The Excited States of the H³ and He⁴ Nuclei

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Calculations were made with trial wave functions corresponding to such excited states of H³ and He⁴ which, in the Hartree approximation, would contain one or two particles excited into the p shell. If only one particle is excited there is no state which would be completely symmetrical in space coordinates and if two particles are excited only an S and a D state have this property. On account of the supposed predominance of the Majorana exchange forces in the nuclear Hamiltonian, these should be the lowest excited states in H³ while in He⁴ the P state in which only one particle would be excited in the Hartree picture lies close to these. No stable excited states are indicated for either of the nuclei when the forces are of the exchange type unless the range of force is much greater than the accepted range. Several improved forms of the D state wave function were tried but they failed to lower the energy sufficiently to make the states stable.

THE normal states of the nuclear three and four particle systems have received considerable attention. The results, although not entirely consistent, indicate that to a fair approximation, at least for these light nuclei, the forces between all pairs of particles are equal when the Coulomb repulsion between protons is neglected. It is then of interest to investigate on such a model the nature and positions of the excited states of these nuclei. The exchange properties and range of the forces will obviously play an important role in their determination, since the different types of excited states possess various types of symmetry. Calculations were first made for He⁴ by Feenberg[1] who found that one might expect a stable P state if the forces are of the ordinary type with a range greater than 2×10^{-13} cm. Feenberg did not investigate in detail the existence of such a state under the assumption of exchange forces. Margenau[2] has pointed out later that this state becomes unstable when the exchange operators are introduced into the Hamiltonian.

There are several ways in which the results of the calculation might be verified experimentally. In the case of H³ (or He³) a state with negative energy but which is unstable with respect to disintigration into a deuteron and a neutron (or a proton) may make its presence known either by a resonance peak in the scattering cross section curve or by an asymmetry in the scattering when the energy of the incident neutrons (or protons) is such that the total energy of the system is equal to the energy of this state. A similar effect may be expected in the scattering of deuterons by deuterons if He⁴ possesses an excited state with negative energy greater than twice the deuteron energy. If in a nuclear reaction an H³, He³, or He⁴ nucleus is formed in a state above the ground state, γ-rays will be observed. Among the products of the reaction there will also be present a group of particles with range less than that which corresponds to the case where all the reaction products are formed in the ground states. γ-rays have been observed[3] in the reaction $Li^7 + H^1 = 2He^4$. This may be interpreted as emission from an excited state of He⁴ although it now appears more probable that the γ-rays are emitted by Be⁸ nuclei produced in an excited state. In the reaction $H^2 + H^2 = He^3 + n$ Bonner[4] has observed a group of recoil protons with range less than the maximum range. This may be due to some of the He³ nuclei which are left in an excited state. As has been mentioned above and as will be seen later the theory fails to predict stable excited states for either of these nuclei.

TYPES OF STATES

The calculations have been made by use of the symmetric interaction potential

$$V_{ij} = [(1-g)P_{ij}^M + gP_{ij}^H]J(r_{ij})$$

between each pair of particles with $J(r_{ij})$

[1] E. Feenberg, Phys. Rev. **49**, 328 (1936).

[2] H. Margenau, Phys. Rev. **53**, 198 (1938).

[3] Crane, Delsasso, Fowler and Lauritson, Phys. Rev. **48**, 125 (1935).

[4] T. W. Bonner, Phys. Rev. **52**, 685 (1937). Also Washington meeting of the American Physical Society, April, 1938, Abstract 73.

$= -A \exp(-\alpha r_{ij}^2)$. A is determined as a function of the range of forces $1/\sqrt{\alpha}$ by numerical integration of the deuteron radial equation.[5] g was taken to be 0.235. In units of length equal to $\hbar/(Mmc^2)^{\frac{1}{2}}$ and energy equal to mc^2 the Hamiltonian, if Coulomb repulsion terms are neglected, takes the form

$$-\tfrac{1}{2}\sum_i \Delta_i + \sum_{i>j} V_{ij}.$$

Calculations were made for both nuclei with P, D, and S state wave functions. The wave functions corresponding to these states are of the form

$$\psi = q \cdot f \cdot s.$$

q is a function of the space coordinates and has the symmetry required for a P, D, or S state wave function. f is a function of the interparticle distances only and becomes small for large separations of any pair of particles. s is the spin wave function and is symmetric or antisymmetric in the like particles according as $q \cdot f$ is antisymmetric or symmetric in these particles. The function q which represents a P state corresponds to a state in which, in the Hartree approximation, a single particle is excited into the p shell while for S and D states it corresponds to the excitation of two particles. One might then expect the P level to lie below the S and D levels. However, this is offset by the fact that the space parts of the S and D wave functions can be made entirely symmetrical in all the particles while this is not possible in the case of the P

state. The Majorana force is supposedly the predominant force in the interaction between nuclear particles and a Majorana interchange of two particles will introduce repulsion terms only in the energy expression for the P state. It thus seems plausible that, while for ordinary forces the P level will be the first one above the ground level, it will lie above the S and D levels for exchange forces. This is what the results of the calculation indicate.

