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# Internal Friction in Solids

## II. General Theory of Thermoelastic Internal Friction

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Stress inhomogeneities in a vibrating body give rise to fluctuations in temperature, and hence to local heat currents. These heat currents increase the entropy of the vibrating solid, and hence are a source of internal friction. The general theory of this internal friction is here developed. The simplest example of stress inhomogeneity is that occurring in the transverse vibrations of reeds and wires. Explicit formulae are obtained for reeds and wires, and the effect is calculated of crystal orientation in single crystal specimens. Microscopic stress inhomogeneities arise from imperfections, such as cavities, and from the elastic anisotropy of the individual crystallites. The internal friction due to spherical cavities is calculated. The internal friction due to elastic anisotropy is investigated for cubic metals, and is found to be greatest for lead, least for aluminum and tungsten.

### §1. INTRODUCTION

In a recent paper[1] the writer investigated theoretically that part of the internal friction of a vibrating reed which arises from the flow of heat back and forth across the reed. On comparing the calculated values of this thermoelastic internal friction with the experimental values of internal friction for longitudinal vibrations in rods, he predicted that over a wide frequency band the internal friction in reeds due to this thermoelastic effect was of a larger order of magnitude than that due to all other causes. In the succeeding paper an experimental verification of this prediction will be presented. The striking agreement of prediction with experiment renders it opportune to investigate more thoroughly the thermoelastic effect. Such an investigation is here undertaken.

This paper begins with a generalization of the analysis given in reference 1 for transverse vibrations (§2). It is found that for a rod of arbitrary cross section vibrating transversely, with frequency $\nu$,

$$Q^{-1}=\frac{E_S-E_T}{E_S}\cdot\frac{\nu_0\nu}{\nu_0^2+\nu^2}. \tag{1}$$

Here $E_S$ is the adiabatic Young's modulus, $E_T$ the isothermal modulus, both measured along the axis of the wire. In the case of a rectangular cross section (reed), the frequency for maximum $Q^{-1}$, $\nu_0$, is given by

$$\nu_0=(\pi/2)Dd^{-2}.$$

Here $D$ is the thermal diffusion constant of the material, $d$ is the width of the rod in the plane of vibration. In the case of circular cross section (wire),

$$\nu_0=0.539Da^{-2},$$

where $a$ is the radius of the rod, or wire. Eq. (1) is valid for both isotropic and anisotropic rods. Tables are given to show the dependence of $(E_S-E_T)/E_S$ upon crystal orientation in single crystal rods. It is predicted that the internal friction in single crystal zinc rods will vary radically with crystal orientation, differing by a factor of seven for parallel and normal orientation.

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[1] C. Zener, Phys. Rev. **52**, 230 (1937).

The thermoelastic effect is present wherever there are inhomogeneities of stress. . These inhomogeneities may be inherent in the type of vibration, as in the transverse vibrations of reeds and wires, or they may be caused by cavities, by a mixture of two phases, by the random orientation of the individual crystallites, etc. Whatever the cause of the stress inhomogeneity, the thermoelastic effect may be investigated from two viewpoints. We may seek the rate of dissipation of the mechanical energy of vibration. This dissipation is due to the slight difference in phase between the stresses and the corresponding strains. On the other hand, we may seek the rate of generation of heat. Stress inhomogeneities give rise to heat currents, which increase entropy, and an increase of entropy is inevitably associated with generation of heat. In §3 both of these points of view are investigated. The mechanical analysis for the dissipation of vibrational energy, and the thermodynamical analysis for the generation of heat, both lead to the same general expression for the $Q^{-1}$ of a cubic metal with no internal strains, namely

$$Q^{-1}=[(C_p-C_v/)C_v]\cdot\mathcal{R}\cdot\sum f_k[\nu_k\nu/(\nu_k^2+\nu^2)], \quad (2)$$

with $$\sum f_k=1.$$

$C_p$ and $C_v$ are the specific heats at constant pressure and constant volume, respectively. The factor $\mathcal{R}$ is that fraction of the total strain energy which is associated with fluctuations in dilation.

Equation (2) may be most readily interpreted by regarding $Q^{-1}$ not as a function of frequency $\nu$, but of $x=\log_{10}\nu$. We then obtain

$$Q^{-1}=\tfrac{1}{2}[(C_p-C_v)/C_v]\cdot\mathcal{R}\cdot\sum f_k\mathfrak{B}(x-x_k), \quad (3)$$

where $$\mathfrak{B}(x)=\operatorname{sech}\{(\log_e 10)x\}.$$

The $\mathfrak{B}$ function, first introduced by Bennewitz and Rötger,[2] is plotted in Fig. 1. It is symmetrical about its maximum at $x=0$. Each term in the summation is thus a $\mathfrak{B}$ function with the weight $f_k$, shifted so that its maximum is at $x_k$. The associated $\nu_k$ is interpreted as the reciprocal of a time of relaxation for the establishment of temperature equilibrium across a distance $L_k$. From the diffusion equation one finds $\nu_k\approx D/L_k^2$.

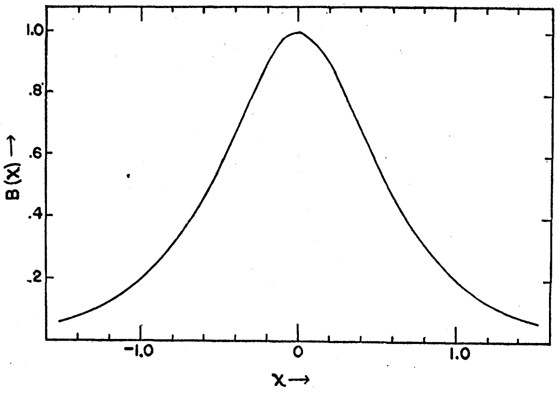

FIG. 1. Plot of the $\mathfrak{B}$ function.

