The Electrical Oscillations of a Prolate Spheroid. Paper I

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Differential equation. Solutions in spheroidal coordinates of the differential equation of the electromagnetic field are obtained which are valid all the way from the surface of a prolate spheroid of eccentricity unity to infinity. *Free oscillations.* The wave-length λ and logarithmic decrement δ of the fundamental free oscillations are computed for all eccentricities. *Forced oscillations.* The wave-length λ, current I, and phase ϕ at resonance are computed for six eccentricities near unity for an antenna stimulated by a uniform field varying sinusoidally with the time. Three resonance curves are computed, the radiation resistance at resonance is calculated for six eccentricities and off resonance for two. Expressions for the entire field at all points are given, and reduced to explicit form for the radiation field. An expression is obtained for the mean rate of radiation.

THE problem of the electrical oscillations of a perfectly conducting prolate spheroid is of great interest because, in the case where the eccentricity is nearly equal to unity, it represents to a high degree of precision the problem of the straight wire antenna. The first attack on this problem was made by Abraham,[1] who succeeded in finding the frequencies and decrements of the free oscillations in the limiting case of eccentricity unity, and in obtaining formulas which give rough approximations to the frequencies and decrements for eccentricities a little less than unity. Macdonald[2] also has obtained an expression for the fundamental frequency of the free oscillations of a straight antenna of negligible diameter by treating it as the limiting case of a cone of very small vertical angle. Obviously this is a very poor physical approximation, and therefore it is not surprising that Macdonald's value of the frequency differs from Abraham's by over twenty percent, and from the experimental value by nearly as much. Recently Stratton[3] has investigated the solution of the scalar wave equation in spheroidal coordinates, but his differential equation is not the one to which the problem here considered leads. Apparently no one has made much progress in solving the problem of the forced oscillations of a prolate spheroid, which are, of course, of far more practical importance than the free oscillations in the application to the antenna.

[1] Abraham, Ann. d. Physik **66**, 435 (1898); **2**, 32 (1900).
[2] Macdonald, *Electric Waves* (Cambridge University Press, 1902).
[3] Stratton, Proc. Nat. Acad. Sci. **21**, 51 and 316 (1935).

The object of this paper is to obtain solutions of the differential equation of the electrical oscillations of a prolate spheroid which will describe the electromagnetic field with an accuracy greater than that of experimental measurement all the way from the surface of the most eccentric spheroidal conductor to infinity, and will permit the discussion not only of the free oscillations, but as well of the oscillations forced by a uniform oscillating electric field, such as that of an exciting electromagnetic wave of length long compared with the shorter axis of the spheroid. Although the methods developed are applicable to all the harmonics, we shall limit the analysis in this paper to the fundamental or first harmonic and, to the approximation to which they are needed for the solution of the forced case, to the third and fifth harmonics. The spheroid will be supposed to be perfectly conducting, the only resistance taken into account being that of radiation. Heaviside-Lorentz symmetrical units are used throughout the analysis.

THE DIFFERENTIAL EQUATION

Let x, y, z be a right-handed set of rectangular coordinates with the x axis along the long axis of the prolate spheroid, and put $\rho^2 \equiv y^2+z^2$. We introduce the prolate spheroidal coordinates ξ, η, ϕ by the transformation

$$x=f\xi\eta, \tag{1}$$

$$\rho=f\{(1-\xi^2)(\eta^2-1)\}^{\frac{1}{2}}, \tag{2}$$

$$\tan\phi=z/y, \tag{3}$$

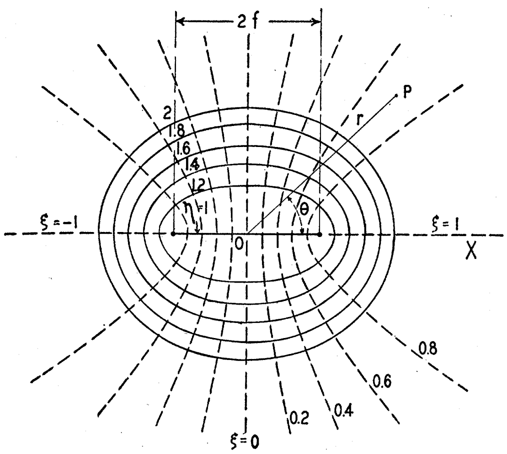

FIG. 1. Spheroidal coordinates.

