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The Multiplicative Theory of Showers as Applied to Large Bursts of Cosmic-Ray Ionization *

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In order to decide whether the multiplicative theory of showers offers an adequate explanation of the experimental data on large showers, or whether it is necessary to suppose that they are produced by some other process, such as has been outlined by Heisenberg, the following calculations are undertaken. The energy spectrum of the electrons and photons incident upon a small thickness of material is calculated from the observed frequency distribution in size of bursts of ionization produced by large showers, the effect of fluctuations being taken into account. The energy distribution obtained is of the form $E^{-\alpha}$, where α is 2.6 for energies of the order of 10^9 volts, and decreases slowly with increasing energy, in agreement with the energy distribution calculated by Heitler to explain the variation of cosmic-ray intensity with altitude. The maximum number of electrons necessary to give the observed frequency of bursts is less than 0.5 percent of the total number of particles observed in a cloud chamber at sea level in the energy range between 10^9 and 10^{10} volts, which is in good accord with the cloud chamber observations. This calculated incident energy spectrum is utilized to calculate the number and frequency distribution of large bursts for large thicknesses of material. These calculated values differ considerably from the experimental ones, but this difference is probably to be ascribed to the effect of the penetrating cosmic rays, and is not to be regarded as evidence of a breakdown of the cascade theory. The experiments on the absorption of a shower are also shown to be in harmony with the theoretical estimates. It is concluded that no mechanism involving the production of many shower particles in a single act need be invoked to explain the occurrence of large showers, but that the ordinary multiplicative processes are entirely adequate when proper account is taken of the fluctuations.

INTRODUCTION

THE theory of the growth and decay of the secondary radiations which accompany a high energy electron in its passage through matter has been worked out in some detail by Carlson and Oppenheimer[1] and by Bhabha and Heitler.[2] This theory has been applied, with considerable success, to explain the variation of cosmic radiation with latitude and altitude by Heitler[3] and Nordheim.[4] The properties of small showers of rays are also well accounted for in this way. It remains to show whether the large showers, such as produce bursts of ionization, or Hoffmann Stösse, are consistent with this mechanism of production or not. It has been realized for some time that the formation of a large shower is not the result of a single act, but that some sort of a multiplicative process must be involved.[5] To what extent this "production of

* A preliminary report of this paper was presented at the New York meeting of the American Physical Society, February, 1938.

[1] J. F. Carlson and J. R. Oppenheimer, Phys. Rev. **51**, 220 (1937).

[2] H. J. Bhabha and W. Heitler, Proc. Roy. Soc. **A159**, 432 (1937).

[3] W. Heitler, Proc. Roy. Soc. **A161**, 261 (1937).

[4] L. W. Nordheim, Phys. Rev. **51**, 1110 (1937).

[5] C. G. Montgomery, Phys. Rev. **45**, 62 (1934); G. L. Locher, Phys. Rev. **44**, 779 (1933); C. G. Montgomery, D. D. Montgomery and W. F. G. Swann, Phys. Rev. **47**, 512 (1935).

one shower by another" is important has not been clear. On the other hand, Heisenberg[6] has suggested a mechanism whereby a large number of electrons would be produced in the interaction with a single nucleus. It is the purpose of this paper to inquire whether some such mechanism as is postulated by Heisenberg is necessary to explain the experimental properties of large showers, or whether successive pair formation by the radiations resulting from the slowing down of high energy electrons is sufficient.

Calculation of the Energy Distribution of Incident Rays

Suppose we have a piece of heavy material, say lead, placed above an ionization chamber, and we make observations of the rates of occurrence of the bursts of ionization as a function of the size of the burst. We suppose that the ionization produced is proportional to the number of electrons which make up the shower. Then it has been pointed out[7] that the frequency of showers, $R(N)dN$, containing numbers of rays between N and $N+dN$ can be well represented by the empirical expression

$$R(N)=A/N^3.$$

This expression is valid over quite a large range of N, and its form is almost independent of the thickness of the shower producing material. The value of the exponent necessary to obtain a best fit of the data varies somewhat with the observations considered, but we may take it as integral for simplicity.