A particularly simple type of stress inhomogeneity is formed by a spherical cavity in a vibrating solid. During vibrations these cavities will cause comparatively large local heat currents. The internal friction resulting therefrom is calculated in §4.

The elastic anisotropy of single crystals, combined with their random orientation, renders it impossible for strains which are macroscopically homogeneous to be also microscopically homogeneous. Hence in a vibrating polycrystalline solid heat currents flow back and forth between adjacent crystallites. The internal friction resulting therefrom may be roughly guessed at from Eq. (3) by replacing the summation by a single $\mathfrak{B}$ function, with the associated $\nu_k$ equal to $D/L^2$, where $L$ is the mean diameter of the crystallites. The ratio $\mathcal{R}$ is estimated in §5 for various polycrystalline metals. It is found that the $Q^{-1}$ for aluminum is lower, for lead is higher, than for any other common metal. This is in agreement with the recent careful experiments of Wegel and Walther,[3] and of Förster and Körster,[4] who find that aluminum has the least, lead the greatest, internal friction of any nonferromagnetic metal investigated.

Two experimental methods are commonly used to measure internal friction. In the first, one determines the number $N$ of oscillations required to reduce the amplitude of free oscillation to $1/e$'th its initial value. The quantity $\delta=1/N$ is called the logarithmic decrement of the solid. In

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[2] K. Bennewitz and H. Rötger, Physik. Zeits. **37**, 578 (1936).

[3] R. L. Wegel and H. Walther, Physics **6**, 141 (1935).

[4] F. Förster and W. Köster, Zeits. f. Metallkunde **29**, 116 (1937).

the second method, one measures the width $\Delta\nu$ of a resonance curve at half-maximum (plot of square of amplitude against frequency). On dividing this width by the frequency at the maximum, one obtains a second dimensionless measure of internal friction. Internal friction may be interpreted theoretically from two points of view. If we regard the elastic modulus as complex, $M+im$, then the ratio $m/M$ is a dimensionless measure of internal friction. Again, if we denote by $E$ the energy of vibration, and by $\Delta E$ the loss per cycle of the energy of vibration, then $\Delta E/E$ is also a dimensionless measure of internal friction. These theoretical measures are related to each other, and to the experimental measures, by the equation

$$Q^{-1}=(1/2\pi)\cdot(\Delta E/E)=m/M.$$

## §2. TRANSVERSE VIBRATIONS OF REEDS AND WIRES

When the width of a rod (wire or reed) is small compared with the wave-length in transverse vibrations, the only stress of importance is the tensile stress along the axis of the rod. This simple stress system enables us to obtain a particularly simple expression for the thermoelastic internal friction of thin rods in transverse vibrations.

When the $x$ axis is taken to lie parallel to the axis of the rod, this tensile stress is denoted by $X_x$. We consider $X_x$, as well as the strain $e_{xx}$ and temperature fluctuation $\Delta T$, to be a harmonic function of time. Hence by substituting

$$X_x=(\partial X_x/\partial e_{xx})_T e_{xx}+(\partial X_x/\partial T)_{e_{xx}}\Delta T$$

in the expression for the energy loss per cycle per unit length,

$$\Delta E=\nu^{-1}\text{ time average of }\int X_x(de_{xx}/dt)d\sigma,$$

we obtain

$$\Delta E=\nu^{-1}\zeta\text{ time average of }\int \Delta T(de_{xx}/dt)d\sigma.$$

In the partial derivative,

$$\zeta=(\partial X_x/\partial T)_{e_{xx}},$$

as well as in all the partial derivatives in this paragraph, the stresses $Y_y$ and $Z_z$ are held constant. The integration is over the cross section of the rod. The temperature fluctuation $\Delta T$ is completely determined by the diffusion equation

$$\partial\Delta T/\partial t=D\nabla^2\Delta T+\eta(\partial e_{xx}/\partial t),$$

and by the appropriate boundary condition. $D$ is the thermal diffusion coefficient of the material, and

$$\eta=(\partial T/\partial e_{xx})_Q.$$

Corresponding to the physical condition that no heat flow across the surface of the rod, this boundary condition specifies that the normal component of the gradient of $\Delta T$ vanish at the surface.

The energy of vibration per unit length is equal to the maximum strain energy per unit length, $E$. We may write this energy in the form

$$\tfrac{1}{2}\epsilon\int e_{xx}^2 d\sigma. \tag{4}$$

The modulus $\epsilon$ may be called the effective Young's modulus. It is the adiabatic modulus $E_s$ for rapid vibrations, the isothermal modulus $E_T$ for slow vibrations. Since these two moduli differ from one another by less than one percent, very little error is made in

$$Q^{-1}=(1/2\pi)(\Delta E/E)$$

by replacing $\epsilon$ by either $E_S$ or $E_T$ throughout the entire frequency range.

By using the evaluation of $\Delta E$ given in the appendix, and by replacing $E$ by (4), we obtain

$$Q^{-1}=(\zeta\eta/\epsilon)\sum f_k[\nu_k\nu/(\nu_k^2+\nu^2)], \tag{5}$$

with $\sum f_k=1.$

The constants $f_k$ and $\nu_k$ are obtained from the eigenfunctions and eigenwert of the differential equation (a-5). (See Appendix.) Both $f_k$ and $\nu_k/D$ are functions only of the shape of the rod. On the other hand, the factor $(\zeta\eta/\epsilon)$ is independent of the shape of the cross section, but depends only upon the thermoelastic constants of the material. Explicit formulae will now be derived for this factor in terms of these constants, and tables will be given for interesting cases.