The P and D state wave functions are automatically orthogonal to the ground state wave function, the ground state being an S state. The excited S state must be made orthogonal to the ground state since it is not automatically so. However the ground state wave function is itself not known accurately and thus the excited S state can be made orthogonal only to an approximate normal state wave function. In the present work this approximate wave function was taken to be $N \exp[-\tfrac{1}{2}\nu_1(r_{12}^2+r_{13}^2+r_{23}^2)]$ for H^3 and the corresponding function was taken for He^4. The ν_1 was chosen in each case to give the lowest normal state energy. Since these functions are only approximately correct for the ground state the trial wave functions of the excited states will not be quite orthogonal to the real wave functions of the normal state and a certain amount of the ground state may be introduced into the excited state wave function. It is then possible for the calculated energy to lie below the actual energy of the excited state. For this reason the S state results do not constitute upper limits for the actual solution of the Hamiltonian.

<center>H³</center>

The three wave functions used for H^3 are

$${}^4P \quad \psi = N(x_1-x_2)\exp\left[-\tfrac{1}{2}\nu(r_{12}^2+r_{13}^2+r_{23}^2)\right]\alpha_1\alpha_2\alpha_3,$$

$${}^2D \quad \psi = N(x_{12}y_{12}+x_{13}y_{13}+x_{23}y_{23})\exp\left[-\tfrac{1}{2}\nu(r_{12}^2+r_{13}^2+r_{23}^2)\right]\frac{1}{\sqrt{2}}\alpha_3(\alpha_1\beta_2-\alpha_2\beta_1),$$

$${}^2S \quad \psi = N[6-(\nu+\nu_1)(r_{12}^2+r_{13}^2+r_{23}^2)]\exp\left[-\tfrac{1}{2}\nu(r_{12}^2+r_{13}^2+r_{23}^2)\right]\frac{1}{\sqrt{2}}\alpha_3(\alpha_1\beta_2-\alpha_2\beta_1)$$

where 1, 2 are neutrons and 3 is a proton. N is the normalizing factor. α and β are spin wave functions corresponding to orientation of the spin along the positive and negative z axis, respectively. In the S wave function ν_1 is the value of ν for which $\exp[-\tfrac{1}{2}\nu(r_{12}^2+r_{13}^2+r_{23}^2)]$ gives the lowest ground state energy. Since $\mathbf{r}_{23}=\mathbf{r}_{13}-\mathbf{r}_{12}$ the integration can be performed over $d\tau_{12}d\tau_{13}$. This automatically elimi-

[5] E. Feenberg and S. S. Share, Phys. Rev. 50, 254 (1936). Table II.

TABLE I. *Energy values, in units of mc², of the ecxited states of H³ as a function of the width parameter ν of the wave function. It is clear how the energy is varying without filling in the blank spaces.*

$\alpha = 10$	ν	$E_{\mathrm{ord}}(^4P)$	$E_{\mathrm{exc}}(^4P)$	$E_{\mathrm{ord}}(^2D)$	$E_{\mathrm{exc}}(^2D)$	$E_{\mathrm{ord}}(^2S)$	$E_{\mathrm{exc}}(^2S)$
	0.5	1.5	1.9			0.7	0.9
	1.0	1.9	2.8	0.7	2.0		
	3.0	0.8	8.9	2.5	7.2	2.0	3.3
	4.0	0.0	13.1			3.2	6.7
	5.0	−0.4		5.8	13.2	4.6	9.8
	7.0	1.3	28.9	10.0	20.1	8.1	18.6
	10.0	5.6		23.3	35.4	17.7	33.4
$\alpha = 20$							
	0.5	2.1	2.3			0.9	1.1
	1.0	3.5	3.8	2.0	3.3	1.5	1.9
	3.0	5.9	9.6	6.1	10.6	3.9	5.4
	5.0	7.0	16.9	17.5	22.6	6.8	10.3
	7.0	7.7	25.1			14.5	21.1
	10.0	8.8	39.5				
	12.0	10.6					

nates the coordinates of the center of gravity. The energy expressions are given below, except for the *S* state which is a rather long expression, for both ordinary forces, taken without spin dependence, and exchange forces.