The cascade theory of showers is given in the most convenient form for calculation by Bhabha and Heitler. They give, in graphical form, the expected number, ϵ, of electrons, positive and negative, of energy greater than a critical value, E_c, ($E_c=10^7$ volts for lead) which emerge from a plate of a given thickness when an incident electron of energy E falls upon the plate. Let us first consider the case of a lead plate 1.2 cm thick. The actual number of rays, N, which emerge from the plate will not, of course, in general be equal to this expected number, ϵ. Furry[8] has shown that, for thin pieces of material,

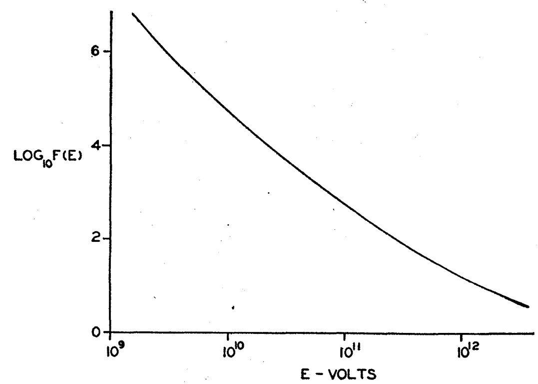

Fig 1. The energy distribution of electrons and photons necessary to produce the observed distribution in size of large showers.

the probability of finding N rays when ϵ rays are expected is given by the expression $(1-1/\epsilon)^{N-1}/\epsilon$. Let us suppose that the number of ion pairs observed in our chamber is proportional to the number of rays above the critical energy. Then if we have a number of electrons $F(E)dE$ of energy between E and $E+dE$ falling upon the lead plate, the number of showers containing N rays will be given by

$$R(N)=A/N^3=\int_{E_c}^{\infty}\epsilon^{-1}(1-\epsilon^{-1})^{N-1}F(E)dE.$$

We thus have an integral equation which can be solved for $F(E)$. An approximate solution of this can easily be obtained. We note that, for a given thickness, ϵ is a function of E only, so that a change of variable gives us

$$R(N)=\int_{1}^{\infty}\epsilon^{-1}(1-\epsilon^{-1})^{N-1}f(\epsilon)d\epsilon.$$

If we assume $f(\epsilon)$ is of the form B/ϵ^s, the integral may be evaluated directly, and we have

$$R(N)=B\Gamma(s)\Gamma(N)/\Gamma(s+N).$$

Now if we equate $B\Gamma(s)=A$ and $s=3$, we obtain

$$R(N)=A/N(N+1)(N+2),$$

which, for large N, is indistinguishable from the experimental A/N^3, and hence $f(\epsilon)=A/2\epsilon^3$ is the desired solution.

We may then immediately apply Bhabha and Heitler's calculations, and find which $F(E)$ corresponds to $f(\epsilon)=A/2\epsilon^3$. Fig. 1 gives the

[6] W. Heisenberg, Zeits. f. Physik **101**, 533 (1936).
[7] C. G. Montgomery and D. D. Montgomery, Phys. Rev. **48**, 786, 969 (1935).
[8] W. H. Furry, Phys. Rev. **52**, 569 (1937).

values of $F(E)$ so calculated. This energy distribution may be considered to be proportional to $E^{-\alpha}$ where α is 2.6 for $E=10^9$ volts, and α decreases with increasing E to a value of 2.1 for $E=10^{11}$ volts. This decrease in α cannot continue indefinitely, of course, since the total energy of the distribution would diverge. It may be shown, however, that if the function $R=A/N^3$ is a good representation of the experimental data from $N=10$ to $N=1000$, then the function $F(E)$ is determined as above from 10^9 to 10^{12} volts. The $F(E)$ calculated here has been spoken of as if it were a distribution of electrons which produced the observed showers. If, on the other hand, we had carried through the calculations assuming that all the observed showers were produced by photons, we would have been led to the same $F(E)$, which, however, would then represent an energy distribution of photons, since the behavior of a high energy photon is almost identical with that of an electron. Actually we know that both electrons and photons produce the showers, and we must regard $F(E)$ as made up of both entities in amounts of comparable importance. Since the shower producing effects are independent of the proportion of electrons and photons, we cannot utilize the shower phenomena to give us any information regarding this proportion.

We may compare our calculated form of $F(E)$ with the energy distribution of electrons at sea level obtained in another way. Heitler,[3] in order to explain the variation of the soft component of the cosmic radiation with elevation, assumes a distribution of electrons entering the earth's atmosphere. This results in an energy distribution of electrons at sea level which is of the form $E^{-2.5}$ approximately.[9] Thus there is good agreement between the two methods of calculation, which are based on quite independent data.

We have determined the form of the energy distribution of the shower producing rays; let us now determine the number of them necessary to produce the observed number of showers. The quantity measured as the size of a shower is the number of ion pairs produced. Since N is the number of rays above the critical energy, E_c, and since there are roughly an equal number of rays above and below the critical energy,[10] we should choose an "effective specific ionization" approximately twice the specific ionization of a single electron. Now there is considerable uncertainty in this quantity, but the maximum value of the constant A will be given by the minimum value of the specific ionization. We shall take this to be 30 ion pairs per cm. We choose for illustration the rate of occurrence of bursts greater than 1.5×10^6 ion pairs produced in a magnesium chamber containing about 15 atmospheres of nitrogen and covered by one centimeter of lead.[11] We are thus dealing with showers containing of the order of 100 rays. These data result in an upper limit for A of 9.8×10^{-3} cm^{-2} sec.$^{-1}$. If we now consider the region of the energy spectrum from 10^9 to 10^{10} volts, we should need only 4.0×10^{-5} electrons and photons per sq. cm per second to account for all the showers produced. The total number of charged particles observed in a cloud chamber in this energy range may be taken from the observations of Blackett,[12] and is 950×10^{-5} cm^{-2} sec^{-1}. Thus at most only 0.5 percent of the particles in this energy range observed in a cloud chamber need be electrons. That there are some electrons is shown by the fact that showers have been observed, in cloud chamber photographs, whose total energies are greater than 10^9 volts[13]. Thus we have good consistency with cloud chamber observations. Similarly even the largest bursts which have been reported can be produced electrons or photons whose number is a negligibly