We first replace $\epsilon$ by the adiabatic modulus.

The factor $(\zeta\eta/\epsilon)$ may then be written explicitly as

$$(\partial X_x/\partial T)_{e_{xx}}(\partial T/\partial e_{xx})_S/(\partial X_x/\partial e_{xx})_S,$$

or in the obviously equivalent form

$$(\partial X_x/\partial T)_{e_{xx}}(\partial T/\partial X_x)_S. \tag{6}$$

This expression may now be transformed in two ways. One transformation leads to the elegant expression $(E_S-E_T)/E_S$, the other leads to an expression containing only experimentally known constants.

Since only two of the four variables $T$, $X_x$, $e_{xx}$, $S$ are independent, we may apply to Eq. (6) the following standard formula in partial differentiation:

$$(\partial x_1/\partial x_2)_{x_4}(\partial x_2/\partial x_1)_{x_3}=1-\frac{(\partial x_2/\partial x_3)_{x_1}}{(\partial x_2/\partial x_3)_{x_4}}.$$

Correlating $x_1$, $x_2$, $x_3$, $x_4$ with $T$, $X_x$, $e_{xx}$, $S$ respectively, we obtain

$$(\zeta\eta/\epsilon)=(E_S-E_T)/E_S. \tag{7}$$

Again, we may transform (6) using the second law of thermodynamics. In the type of vibrations here considered, $X_x$ is the only nonvanishing stress. Hence the total static energy density $\mathcal{E}$, as well as $X_x\, e_{xx}$, is a single valued function of any two of the variables $T$, $X_x$, $e_{xx}$, $S$. A small increment in $\mathcal{E}-X_xe_{xx}$, namely

$$T\,dS-e_{xx}dX_x,$$

is thus a perfect differential. It follows that

$$(\partial T/\partial X_x)_S=-(\partial e_{xx}/\partial S)_{X_x}=-(T/C_p)(\partial e_x/\partial T)_{X_x}.$$

Substituting this equation into (6), and replacing

$$(\partial X_x/\partial T)_{e_{xx}} \quad \text{by} \quad -(\partial X_x/\partial e_{xx})_T(\partial e_{xx}/\partial T)_{X_x},$$

we obtain

$$(\zeta\eta/\epsilon)=TE_T\alpha^2/C_p. \tag{8}$$

Here $\alpha$ is the linear thermal expansion coefficient, $E_T$ is the isothermal Young's modulus, both *measured along the axis of the rod.* $C_p$ is the specific heat, at constant pressure, per unit volume. Comparing Eqs. (7) and (8), we obtain the well-known formula

$$(E_S-E_T)/E_S=TE_T\alpha^2/C_p. \tag{9}$$

If the rod is isotropic, still another expression for $(\zeta\eta/\epsilon)$ may be obtained. When $\epsilon$ is replaced by the isothermal modulus, the appendix of reference 1 shows that

$$(\zeta\eta/\epsilon)=\tfrac{1}{3}(1-2\sigma)(C_p-C_v)/C_v,$$

where $\sigma$ is here Poisson's ratio.

Table I gives the ratio $(E_S-E_T)/E_S$ for various polycrystalline metals, calculated from Eq. (9). Since $E_T$ is affected by fiber structure, which is always present to some extent in cold worked specimens, the ratio for any particular sample may vary considerably from Table I. Since $(E_S-E_T)/E_S$ varies as the square of $\alpha$, it is especially small in those alloys with small coefficients of thermal expansion, such as invar.

In rods of single crystals, $(E_S-E_T)/E_S$ will depend upon the relative orientation of the crystal with respect to the axis of the rod. In single metal crystals with cubic symmetry, $\alpha$ is independent of orientation. Young's modulus is greatest along the [111] axes, least along the [100] axes.[5] Table II has been constructed by substituting into Eq. (9) the maxima and minima values of $E_T$ given in reference 5.

In single metal crystals with hexagonal symmetry, both $\alpha$ and $E_T$ depend only upon the angle $\theta$ between the principal axis of the crystal

TABLE I. *Values of $(E_S-E_T)/E_S$ for polycrystalline metals at 20°C.*

| METAL | $(E_S-E_T)/E_S$ | METAL | $(E_S-E_T)/E_S$ |
|---|---|---|---|
| invar | 0.00002 | Ni | 0.0029 |
| W | .0008 | Cu | .0030 |
| Bi | .0014 | Ag | .0034 |
| Pl | .0015 | Sn | .0040 |
| Au | .0017 | Al | .0046 |
| Sb | .0018 | Mg | .0050 |
| Pd | .0020 | Zn | .0080 |
| Pb | .0024 | Cd | .010 |
| Fe | .0024 | | |

TABLE II. *Values of $(E_S-E_T)/E_S$ at 20°C when crystal axes [111] and [100] are parallel to axis of rod.*

| METAL | DIRECTION PARALLEL TO ROD<br>[111] | <br>[100] |
|---|---|---|
| Cu | 0.0046 | 0.0016 |
| Ag | .0050 | .0019 |
| Au | .0028 | .0010 |
| Al | .0060 | .0050 |
| Fe($\alpha$) | .0024 | .0011 |

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[5] E. Schmid and W. Boas, *Kristallplastizität* (Springer, 1935), p. 202.