where $-1 \leqslant \xi \leqslant 1$, $1 \leqslant \eta < \infty$, $0 \leqslant \phi < 2\pi$. The coordinate surfaces $\xi = \text{const.}$ constitute a set of confocal hyperboloids of revolution about the x axis, and the coordinate surfaces $\eta = \text{const.}$ are a set of confocal prolate spheroids of eccentricity $1/\eta$. The constant f represents the semi-interfocal distance, which is approached by the semi-major axis a of the prolate spheroids as η approaches unity. The unit vectors $\boldsymbol{\xi}_1$, $\mathbf{n}_1$, $\boldsymbol{\phi}_1$, in the directions of increasing ξ, η, ϕ, respectively, constitute a right-handed set in the order named. The radius vector r, whose square is $r^2 = f^2(\eta^2 + \xi^2 - 1)$, becomes effectively equal to $f\eta$ at great distances from the origin O, and the cosine of the angle θ which the radius vector makes with the x axis approaches ξ when r becomes very great. A section through the x axis of the coordinate surfaces is shown in Fig. 1.

We are interested in fields whose time dependence is specified by the factor $\exp(-i\omega t)$. Evidently the only nonvanishing field components are E_ξ, E_η, H_ϕ. If we put $A \equiv \rho H_\phi$, it has been shown by Abraham[1] that the field equations give

$$E_\xi = i\left(\frac{\mu}{\kappa}\right)^{\frac{1}{2}} \frac{1}{\epsilon f} \frac{1}{\{(1-\xi^2)(\eta^2-\xi^2)\}^{\frac{1}{2}}} \frac{\partial A}{\partial \eta}, \quad (4)$$

$$E_\eta = -i\left(\frac{\mu}{\kappa}\right)^{\frac{1}{2}} \frac{1}{\epsilon f} \frac{1}{\{(\eta^2-1)(\eta^2-\xi^2)\}^{\frac{1}{2}}} \frac{\partial A}{\partial \xi}, \quad (5)$$

$$H_\phi = \frac{1}{f} \frac{1}{\{(1-\xi^2)(\eta^2-1)\}^{\frac{1}{2}}} A, \quad (6)$$

where A satisfies the differential equation

$$(1-\xi^2)\partial^2 A/\partial\xi^2 + (\eta^2-1)\partial^2 A/\partial\eta^2 + \epsilon^2(\eta^2-\xi^2)A = 0. \quad (7)$$

Here κ and μ are respectively the permittivity and permeability of the medium surrounding the spheroid, and

$$\epsilon \equiv (\omega/v)f = 2\pi f/\lambda, \quad (8)$$

where $v \equiv c/(\kappa\mu)^{\frac{1}{2}}$ is the normal wave velocity in the medium, c represents the velocity of light in vacuum, and λ is the wave-length.

The variables are separable in (7), and if we put $A(\xi, \eta) = X(\xi)Y(\eta)$ we obtain the two ordinary differential equations

$$(1-\xi^2)d^2X/d\xi^2 + (\alpha - \epsilon^2\xi^2)X = 0, \quad (9)$$

$$(\eta^2-1)d^2Y/d\eta^2 + (-\alpha + \epsilon^2\eta^2)Y = 0, \quad (10)$$

where α is the constant of separation.

SOLUTION OF THE EQUATION IN ξ

First we solve (9). Putting

$$X(\xi) = (1-\xi^2)^{\frac{1}{2}} u(\xi), \quad (11)$$

we find

$$\frac{d}{d\xi}\left\{(1-\xi^2)\frac{du}{d\xi}\right\} - \frac{u}{1-\xi^2} + \alpha u = \epsilon^2\xi^2 u, \quad (12)$$

from which we infer that the characteristic solutions $u_1, u_2, \cdots u_l, \cdots$ form an orthogonal set of functions. We express the characteristic values and the characteristic functions as power series in ϵ^2, *viz.*

$$\alpha_l = \alpha_{l0} + \alpha_{l1}\epsilon^2 + \alpha_{l2}\epsilon^4 + \alpha_{l3}\epsilon^6 + \alpha_{l4}\epsilon^8 + \cdots,$$

and

$$u_l(\xi) = u_{l0}(\xi) + u_{l1}(\xi)\epsilon^2 + u_{l2}(\xi)\epsilon^4 + u_{l3}(\xi)\epsilon^6 + \cdots.$$

Evidently

$$\alpha_{l0} = l(l+1), \quad l = 1, 2, 3 \cdots,$$

and u_{l0} is the associated Legendrian polynomial

$$u_{l0} = P_{l1}(\xi) = (1-\xi^2)^{\frac{1}{2}}(d/d\xi)P_l(\xi),$$

where $l = 1$ represents the fundamental or first harmonic, $l=2$ the second harmonic, $l=3$ the third harmonic, and so forth.