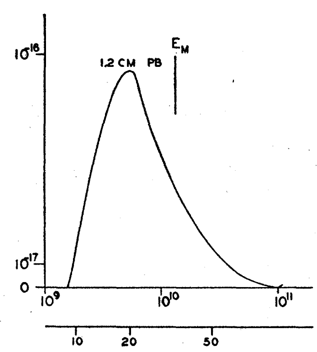

FIG. 2. The contribution to the number of 100 ray showers by the rays of various energies. The vertical line, E_M, marks the median energy.

[9] Reference 3, page 276.

[10] H. J. Bhabha, Proc. Roy. Soc. **A164**, 257 (1938).

[11] C. G. Montgomery and D. D. Montgomery, Phys. Rev. **48**, 786 (1935).

[12] P. M. S. Blackett, Proc. Roy. Soc. **A159**, 1 (1937).

[13] C. D. Anderson and S. H. Neddermeyer, Phys. Rev. **50**, 263 (1936).

small proportion of the total number of particles. We see that it is unnecessary to invoke any special mechanism for the production of large showers such as suggested by Heisenberg.

It is interesting to see what energies are effective in producing showers of a given size. Fig. 2 gives the contribution of each energy to the total number of showers for $N=100$. The lower scale of abscissae gives the expected number of rays, ϵ, for each energy. We note the striking fact that the rays which are most effective in producing 100 ray showers would only be expected to produce about 20 rays, and that the rays for which we expect 100 shower particles give a wholly negligible contribution. Table I shows some characteristics of this distribution. Thus all the observed showers of 100 rays are fluctuations from expectation.

It might seem from a cursory examination of the problem that since $f(\epsilon)$ and $R(N)$ are the same functions of their respective variables, the effect of the fluctuations is only to introduce a small constant factor. However, this is quite illusory since, in the absence of the fluctuation phenomena, the rays which would be effective in producing 100 ray showers would not lie in the energy range between 10^9 and 10^{10} volts, but would have considerably higher energies in the neighborhood of 5×10^{11} volts. Thus in the absence of fluctuations we should need, in order to account for the observed number of bursts, considerably more electrons of energy around 10^9 volts than all the particles of this energy observed. It is the neglect of the fluctuations which has led many investigators,[14] ourselves included,[15] to the view that it is necessary to have some mechanism whereby many shower particles are produced in a single act to account for the large showers from small thicknesses. However, when the fluctuations are properly taken into account, the ordinary cascade picture is entirely adequate.

TABLE I. *Characteristics of distribution in Fig.* 2.

	Energy	ϵ
Most probable	5.0×10^9 volts	20
Median	1.2×10^{10}	35
Mean	4.3×10^{10}	51

[14] Reference 10, page 265; H. Euler, Physik. Zeits. **38**, 943 (1937).

[15] C. G. Montgomery and D. D. Montgomery, Phys. Rev. **50**, 490 (1936).

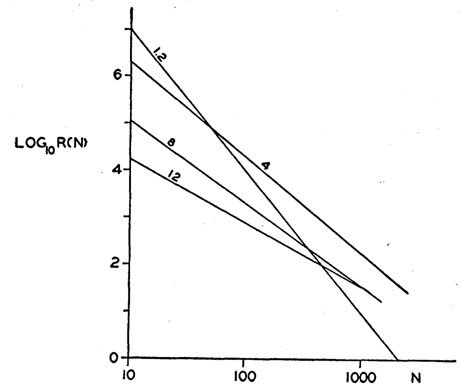

FIG. 3. The variation of frequency distribution in size of showers with thickness of the producing material. The numbers indicate the thickness of lead in centimeters.

The Variation of the Production of Large Showers with Thickness of the Material

We may use the energy spectrum of the incident rays derived above to calculate how many showers would be expected at large thicknesses of material. However, we enter here on more uncertain ground. Although the effect of the fluctuations is all important at small thicknesses, its importance should become less and less as the thickness increases. In the absence of a more accurate knowledge of the effect of fluctuations, the best approximation possible at the present time would appear to be to neglect the fluctuations entirely for large thicknesses. The calculation of the number of showers is then quite straightforward, and we give only the results. Fig. 3 shows the distributions in shower sizes to be expected at several thicknesses of lead. The lines in the figure have been drawn as straight, although they have a slight curvature concave upward.