TABLE III. *Variation of* $10^3\,(E_S-E_T)/E_S$ *with* $\theta$ *in zinc.*

| $T \backslash \theta$ | 0° | 10° | 20° | 30° | 40° | 50° | 60° | 70° | 80° | 90° |
|---|---|---|---|---|---|---|---|---|---|---|
| 100°K | 6.6 | 6.6 | 6.4 | 5.8 | 5.0 | 3.4 | 1.82 | 0.64 | 0.16 | 0.04 |
| 200°K | 11.2 | 11.4 | 11.2 | 10.6 | 9.6 | 7.6 | 5.2 | 2.6 | 1.4 | 1.0 |
| 300°K | 15.4 | 15.8 | 15.6 | 15.2 | 14.0 | 11.8 | 8.2 | 5.0 | 3.0 | 2.4 |

TABLE IV. *Variation of* $10^3(E_S-E_T)/E_S$ *with* $\theta$ *in cadmium.*

| $T \backslash \theta$ | 0° | 10° | 20° | 30° | 40° | 50° | 60° | 70° | 80° | 90° |
|---|---|---|---|---|---|---|---|---|---|---|
| 100°K | 5.7 | 5.6 | 5.4 | 4.9 | 4.2 | 3.3 | 2.4 | 1.6 | 1.2 | 1.0 |
| 200°K | 9.5 | 9.3 | 8.9 | 8.2 | 7.3 | 6.0 | 4.7 | 3.4 | 2.6 | 2.4 |
| 300°K | 12.0 | 12.0 | 11.6 | 11.0 | 10.2 | 9.1 | 7.8 | 6.2 | 5.3 | 5.0 |

and the axis of the rod. Taking the data for $E_T$ from reference 6, and the data for $\alpha$ from reference 7, we have constructed Table III for zinc and Table IV for cadmium. In a zinc rod whose axis is normal to the principal axis of the crystal, $(E_S-E_T)/E_S$ disappears slightly below 100°K. This is because the linear thermal expansion coefficient normal to the principal axis changes sign at this temperature.

The constants $f_k$ and $\nu_k$ in Eq. (5) have already been evaluated for a rod of rectangular cross section (reed) in reference 1. If $d$ is the width of the rod in the plane of vibration,

$$f_0, f_1, f_2 \cdots = 0.986,\ 0.012,\ 0.0016,\ \cdots \tag{10}$$
$$\nu_0, \nu_1 \cdots = (\pi/2)Dd^{-2},\ (9\pi/2)Dd^{-2},\ \cdots.$$

We shall now evaluate these constants for a rod of circular cross section (wire) of radius $a$.

Following the analysis given in the appendix, we must first find the eigenfunctions $U_k$ and the eigenwert $\nu_k$ of the differential equation (a-5), where in the present case $\nabla^2$ is the two dimensional Laplace operator. These eigenfunctions must satisfy the boundary condition

$$(\partial/\partial r)U_k = 0 \quad \text{at} \quad r = a. \tag{11}$$

We next expand $e_{xx}$ in terms of these eigenfunctions. $f_k$ is then given by

$$\left(\int e_{xx} U_k d\sigma\right)^2 \Big/ \int e_{xx}^2 d\sigma \int U_k^2 d\sigma.$$

The eigenfunctions of Eq. (a-5) and of the boundary condition (11) are of the form

$$\cos(m\varphi) J_m\{(2\pi\nu/D)^{\frac{1}{2}} r\},$$

where $m$ is an integer, and where $\nu$ is one of the roots of the equation

$$J_m'\{(2\pi\nu/D)^{\frac{1}{2}} a\} = 0.$$

Since $\qquad e_{xx} = e_0 r \cos\varphi,$

only those eigenfunctions contribute to the summation in (5) for which $m=1$. From tables[8] of Bessel functions we find the first few roots of

$$(d/dq) J_1(q) = 0$$

to be $\quad q_0, q_1, q_2 \cdots = 1.84,\ 5.33,\ 8.53 \cdots. \qquad (12)$

Setting $q = (2\pi\nu/D)^{\frac{1}{2}} a$, we find the first few allowed values of $\nu$ to be

$$\nu_0, \nu_1, \cdots = 0.539 Da^{-2},\ 4.53 Da^{-2} \cdots.$$

We may now write $f_k$ explicitly as

$$f_k = \frac{\left(\int_0^a r^2 J_1(q_k r/a) dr\right)^2}{\int_0^a r^3 dr \int_0^a r J_1^2(q_k r/a) dr}.$$

With the aid of the formulae[9]

$$\int q^2 J_1(q) dq = q^2 J_2(q)$$

and $\quad \int q J_1^2(q) dq = \frac{1}{2} q^2 \{J_1^2(q) - J_0(q) J_2(q)\},$

we obtain

$$f_k = \frac{8 J_2^2(q_k)}{q_k^2 \{J_1^2(q_k) - J_0(q_k) J_2(q_k)\}}.$$

We now obtain the numerical value of $f_k$ by substituting into this equation the values of $q_k$ given by Eq. (12). The $J_0(q_k)$ and $J_1(q_k)$ can be obtained with sufficient accuracy by interpolation from the usual tables of Bessel functions, $J_2(q_k)$ from the tables in reference 9, p. 252. We obtain

$$f_0, f_1, f_2, \cdots = 0.988,\ 0.010,\ 0.0013 \cdots.$$

Hence with a circular wire, as with a reed, we are justified in replacing $f_0$ by unity, and all the other $f_k$'s by zero. This approximation, which

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[6] E. Gruneisen and E. Goens, Zeits. f. Physik **26**, 245 (1924).

[7] E. Gruneisen and E. Goens, Zeits. f. Physik **29**, 147, 148 (1924).

[8] E.g., G. N. Watson's *Theory of Bessel Functions* (Cambridge Press, 1922), pp. 666–695.

[9] E. Jahnke and F. Emde, *Funktionentafeln* (Teubner, 1933) pp. 213–214.

eliminates all except the first term in the summation of Eq. (5), will probably be valid for all rods with a regular cross section.