By the usual perturbation methods we find for the fundamental

$$\alpha_1=2+\frac{1}{5}\epsilon^2-\frac{4}{5^3\cdot 7}\epsilon^4+\frac{8}{3\cdot 5^5\cdot 7}\epsilon^6-\frac{124}{5^6\cdot 7^3\cdot 11}\epsilon^8+\cdots, \tag{13}$$

$$\begin{aligned}u_1=P_{11}(\xi)&-\frac{1}{3\cdot 5^2}P_{31}(\xi)\epsilon^2\\&+\left\{\frac{2}{3^2\cdot 5^4}P_{31}(\xi)+\frac{1}{3^2\cdot 5^2\cdot 7^2}P_{51}(\xi)\right\}\epsilon^4\\&-\left\{\frac{31}{3\cdot 5^5\cdot 7^2\cdot 11}P_{31}(\xi)+\frac{4}{3\cdot 5^4\cdot 7^2\cdot 13}P_{51}(\xi)\right.\\&\left.+\frac{1}{3^4\cdot 5\cdot 7^2\cdot 11\cdot 13}P_{71}(\xi)\right\}\epsilon^6+\cdots.\end{aligned} \tag{14}$$

For the third harmonic we get

$$\alpha_3=12+\frac{7}{3\cdot 5}\epsilon^2+\frac{152}{3^4\cdot 5^3\cdot 11}\epsilon^4+\cdots, \tag{15}$$

$$\begin{aligned}u_3=P_{31}(\xi)&+\left\{\frac{6}{5^2\cdot 7}P_{11}(\xi)-\frac{2}{3^3\cdot 7}P_{51}(\xi)\right\}\epsilon^2\\&-\left\{\frac{4}{5^4\cdot 7}P_{11}(\xi)-\frac{4}{3^6\cdot 5\cdot 7\cdot 13}P_{51}(\xi)\right.\\&\left.-\frac{5}{3^2\cdot 7\cdot 11^2\cdot 13}P_{71}(\xi)\right\}\epsilon^4+\cdots,\end{aligned} \tag{16}$$

and for the fifth harmonic

$$\alpha_5=30+\frac{19}{3\cdot 13}\epsilon^2+\cdots, \tag{17}$$

$$u_5=P_{51}(\xi)+\left\{\frac{5}{3^3\cdot 11}P_{31}(\xi)-\frac{15}{11\cdot 13^2}P_{71}(\xi)\right\}\epsilon^2+\cdots. \tag{18}$$

Solution of the Equation in η for Large Values of the Variable

Since we are interested in a diverging wave system the boundary condition at infinity demands a solution of (10) of the form

$$Y(\eta)=v(\eta)e^{i\epsilon\eta}, \tag{19}$$

where $v(\eta)$ is a series in descending powers of η. If we put $z\equiv i/\epsilon\eta$ the differential equation for v becomes

$$z^2\frac{d^2v}{dz^2}+2(z+1)\frac{dv}{dz}-\frac{\alpha-\epsilon^2}{1+\epsilon^2z^2}v=0. \tag{20}$$

We expand the denominator of the last term by the binomial theorem and look for a solution of the form

$$v=1+a_1z+a_2z^2+a_3z^3+\cdots,$$

in which we use the characteristic value α_l already obtained for the particular harmonic under consideration.

After some labor we obtain for the fundamental the two way infinite series

$$\begin{aligned}Y_1(\eta)=e^{i\epsilon\eta}\Bigg[1&+\frac{i}{\epsilon}\left\{1-\frac{2}{5}\epsilon^2-\frac{2}{5^3\cdot 7}\epsilon^4+\frac{4}{3\cdot 5^5\cdot 7}\epsilon^6-\frac{62}{5^6\cdot 7^3\cdot 11}\epsilon^8+\cdots\right\}\frac{1}{\eta}\\&+\frac{1}{5}\left\{1-\frac{69}{5^2\cdot 7}\epsilon^2-\frac{62}{3\cdot 5^4\cdot 7}\epsilon^4+\frac{3943}{3\cdot 5^5\cdot 7^3\cdot 11}\epsilon^6+\cdots\right\}\frac{1}{\eta^2}\\&+\frac{1}{5}\frac{i}{\epsilon}\left\{1-\frac{94}{5^2\cdot 7}\epsilon^2+\frac{2014}{3^2\cdot 5^4\cdot 7}\epsilon^4+\frac{34{,}588}{3\cdot 5^5\cdot 7^3\cdot 11}\epsilon^6+\cdots\right\}\frac{1}{\eta^3}\\&+\frac{3}{5\cdot 7}\left\{1-\frac{289}{3^3\cdot 5^2}\epsilon^2+\frac{75{,}914}{3^3\cdot 5^4\cdot 7^2\cdot 11}\epsilon^4+\cdots\right\}\frac{1}{\eta^4}\end{aligned}$$