$$^4P \quad E_{\mathrm{ord}} = 6\nu - 3^{5/2}A\,\nu^{3/2}(3\nu+\alpha)/(3\nu+2\alpha)^{5/2}, \quad E_{\mathrm{exc}} = 6\nu - 3^{5/2}A\,\nu^{3/2}\alpha/(3\nu+2\alpha)^{5/2},$$

$$^2D \quad E_{\mathrm{ord}} = \frac{15}{2}\nu - \frac{3^{5/2}A\,\nu^{3/2}}{(3\nu+2\alpha)^{7/2}}(9\nu^2+2\alpha^2+6\alpha\nu), \quad E_{\mathrm{exc}} = \frac{15}{2}\nu - \frac{3^{5/2}(1-g)A\,\nu^{3/2}}{(3\nu+2\alpha)^{7/2}}(9\nu^2+2\alpha^2+6\alpha\nu).$$

The numerical results are shown in Table I. It is seen that, as was expected, the *P* state lies lower than the other states for ordinary forces and higher for exchange forces. It is also seen that the energy is lower for the longer range of forces. This is opposite to the way in which the ground state energy varies with the range of forces and can be explained by the fact that the excited states are spread out as compared to the ground state, and for this reason the potential energy curve should not fall off too rapidly if an appreciable potential energy is to be obtained. Except in the case of the *P* state for an ordinary force of somewhat too long range, there is no indication of a minimum even for positive values of the energy. This would lead one to believe that probably no stable excited state of H³ exists. It may of course be that these wave functions are very poor approximations to the correct ones. It seems likely, however, that they are not so poor that there would not be some indication of a minimum in the energy curve, at least for the longest range of forces.

The three particle system consisting of a free incident neutron and a deuteron has a total internal energy which is negative if the kinetic energy of the relative motion of the neutron and deuteron is less than the deuteron binding energy. It is thus always possible to construct a state with negative energy by choosing a wave function which represents a neutron moving in the field of a deuteron and which diminishes quite slowly with increasing separation between the neutron and deuteron. A possible form of such a wave function[6] consists of a symmetrized sum of three terms of the type $(y_{13}+y_{23})(z_{13}+z_{23})f(r_{12}) \exp\left[-a(r_{13}^2+r_{23}^2)\right]$, where $f(r_{12})$ is the exact deuteron wave function. This corresponds to a *D* state of the system and it was thought it might represent an attraction between the neutron and deuteron, i.e. that the energy of the system would be lower than the deuteron energy. There would then exist a stable excited state of H³. If we let

$$\psi_{12} = (y_{13}+y_{23})(z_{13}+z_{23})f(r_{12}) \exp\left[-a(r_{13}^2+r_{23}^2)\right]$$

and $\qquad\qquad \Psi = (\psi_{12}+\psi_{13}+\psi_{23}),$

then $\qquad\qquad E = (\Psi, H\Psi)/(\Psi, \Psi) = (\Psi, H\psi_{12})/(\Psi, \psi_{12}),$

[6] J. A. Wheeler, Phys. Rev. **52**, 1083 (1937).

where the potential energy term must be multiplied by $1-g$ to take into account the spin exchanges. Since ψ_{12} contains the deuteron wave function as a factor and since among the terms of H there are $-\frac{1}{2}(\Delta_1+\Delta_2)+V(r_{12})$ the energy expression can be written as

$$E=E_D+E',$$

where E_D is the deuteron energy $(-4.3\ mc^2)$ and

$$E'=(\Psi, (H-H_{12})\psi_{12})/(\Psi, \psi_{12}),$$

in which $\quad H_{12}\psi_{12}= V_{12}\psi_{12}-\frac{1}{2}\exp\left[-a(r_{13}{}^2+r_{23}{}^2)\right](y_{13}+y_{23})(z_{13}+z_{23})(\Delta_1+\Delta_2)f(r_{12}).$

A negative value of E' would indicate an attraction between the deuteron and the neutron. There is no analytical expression for $f(r_{12})$, for the error function potential, and to facilitate calculation it was taken to be $\exp(-br_{12}{}^2)$ where b was chosen so as to give the lowest energy for the deuteron energy. E' was found to be positive and approached zero as a approached zero. This indicates that the deuteron and neutron repel each other when they are in a D state. No similar calculations were made for the P or S states.