## §3. General Theory

In the expression for the internal friction of a solid,

$$Q^{-1}=(1/2\pi)(\Delta E/E), \tag{13}$$

$\Delta E$ may be interpreted either as the mechanical work $\Delta W$ required per cycle to keep the solid in steady vibration, or as the increment per cycle of the heat content $\Delta Q$. We shall investigate $\Delta E$ from both points of view, directly calculating both $\Delta W$ and $\Delta Q$.

Consider the vibrations of the solid to be a simple harmonic function of time. In each element of volume $dv$ a set of coordinate axes may then be found with respect to which the shearing strains $e_{yz}$, $e_{zx}$, $e_{xy}$ remain zero. The work performed upon an element of volume by giving the three tensile strains small increments is

$$dwdv=(X_x de_{xx}+Y_y de_{yy}+Z_z de_{zz})dv.$$

$dw/dt$ will be equal to the sum of three terms of the type $X_x de_{xx}/dt$. The net work performed upon the element of volume during one cycle is thus equal to the sum of three terms of the type

$$\nu^{-1} \text{ time average of } X_x(de_{xx}/dt)dv.$$

In order to evaluate these time averages, express each stress as the sum of two terms. The first term is the value the stress would have if the temperature remained constant at its average value $T_0$. This term is in phase with the strains, and hence contributes nothing to the above time average. The second term is due to the deviation $\Delta T$ of the temperature from its average value. In the time average the stress $X_x$ may thus be replaced by $(\partial X_x/\partial T)\Delta T$, where in the partial derivative it is understood that all the strains are kept constant. We now introduce a simplification by assuming that the elastic and thermal constants of the element of volume have cubic symmetry. This assumption not only limits us to solids with cubic symmetry, but also to solids which have no internal strains other than that due to vibrations. This simplification enables us to replace each of the three derivatives of the type $(\partial X_x/\partial T)$ by $-(\partial p/\partial T)_\theta$, where $p$ refers to a hydrostatic pressure, and $\theta$ refers to the dilation, $e_{xx}+e_{yy}+e_{zz}$. We now obtain the net work performed upon the whole solid per cycle by integrating $dwdv$ over the solid.

$$\Delta W=-\nu^{-1}(\partial p/\partial T)_\theta \text{ time average of } \int \Delta T(d\theta/dt)dv. \tag{14}$$

The temperature fluctuation $\Delta T$ is completely determined by the differential equation

$$d\Delta T/dt=D\nabla^2\Delta T+(\partial T/\partial\theta)_Q(d\theta/dt) \tag{15}$$

and by the boundary condition that the normal component of the gradient of $\Delta T$ vanish at the surface of the solid.

We shall now calculate the rise in heat content per cycle, $\Delta Q$, using only thermodynamical considerations. The rate at which the entropy of an elementary volume $dv$ changes with time is equal to $T^{-1}$ times the rate at which heat flows into the element of volume:

$$T^{-1}\sigma\nabla^2 T dv,$$

where $\sigma$ is the thermal conductivity of the substance. Since the fluctuations in temperature are very small, we may write $\Delta Q$ as $T_0^{-1}\Delta S$, where $\Delta S$ is the entropy change of the whole solid per cycle. Summing the entropy changes in all the elementary volumes, we obtain

$$\Delta Q=\nu^{-1}\sigma T_0 \text{ time average of } \int T^{-1}\nabla^2 T dv.$$

In the integrand $T^{-1}$ is now replaced by $T_0^{-1}-T_0^{-2}\Delta T$. Eq. (15), valid only for solids with cubic symmetry, is now used to replace $\nabla^2 T$ in the integrand by

$$D^{-1}\{d(\Delta T)/dt-(\partial T/\partial\theta)_Q d\theta/dt\}.$$

The only term in the integrand which does not vanish upon taking the time average is $T_0^{-2}\Delta T(\partial T/\partial\theta)_Q d\theta/dt$. Using the relation

$$\sigma/D=C_v,$$

where $C_v$ is the specific heat per unit volume, and using the thermodynamic formula

$$T^{-1}C_v(\partial T/\partial\theta)_Q=-(\partial p/\partial T)_\theta,$$

we obtain finally

$$\Delta Q = -\nu^{-1}(\partial p/\partial T)_\theta \text{ time average of } \int \Delta T(d\theta/dt)dv. \quad (16)$$

We have, from thermodynamical considerations, obtained an expression for $\Delta Q$ which is identical with the $\Delta W$ obtained from mechanical considerations.

On comparing Eq. (14) or (16), and Eq. (15) with the integral solved in the appendix, we find for $\Delta E(=\Delta W=\Delta E)$

$$\Delta E = -\pi(\partial p/\partial T)_\theta(\partial T/\partial\theta)_Q \int \Theta^2 dv \times \sum f_k[\nu_k\nu/(\nu_k{}^2+\nu^2)],$$

with $\sum f_k = 1.$

Here $\Theta$ is the maximum value of the dilation $\theta$ during a cycle. The energy associated with this maximum dilation is

$$-\tfrac{1}{2}(\partial p/\partial\theta)_T \int \Theta^2 dv.$$

This expression is exact only for slow isothermal vibrations. However, only very little error is made by using the isothermal differential for all frequencies. Let $\mathcal{R}$ denote the ratio of this maximum strain energy associated with dilation to the maximum of the total strain energy, $\Delta E$. We obtain

$$\Delta E/E = 2\pi\{(\partial p/\partial T)_\theta(\partial T/\partial\theta)_Q/(\partial p/\partial\theta)_T\}\mathcal{R} \times \sum f_k[\nu_k\nu/(\nu_k{}^2+\nu^2)].$$