(*Formula continued on next page.*)

$$+\frac{3}{5\cdot 7}\frac{i}{\epsilon}\left\{1-\frac{46}{3\cdot 5^2}\epsilon^2+\frac{761{,}914}{3^3\cdot 5^4\cdot 7^2\cdot 11}\epsilon^4+\cdots\right\}\frac{1}{\eta^5}$$
$$+\frac{1}{3\cdot 7}\left\{1-\frac{371}{3\cdot 5^2\cdot 11}\epsilon^2+\cdots\right\}\frac{1}{\eta^6}$$
$$+\frac{1}{3\cdot 7}\frac{i}{\epsilon}\left\{1-\frac{182}{5^2\cdot 11}\epsilon^2+\cdots\right\}\frac{1}{\eta^7}$$
$$+\frac{1}{3\cdot 11}\left\{1+\cdots\right\}\frac{1}{\eta^8}$$
$$+\frac{1}{3\cdot 11}\frac{i}{\epsilon}\left\{1+\cdots\right\}\frac{1}{\eta^9}+\cdots\Bigg]. \quad (21)$$

The calculation of this series gives us an absolute check on the correctness of the previously computed characteristic value (13), for the successive terms in the series for α_1 are of just the magnitude necessary to cause each alternate coefficient of z to start with a power of ϵ two greater than the previous coefficient. Shortly we shall obtain an absolute check on the correctness of every coefficient in (21).

Similarly, we obtain for the third harmonic

$$Y_3(\eta)=e^{i\epsilon\eta}\Bigg[1$$
$$+6\frac{i}{\epsilon}\left\{1-\frac{2}{3^2\cdot 5}\epsilon^2+\frac{38}{3^5\cdot 5^3\cdot 11}\epsilon^4+\cdots\right\}\frac{1}{\eta}$$
$$-15\frac{1}{\epsilon^2}\left\{1-\frac{22}{3^2\cdot 5^2}\epsilon^2+\frac{398}{3^5\cdot 5^4}\epsilon^4+\cdots\right\}\frac{1}{\eta^2}$$
$$-15\frac{i}{\epsilon^3}\left\{1-\frac{8}{5^2}\epsilon^2+\frac{3242}{3^3\cdot 5^4\cdot 11}\epsilon^4+\cdots\right\}\frac{1}{\eta^3}$$
$$-10\frac{1}{\epsilon^2}\left\{1-\frac{1667}{2\cdot 3^3\cdot 5^2\cdot 11}\epsilon^2+\cdots\right\}\frac{1}{\eta^4}$$
$$-10\frac{i}{\epsilon^3}\left\{1-\frac{2521}{3^3\cdot 5^2\cdot 11}\epsilon^2+\cdots\right\}\frac{1}{\eta^5}$$
$$-\frac{75}{11}\frac{1}{\epsilon^2}\left\{1+\cdots\right\}\frac{1}{\eta^6}$$
$$-\frac{75}{11}\frac{i}{\epsilon^3}\left\{1+\cdots\right\}\frac{1}{\eta^7}+\cdots\Bigg], \quad (22)$$

and for the fifth harmonic

$$Y_5(\eta)=e^{i\epsilon\eta}\Bigg[1+15\frac{i}{\epsilon}\{1+\cdots\}\frac{1}{\eta}-105\frac{1}{\epsilon^2}\{1+\cdots\}\frac{1}{\eta^2}$$
$$-420\frac{i}{\epsilon^3}\{1+\cdots\}\frac{1}{\eta^3}+945\frac{1}{\epsilon^4}\{1+\cdots\}\frac{1}{\eta^4}$$
$$+945\frac{i}{\epsilon^5}\{1+\cdots\}\frac{1}{\eta^5}+\cdots\Bigg], \quad (23)$$

to more than the needed degree of approximation.