<center>He⁴</center>

The trial wave functions for the excited states of He^4 are

$$^1P \quad \psi=N(x_1+x_2-x_3-x_4)\exp\left[-\tfrac{1}{2}\nu(r_{12}{}^2+r_{13}{}^2+r_{14}{}^2+r_{23}{}^2+r_{24}{}^2+r_{34}{}^2)\right]\frac{1}{\sqrt{4}}(\alpha_1\beta_2-\alpha_2\beta_1)(\alpha_3\beta_4-\alpha_4\beta_3),$$

$$^3P \quad \psi=N(x_1-x_2)\exp\left[-\tfrac{1}{2}\nu(r_{12}{}^2+r_{13}{}^2+r_{14}{}^2+r_{23}{}^2+r_{24}{}^2+r_{34}{}^2)\right]\frac{1}{\sqrt{2}}\alpha_1\alpha_2(\alpha_3\beta_4-\alpha_4\beta_3),$$

$$^1D \quad \psi=N(x_{12}y_{12}+x_{13}y_{13}+x_{14}y_{14}+x_{23}y_{23}+x_{24}y_{24}+x_{34}y_{34})\exp\left[-\tfrac{1}{2}\nu(r_{12}{}^2+r_{13}{}^2+r_{14}{}^2+r_{23}{}^2+r_{24}{}^2+r_{34}{}^2)\right]$$
$$\cdot\frac{1}{\sqrt{4}}(\alpha_1\beta_2-\alpha_2\beta_1)(\alpha_3\beta_4-\alpha_4\beta_3),$$

$$^1S \quad \psi=N[9-(\nu+\nu_1)(r_{12}{}^2+r_{13}{}^2+r_{14}{}^2+r_{23}{}^2+r_{24}{}^2+r_{34}{}^2)]\exp\left[-\tfrac{1}{2}\nu(r_{12}{}^2+r_{13}{}^2+r_{14}{}^2+r_{23}{}^2+r_{24}{}^2+r_{34}{}^2)\right]$$
$$\cdot\frac{1}{\sqrt{4}}(\alpha_1\beta_2-\alpha_2\beta_1)(\alpha_3\beta_4-\alpha_4\beta_3).$$

1,2 are neutrons and 3,4 are protons. As in the case of H^3 the ν_1 in the S wave function was chosen so that $\exp\left[-\tfrac{1}{2}\nu_1(r_{12}{}^2+r_{13}{}^2+r_{13}{}^2+r_{23}{}^2+r_{24}{}^2+r_{34}{}^2)\right]$ gives the lowest energy for the ground state. The energy expressions for exchange potentials are, except for the S state,

$$^1P \quad E=11\nu-\frac{2^{3/2}A\nu^{3/2}(4\nu+4\alpha-8\nu g-5\alpha g)}{(2\nu+\alpha)^{5/2}},$$

$$^3P \quad E=11\nu-\frac{2^{1/2}A\nu^{3/2}(8\nu+8\alpha-12\nu g-7\alpha g)}{(2\nu+\alpha)^{5/2}},$$

$$^1D \quad E=13\nu-\frac{2^{7/2}A(1-g)\nu^{3/2}(6\nu^2+4\alpha\nu+\alpha^2)}{(2\nu+\alpha)^{7/2}}.$$

The numerical results are given in Table II. A negative energy minimum for the S state is observed which by interpolation is about $-4.9\ mc^2$ at $\alpha=16$. The state would be unstable with respect to disintigration into either two deuterons, each with energy $\frac{1}{2}(2\times4.3-4.9)=1.9\ mc^2$, or into He^3 and a proton. The D and P state energies lie fairly close together; neither of them have a negative energy minimum. For reasons mentioned previously it was expected that the correct D state energy would lie below that of the P state. Several attempts were made to improve the D state wave function. Calcu-

TABLE II. *Energy values, in units of mc^2, of the excited states of He^4 as a function of the width parameter ν of the wave function. It is clear how the energy is varying without filling in the blank spaces.*