In the appendix of reference 1 it is shown that the bracketed expression is equal to $(C_p-C_v)/C_v$. Since $\Delta E/E=2\pi Q^{-1}$, we obtain finally

$$Q^{-1}=[(C_p-C_v)/C_v]\mathcal{R}\sum f_k[\nu_k\nu/(\nu_k{}^2+\nu^2)]. \quad (17)$$

## §4. INTERNAL FRICTION CAUSED BY SMALL CAVITIES

In this paragraph Eq. (17) is used to calculate the internal friction caused by stress inhomogeneities about a spherical cavity in a vibrating isotropic solid. In this calculation the description of the strains about a spherical cavity is taken from Love.[10]

Since a general strain may be represented as the superposition of a pure dilation and of a pure shear, we need only consider the two special cases where the medium, in the absence of the cavity, has a pure dilation, and where it has a pure shear. When a spherical cavity in a medium of infinite extent is subjected to hydrostatic pressure, the strains about the cavity are pure shears, the dilation being exactly zero (reference 10, p. 185). One thus concludes that when a small spherical cavity is placed in a medium with pure dilation, the only strains introduced by the cavity are pure shears. Hence such cavities give rise to no thermoelastic damping of vibrations composed only of dilations.

On the other hand, a spherical cavity placed in a medium containing only pure shears introduces local strains which have dilation. From the strain given in reference 10, p. 256, we find this dilation $\Theta$ to be

$$\Theta=[-5\mu s/(9\lambda+14\mu)] \times (a/r)^2 \sin^2\theta \cos\varphi \sin\varphi. \quad (18)$$

Here $a$ is the radius of the cavity; $\mu$ and $\lambda$ are elastic constants related to Poisson's ratio $\sigma$ and Young's modulus $E$ by Eqs. (24), p. 101 of reference 10; $s$ is the shear at large distances from the cavity.

As a first step in calculating the contribution of this cavity to the $Q^{-1}$ of the solid, one obtains the ratio $\mathcal{R}$ of the strain energy of dilation to the total strain energy. Now the strain energy of dilation is

$$\tfrac{1}{2}K\int\Theta^2 dv,$$

where $K$ is the bulk modulus. The total strain energy per unit volume is $\frac{1}{2}\mu s^2$, neglecting the small contribution due to the cavity. The $Q^{-1}$ of the solid due to one cavity per unit volume is obtained by setting

$$\mathcal{R}=\tfrac{1}{2}K\int\Theta^2 dv/\tfrac{1}{2}\mu s^2.$$

Replacing $\mu$, $\lambda$ by their equivalents in $\sigma$ and $E$,

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[10] A. E. H. Love, *Mathematical Theory of Elasticity* (Cambridge Press, 1920).

TABLE V.

| Poisson's ratio, $\sigma$ | 0 | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 | 0.40 | 0.45 | 0.50 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p(\sigma)$ | 1.0 | 1.92 | 1.05 | 1.01 | 0.98 | 0.93 | 0.84 | 0.73 | 0.55 | 0.32 | 0 |

and denoting the volume of the cavity by $v$, we obtain

$$\mathcal{R}=(10v/1764)p(\sigma),$$

where $p(\sigma)=(1-2\sigma)(1+\sigma)/(1-5\sigma/7)^2$,

and is given in Table V as a function of $\sigma$.

After the finding of the ratio $\mathcal{R}$, the next step is to find the constants $f_k$ and $\nu_k$ under the summation in Eq. (17). Following the analysis given in the appendix, we must first find those solutions of Eq. (a-5) whose normal derivatives vanish at the surface of a sphere of radius $a$, and also the associated eigenfunctions. In order that these solutions be quadratically integrable, we shall impose the second boundary condition that the solutions vanish at the surface of a large sphere of radius $R$, where $R \gg a$. We then expand $\Theta$, given by Eq. (18), in terms of these eigenfunctions. $f_k$ is then given by

$$f_k=\Theta_k{}^2\bigg/\int\Theta^2dv,$$

where $\Theta_k$ is the $k$'th coefficient in this expansion. In the final expression for $Q^{-1}$, we shall let the radius $R$ approach infinity. The summation then passes into an integration.

The eigenfunctions of Eq. (a-5) may be written as the product of a surface harmonic and of a function of $r$. Only those eigenfunctions will enter into the expansion of $\Theta$ whose surface harmonics are given by $\sin^2\theta\cos\varphi\sin\varphi$. The radial dependence of these eigenfunctions is given by

$$[d/dy\{y^{-\frac{1}{2}}J_{-5/2}(y)\}]_{y=q}y^{-\frac{1}{2}}J_{5/2}(y)$$
$$-[d/dy\{y^{-\frac{1}{2}}J_{5/2}(y)\}]_{y=q}y^{-\frac{1}{2}}J_{-5/2}(y).$$

Here $\qquad y=qr/a,$

and $\qquad q^2=2\pi\nu_ka^2/D.$

The derivative of this function automatically vanishes at $r=a$ for all values of $q$. The condition that this function vanish at $r=R$ is however satisfied by only a discrete set of $q$'s. The normalization of these eigenfunctions, the evaluation of the coefficients $\Theta_k$, and the passing to the limit $R\rightarrow\infty$, involves lengthy but only standard mathematics, and so will not be here reproduced.

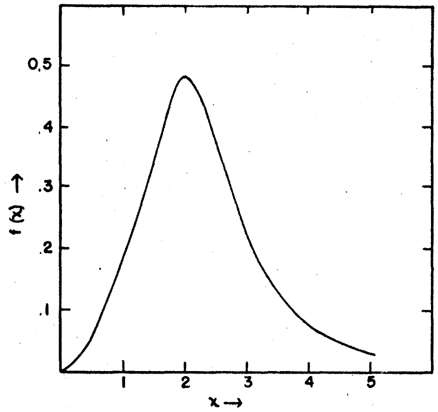

FIG. 2. Plot of $f(x)=18x^2/(81+9x^2-2x^4+x^6)$.