These results differ considerably from experiment.[16] This disagreement may be expressed in this way: for showers of a given size, the predicted decrease at large thicknesses is much too large, and the shapes of the distribution in size curves, that is, the slopes of the lines in Fig. 3, change too much with thickness. The experimental data do show some decrease in slope, but not nearly as large a one as is indicated here. This is usually expressed as a shift of the maxi-

[16] E.g., R. T. Young, Phys. Rev. **52**, 559 (1937); J. K. Bøggild, Diss. Copenhagen, 1937.

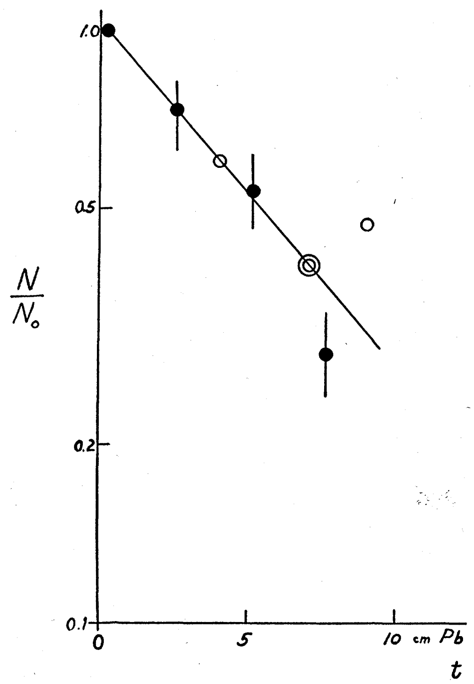

FIG. 4. Observations on the absorption of showers. Open circles: data of H. Nie recalculated by the authors; double circle: authors' measurements, first method; filled circles: authors' measurements, second method.

mum to larger thicknesses in the relation between the number of showers and the thickness of the material, as the size of the showers under consideration increases. These points of disagreement are probably the result of neglecting the effects of the penetrating cosmic radiation, and probably do not represent any fault in the theory of showers produced by electrons or photons. As this point of view has been much discussed previously,[10, 17] we merely wish to emphasize that there need be no discrepancy in the cascade theory of showers here.

THE ABSORPTION OF A SHOWER

Another means of testing the multiplicative theory of showers is to utilize the experiments on the passage of a shower through lead. Such experiments have been performed by Nie[18] and the authors,[19] and lead to results substantially in agreement. Fig. 4 shows the results of these experiments. Again the showers dealt with contain approximately 100 rays. Now, on account of the importance of the fluctuations, we cannot expect to be able to treat this question exactly, but can only obtain a rough estimate of the absorption from the theory. We may represent the experimental points by the equation $N=N_0e^{-\mu t}$, where N is the number of rays in the shower after it has passed through t cm of lead, N_0 the original number of rays ($N_0=100$), and $\mu=\frac{1}{8}$ cm^{-1} of lead. Nordheim[4] has shown that a particular solution of the diffusion equations of Carlson and Oppenheimer is given by

$$P(t, E)=e^{-\mu t}E^{-n},$$

for energies above the critical energy, E_c, with the relation between n and μ:

$$0.4\mu=4/3-1/n-[(\tfrac{2}{3}-1/n)^2+4/3n(n-1)]^{\frac{1}{2}}$$

when μ is measured in cm^{-1} of lead. Now if we suppose that the N_0 rays in the "average" shower have an energy distribution of the form E^{-n}, we can compute what the value of n must be in order that the shower will be absorbed exponentially with the observed coefficient. This leads to the value $n=2.07$. The total energy of the shower is easily seen to be $E=N_0E_c(n-1)/(n-2)$, or for this case, $E=1.5\times10^{10}$ volts. This "average" energy of the shower is equal to the energy of the incident ray, and may be compared with the energies listed in Table I for 100 ray showers. It is seen to be of the correct order of magnitude. Thus, although these calculations are only approximate, here again we obtain agreement with the cascade theory, and find no evidence that any other mechanism is necessary to account for the observed behavior of large showers.

The authors wish to thank Professor W. F. G. Swann for much valuable discussion of these matters.

[17] R. H. Woodward, Phys. Rev. **49**, 711 (1936); H. Euler, ref. 14; R. T. Young and J. C. Street, Phys. Rev. **52**, 552 (1937).

[18] H. Nie, Zeits. f. Physik **99**, 776 (1936).

[19] C. G. Montgomery and D. D. Montgomery, Phys. Rev. **49**, 705 (1936); Zeits. f. Physik **102**, 534 (1936).