$\alpha=10$	ν	$E(^1P)$	$E(^3P)$	$E(^1S)$	$E(^1D)$	$\alpha=20$	$E(^1P)$	$E(^3P)$	$E(^1S)$	$E(^1D)$
	0.5	1.8	1.3				3.0	2.7		3.6
	1.0	2.0	1.0	−0.1	2.2		4.6	3.8	1.0	5.2
	1.5	2.5	0.8					4.5		
	2.0				1.5		6.9	5.1	0.5	7.2
	2.5	4.4	1.6							
	3.0				1.1		9.5		−2.8	7.5
	4.0			−6.2	1.6				−3.3	8.1
	5.0	19.0	14.2	−6.5	3.5		16.1	12.3	−3.9	
	6.0			−5.8	6.4				−3.8	10.0
	7.0			−0.8						
	8.0			6.9					−1.9	16.5
	10.0						49.4		2.9	

lations were made with

$$\psi = N(x_{12}y_{12} \exp\left[-\tfrac{1}{2}\mu r_{12}{}^2\right] + x_{13}y_{13} \exp\left[-\tfrac{1}{2}\mu r_{13}{}^2\right] + x_{14}y_{14} \exp\left[-\tfrac{1}{2}\mu r_{14}{}^2\right] + x_{23}y_{23} \exp\left[-\tfrac{1}{2}\mu r_{23}{}^2\right]$$

$$+ x_{24}y_{24} \exp\left[-\tfrac{1}{2}\mu r_{24}{}^2\right] + x_{34}y_{34} \exp\left[-\tfrac{1}{2}\mu r_{34}{}^2\right] \exp\left[-\tfrac{1}{2}\nu(r_{12}{}^2 + r_{13}{}^2 + r_{14}{}^2 + r_{23}{}^2 + r_{24}{}^2 + r_{34}{}^2)\right]$$

$$\cdot \frac{1}{\sqrt{4}}(\alpha_1\beta_2 - \alpha_2\beta_1)(\alpha_3\beta_4 - \alpha_4\beta_3).$$

A minimum of $-0.1\ mc^2$ was obtained with $\nu=4$ and $\mu=-3$ at $\alpha=16$. The energy surface plotted as a function of ν and μ for $\alpha=16$ is everywhere positive showing a shallow trough at $\mu \sim -3$ and becoming 0 for $\nu=0$ and $\mu=0$. A linear combination of two error functions

$$\psi = (x_{12}y_{12} + x_{13}y_{13} + x_{14}y_{14} + x_{23}y_{23} + x_{24}y_{24} + x_{34}y_{34})(c_1 \exp\left[-\tfrac{1}{2}\nu_1(r_{12}{}^2 + r_{13}{}^2 + r_{14}{}^2 + r_{23}{}^2 + r_{24}{}^2 + r_{34}{}^2)\right]$$

$$+ c_2 \exp\left[-\tfrac{1}{2}\nu_2(r_{12}{}^2 + r_{13}{}^2 + r_{14}{}^2 + r_{23}{}^2 + r_{24}{}^2 + r_{34}{}^2)\right])\frac{1}{\sqrt{4}}(\alpha_1\beta_2 - \alpha_2\beta_1)(\alpha_3\beta_4 - \alpha_4\beta_3)$$

was also tried. The secular equation was solved and the result minimized numerically with respect to ν_1 and ν_2. A minimum of $-1\ mc^2$ in the neighborhood of $\nu_1=6$ and $\nu_2=2$ was obtained for $\alpha=10$. For $\alpha=16$ no negative energies were obtained.

From the results obtained here it appears probable that H^3 and He^4 have no stable excited states. A stable H^3 state would need to have an energy below $-4.3\ mc^2$, while He^4 would need an energy lower than $-15.1\ mc^2$. Although there are a great variety of ways of improving the wave functions it does not seem likely that they could be improved sufficiently to give stable states above the ground state. A considerable increase in the range of forces would be required in order that any of these states should be stable. The discrepancies which have appeared in the ground state calculations seem to necessitate a shorter range of forces than has hitherto been accepted. This would make the states still more unstable.

It should be mentioned that a free neutron or proton may interact with a deuteron in such a way that for certain energies a large scattering cross section results. The interaction might in such a case be described by a potential well surrounded by a potential barrier and the wave function of the incident particle, neutron or proton, will be a plane wave with unit amplitude when it is far from the deuteron. Resonance, i.e. a large scattering cross section, will occur if for a certain energy the portion of the wave function inside the well is much larger than it is for neighboring energies. A similar consideration applies to the four particle system. The wave functions which have been used in this work are probably not suitable for the description of such virtual states and would probably give no information concerning them. In this connection an error function and a simple exponential variation wave function were used to calculate the energy of the deuteron in the singlet state. The curve for the energy plotted as a function of the width parameter of the wave function showed no minimum and no evidence of the deuteron virtual level. The writer wishes to express his thanks to Professor Breit and to Professor Wigner for many helpful suggestions.