The result of this calculation is that the summation

$$\sum f_k[\nu_k\nu/(\nu_k{}^2+\nu^2)]$$

is replaced by the integration

$$\int_0^\infty f(x)[\nu(x)\nu/(\nu^2(x)+\nu^2)]dx.$$

The function $f(x)$ is given by

$$f(x)=\frac{18x^2}{81+9x^2-2x^4+x^6},$$

and is plotted in Fig. 2. It satisfies of course the normalization condition

$$\int_0^\infty f(x)dx=1.$$

The function $\nu(x)$ is given by

$$\nu(x)=(D/2\pi a^2)x^2.$$

## §5. INTERNAL FRICTION DUE TO RANDOM ORIENTATION OF CRYSTALLITES

As mentioned in the introduction, the stress in a vibrating polycrystalline solid will vary from crystallite to crystallite. This stress fluctuation gives rise to heat currents between adjacent crystallites, which in turn give rise to internal

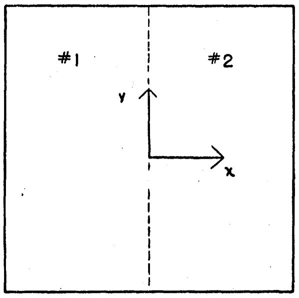

FIG. 3.

friction. The precise dependence of this internal friction upon frequency of vibration cannot be calculated without a detailed knowledge of the grain structure. However, we can predict the frequency $\nu_0$ at which $Q^{-1}$ is a maximum, and we can also predict the area beneath the curve of $Q^{-1}$ plotted against $\log \nu = x$. From the diffusion equation, one can readily see that

$$\nu_0 \approx D/L^2,$$

where $L$ is the mean diameter of the crystallites. From the relation

$$\int_{-\infty}^{\infty} \mathfrak{B}(x)dx = \pi \log_{10} e = 1.36,$$

and from Eq. (3), we obtain

$$\int_{-\infty}^{\infty} Q^{-1}dx = 0.68\mathcal{R}(C_p - C_v)/C_v. \tag{19}$$

In this integral the effect of the detailed structure no longer appears. The factor $(C_p - C_v)/C_v$ may be calculated from the well-known formula

$$(C_p - C_v)/C_v = T\alpha^2/C\chi,$$

where $C$ is the specific heat, for constant volume, per unit volume, $\alpha$ is the volume coefficient of thermal expansion, and $\chi$ is the compressibility. There follows an estimate of that fraction $\mathcal{R}$ of the total strain energy, in cubic metals, which is associated with fluctuations in dilation.

In cubic metals a macroscopic pure dilation implies a uniform dilation of all the crystallites. Hence we need consider only macroscopic pure shears.

In order to estimate that fraction of the energy of a macroscopic pure shear which is associated with microscopic fluctuations in dilation, consider a small cube divided in half. Each half is a crystallite, orientated at random. This system, together with a specification of the coordinate axes, is shown in Fig. 3. By compressing the block along the $y$ axis by a small amount, and extending it along the $x$ axis by the same small amount, a strain is produced which, on the average, has no net dilation. Using superscripts to distinguish the two crystallites, we obtain the following equations for the strains in the two halves of the cube

$$\begin{aligned} \tfrac{1}{2}(e_{xx}{}^1 + e_{xx}{}^2) &= e, \\ e_{yy}{}^1 = e_{yy}{}^2 &= -e, \\ e_{zz}{}^1 = e_{zz}{}^2 &= 0, \end{aligned}$$

all shearing strains $=0$. Denoting by $\Theta$ the dilation of the first crystallite, we may write the first equation of the above set as

$$e_{xx}{}^1 = e + \Theta; \quad e_{xx}{}^2 = e - \Theta.$$

A solution for $\Theta$ is obtained by substituting the stress-strain relations

$$X_x{}^k = C_{11}{}^k e_{xx}{}^k + C_{12}{}^k e_{yy}{}^k, \quad k = 1, 2,$$

in the equation of equilibrium

$$X_x{}^1 = X_x{}^2,$$

and by using the above expressions for the strains. The solution is

$$\Theta = Pe,$$

$$\text{where} \quad P = \frac{(C_{11}{}^2 - C_{11}{}^1) - (C_{12}{}^2 - C_{12}{}^1)}{C_{11}{}^1 + C_{11}{}^2}. \tag{20}$$

The ratio $\mathcal{R}_0$ for the cube is given by

$$\tfrac{1}{2}K\Theta^2/\tfrac{1}{2}G(2e)^2.$$

Here $K$ is the microscopic as well as the macroscopic bulk modulus, and $G$ is the macroscopic modulus of rigidity. On expressing $K$ and $G$ in

