$ the slope of N_ν is equal and opposite to that of N_μ; (c) N_ν falls exponentially to a half-value of $\frac{1}{4}$ at about $t=0.2$.

TABLE I.

y	k	a	b	H
0.2	1.8876	1.485	2.06	0.286
0.4	0.9538	1.207	1.51	0.330
0.6	0.4944	1.063	1.41	0.351
0.8	0.2026	0.957	1.37	0.352
1.0	0.0000	0.857	1.35	0.342
1.2	−0.1479	0.762	1.33	0.320
1.4	−0.2588	0.669	1.30	0.294
1.6	−0.3418	0.581	1.26	0.267
1.8	−0.4055	0.504	1.18	0.243
2.0	−0.4542	0.428	1.08	0.221
2.2	−0.4920	0.366	0.99	0.201
2.4	−0.5215	0.313	0.91	0.185

V

We will now look at Eq. (18) and see if we can find a rational way in which to introduce a low energy cut-off by means of which we can estimate the total size of the shower. This is the procedure suggested by Bhabha and Heitler.[2] We introduce E_c, supposed independent of t, and assume that the energy distribution of particles of energy greater than E_c is not disturbed by the ionization losses, and that because of the ionization losses there are no particles of energy lower than this. Since the initial energy of the shower E_0 must be absorbed by ionization, we must have

$$\int_0^\infty N_0(E_c)dt = E_0/\beta$$

$$= \frac{1}{2\pi i}\int_0^\infty dt \int_c \frac{dy}{y} \frac{\mu e^{(\mu-\sigma)t + \lambda_c y}}{\mu - \nu} \tag{38}$$

with $\lambda_c = \ln (E_0/E_c)$.

These integrals are easy to evaluate if we neglect terms of order β/E_0; then (38) gives

$$E_c = \left[\frac{\mu(1)(-1)}{[\mu(1) - \nu(1)]\mu'(1)}\right]\beta$$

$$= \frac{1}{C_\mu(1, -1)}\beta = 0.467\beta. \tag{39}$$

In accordance with (39), the number of particles in the shower becomes

$$N_0=\frac{(0.467)^{-y_s}\mu_s e^{\epsilon y_s+(\mu_s-\sigma)t}}{[2\pi(\mu_s''y_s^2t+1)]^{\frac{1}{2}}(\mu_s-\nu_s)} \tag{40}$$

with the saddle point determined as before by Eq. (37). This equation, (40), is of the same form as (32) with $C(y, -y)$ replaced by

$$\left[-\frac{\mu'(1)(\mu(1)-\nu(1))}{\mu(1)}\right]^y=\left(\frac{1}{0.467}\right)^y.$$

For $y=0$ and 1, these give identical results, and for $y=2$ they give 3.28 and 4.58, respectively. It should be observed that the proper "cut-off energy" is less than half β.

Calculating the total energy of the shower, as given by (29), and again neglecting terms of order β/E_0, we get for this energy

$$-\frac{C_\mu(1,\ -1)\mu(1)}{[\mu(1)-\nu(1)]\mu'(1)}E_0=E_0. \tag{41}$$

VI

In their work, Carlson and Oppenheimer replace the last two terms of Eq. (5) by an approximate expression which enabled them to find a differential equation for z. The approximation was made in such a manner that the number of particles lost in one energy range appeared in some other, and so that the average energy lost by radiation by the charged particles was gained by the gamma-rays. The approximation distorts the energy distribution among the charged particles so that there are too many very high energy particles, too many low energy particles, and too few of intermediate energy ($\sim\beta$). In Fig. 1 the effect of this distortion on the results for the average number of particles is shown for $E_0=100\beta$. The curve marked I was obtained from the C. O. calculations, and the other, marked II, from Eqs. (36) and (37). Their approximation: (a) slightly lowers the height of

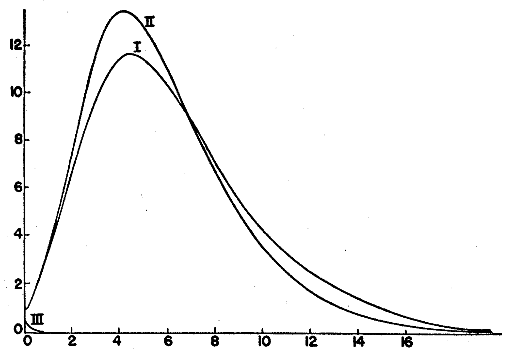

FIG. 1. I is a plot of the number of particles against t as determined from the C. O. calculations. II is a plot of the same as given by Eqs. (36) and (37). III shows the contribution to the shower by N_ν. The calculations were made for $E_0=100\beta$. One unit of t is about 0.4 m water equivalent.

the maximum, (b) moves the maximum about 0.2 m to greater thicknesses, (c) considerably overestimates the size of the shower at great depths. At ten meters depth, near sea level, the C. O. formula gives $N=7.5\times10^{-3}$, and we find $N=4.5\times10^{-3}$, for a 9×10^3 Mev particle with vertical incidence.

The changes in the theoretical curve near the top of the atmosphere help to remove some of the discrepancies found in this region by Bowen, Millikan and Neher[3] between their experimental results, and the calculation of C. O. applied to an isotropic incident distribution. So detailed a comparison can hardly be significant, however, unless it takes into account the large deviations from isotropy to be expected from the effects of the earth's field. Near sea level, on the other hand, our results are about half those of C. O. and thus accentuate still further the discrepancies with the experimental curves, discrepancies clearly associated with the penetrating component.

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[3] Bowen, Millikan and Neher, Phys. Rev. **52**, 80 (1937).