TABLE VI. *Effect of elastic anisotropy in cubic crystals.*

| METAL | Pb | Cu | Ag | Au | Fe | Al | W |
|---|---|---|---|---|---|---|---|
| $\mathcal{R}$ | 0.065 | 0.031 | 0.031 | 0.014 | 0.022 | 0.0009 | $10^{-6}$ |
| $(C_p - C_v)/C_v$ | 0.67 | 0.028 | 0.040 | 0.038 | 0.016 | 0.046 | 0.006 |
| $10^3 \times \int Q^{-1}dx$ | 3.0 | 0.59 | 0.57 | 0.36 | 0.24 | 0.028 | $10^{-5}$ |

terms of Young's modulus and Poisson's ratio $\sigma$, we find

$$\mathcal{R}_0=\frac{1}{6}\frac{1+\sigma}{1-2\sigma}P^2.$$

A good estimate of the ratio $\mathcal{R}$ for the solid would be obtained by averaging $\mathcal{R}_0$ over all orientations of the two crystallites. Since, however, we are at present interested primarily in orders of magnitude, only the following rough average will be performed. The principal axes of the first crystallite are chosen to coincide with the coordinate axes. Further, one of the principal axes of the second crystallite is chosen to coincide with the $z$ axis. Let $\theta$ be the angle which one of the other axes makes with the $x$ axis. Then (see p. 151 of reference 10)

$$\begin{aligned} C_{11}{}^2-C_{11}{}^1&=4\Delta C_{44}\sin^2\theta\cos^2\theta,\\ C_{12}{}^2-C_{12}{}^1&=-4\Delta C_{44}\sin^2\theta\cos^2\theta,\\ C_{11}{}^2+C_{11}{}^1&=2C_{11}+4\Delta C_{44}\sin^2\theta\cos^2\theta. \end{aligned}\tag{21}$$

Here $\Delta C_{44}$ is a measure of the elastic anisotropy of the metal. It is defined by

$$\Delta C_{44}=C_{44}-\tfrac{1}{2}(C_{11}-C_{12}).$$

We now substitute Eqs. (21) in Eq. (20), and average $\mathcal{R}_0$ over $\theta$. We obtain

$$\mathcal{R}=\frac{4}{45}\frac{1+\sigma}{1-2\sigma}\left(\frac{\Delta C_{44}}{C_{11}}\right)^2.$$

Values of this ratio are given in Table VI for various metals. In constructing this table, $\Delta C_{44}$ for Pb is taken from reference 11, $\Delta C_{44}$ for the other metals is taken from reference 5, p. 21.

## APPENDIX

In the text we must evaluate the following type of integral:

$$\Delta E=\beta\nu^{-1}\text{ time average of }\int\Delta T(dG/dt)d\tau.\tag{a-1}$$

Here $G$ is a harmonic function of time $t$, with frequency $\nu$, and is an arbitrary function of the coordinates. $\Delta T$ is also a harmonic function of time, but its dependence upon the coordinates is determined by the equation

$$d(\Delta T)/dt=D\nabla^2\Delta T+\gamma dG/dt,\tag{a-2}$$

and by the condition that the normal component of the gradient of $\Delta T$ vanish at the boundary of the region.

The variable $t$ is eliminated from Eqs. (a-1) and (a-2) by the introduction of complex quantities:

$$\begin{aligned}\Delta T(\mathbf{r},t)&=\text{Real part of }\mathfrak{T}(\mathbf{r})e^{2\pi i\nu t},\\ G(\mathbf{r},t)&=\text{Real part of }\mathfrak{G}(\mathbf{r})e^{2\pi i\nu t}.\end{aligned}$$

We choose the origin of time so as to make $\mathfrak{G}$ real. Then $\mathfrak{T}$ is in general complex. Eqs. (a-1) and (a-2) may now be rewritten as

$$\Delta E=-i\pi\beta\int\mathfrak{G}(\text{imaginary part of }\mathfrak{T})d\tau,\tag{a-3}$$

$$(D\nabla^2-2\pi i\nu)\mathfrak{T}+2\pi i\nu\gamma\mathfrak{G}=0.\tag{a-4}$$

The complex function $\mathfrak{T}$ may formally be expanded in any complete set of normalized functions. We shall choose that particular set of functions $U_0$, $U_1$, $U_2$ $\cdots$ which satisfy the differential equation

$$(D\nabla^2+2\pi\nu_k)U_k=0,\tag{a-5}$$

and also the same boundary condition that $\mathfrak{T}$ satisfies, namely the vanishing of the normal component of their gradients at the surface. By so choosing this boundary condition, we automatically make that expansion which satisfies Eq. (a-4) also satisfy the proper boundary condition.

In the expansion

$$\mathfrak{T}=\sum\mathfrak{C}_jU_j\tag{a-6}$$

the constants $\mathfrak{C}_j$ are in general complex. They are determined by substituting the expansion in Eq. (a-4), multiplying to the left by $U_k$, and integrating over all the solid. Using Eq. (a-5), and the notation

$$G_k=\int\mathfrak{G}U_kd\tau,\tag{a-7}$$

we obtain

$$\mathfrak{C}_k=\beta G_k\nu/(\nu_k+i\nu).\tag{a-8}$$

Now substitute the expansion (a-6) in Eq. (a-3), using the coefficients given by Eq. (a-8). Since

Imaginary part of $\mathfrak{C}_k=i\beta G_k\nu_k\nu/(\nu_k{}^2+\nu^2)$, we obtain

$$\Delta E=\pi\beta\gamma\sum G_k\nu_k\nu/(\nu_k{}^2+\nu^2).\tag{a-9}$$

Since the functions $U_0$, $U_1$, $U_2$ $\cdots$ form a complete orthogonal set, the following relation is satisfied[12]

$$\sum(G_k{}^2/\int G^2d\tau)=1.\tag{a-10}$$

By rewriting Eq. (a-9) as

$$\Delta E=\pi\beta\gamma(\int G^2d\tau)\sum\frac{G_k{}^2}{\int G^2d\tau}\cdot\frac{\nu_k\nu}{\nu_k{}^2+\nu^2},$$

we make the sum of the coefficients of $\nu_k\nu/(\nu_k{}^2+\nu^2)$ equal to unity.

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[11] E. Goens, Wiss. Abhandl. Physik. Techn. Reichsanhalt **20**, 63 (1936).

[12] R. Courant and D. Hilbert, *Methoden der Mathematischen Physik*, (Springer, 1924) p. 36.