While (21) serves for the calculaton of the frequency and decrement of the fundamental free oscillation for eccentricities less than 0.8, the series converge altogether too slowly to be useful for the important cases of eccentricities nearly equal to unity. To treat the latter we must obtain solutions of (10) in the neighborhood of the origin ($\eta=1$).

Solution of the Equation in η for Small Values of the Variable

If we put $t\equiv\eta^2-1$ the differential equation (10) becomes

$$4t(1+t)d^2Y/dt^2+2tdY/dt+\{\epsilon^2(1+t)-\alpha\}Y=0. \quad (24)$$

As this equation is of the second order the complete solution is a linear combination of two independent primitives. We look, therefore, for series solutions in powers of ϵ^2 the coefficients of which are polynomials in positive powers of t.

Denoting the two independent primitives by U and V, we find for the fundamental $(\alpha=\alpha_1)$,

$$U_1=t-\frac{1}{2\cdot 5}t^2\epsilon^2-\left(\frac{1}{2\cdot 5^3\cdot 7}t^2-\frac{1}{2^3\cdot 5\cdot 7}t^3\right)\epsilon^4+\left(\frac{1}{3\cdot 5^5\cdot 7}t^2+\frac{1}{2^2\cdot 3^2\cdot 5^3\cdot 7}t^3-\frac{1}{2^4\cdot 3^3\cdot 5\cdot 7}t^4\right)\epsilon^6+\cdots, \quad (25)$$

$$V_1=U_1\log\frac{(1+t)^{\frac{1}{2}}-1}{(1+t)^{\frac{1}{2}}+1}+(1+t)^{\frac{1}{2}}\left[2+\left(\frac{4}{5}-\frac{1}{5}t\right)\epsilon^2+\left(\frac{284}{5^3\cdot 7}-\frac{151}{5^3\cdot 7}t+\frac{1}{2^2\cdot 5\cdot 7}t^2\right)\epsilon^4+\left(\frac{8632}{3\cdot 5^5\cdot 7}-\frac{115{,}646}{3^4\cdot 5^5\cdot 7}t+\frac{667}{3^4\cdot 5^3\cdot 7}t^2-\frac{1}{2^3\cdot 3^3\cdot 5\cdot 7}t^3\right)\epsilon^6+\cdots\right]. \quad (26)$$

The most general solution of (10) is, therefore,

$$Y_1=A_1U_1+B_1V_1 \quad (27)$$

in the case of the fundamental, where the constants A_1 and B_1 may be functions of the parameter ϵ. Now, in order to satisfy the boundary condition at infinity, (27) must be identical with (21). Therefore we expand the logarithm in a power series and replace t by η^2-1 in the former, and expand the exponential in a power series in the latter, and compare coefficients. In this way we find that

$$A_1=-\tfrac{1}{3}\epsilon^2 a_1, \quad B_1=\tfrac{3}{4}(i/\epsilon)b_1, \quad (28)$$

where

$$a_1\equiv 1-\frac{1}{2\cdot 5^2}\epsilon^2+\frac{187}{2^3\cdot 5^4\cdot 7^2}\epsilon^4-\frac{26{,}021}{2^4\cdot 3^4\cdot 5^6\cdot 7^2}\epsilon^6+\cdots$$
$$=1-0.020{,}000\epsilon^2+0.000{,}763\epsilon^4-0.000{,}026\epsilon^6+\cdots, \quad (29)$$

$$b_1\equiv 1-\frac{19}{2\cdot 5^2}\epsilon^2-\frac{2609}{2^3\cdot 5^4\cdot 7^2}\epsilon^4+\frac{32{,}593}{2^4\cdot 3^4\cdot 5^5\cdot 7^2}\epsilon^6+\cdots$$
$$=1-0.380{,}000\epsilon^2-0.010{,}649\epsilon^4+0.000{,}164\epsilon^6+\cdots. \quad (30)$$

In carrying through this process each term in A_1 and B_1 was calculated by comparing at least *two* different groups of terms in the two solutions, and every term in each solution was used at least once. Therefore the calculation of A_1 and B_1 constituted an absolute check on the accuracy of every numerical coefficient in (21) on the one hand and in (25) and (26) on the other.

Similarly in the case of the third harmonic $(\alpha=\alpha_3)$ we find

$$U_3=t+\frac{5}{2^2}t^2-\left(\frac{1}{3\cdot 5}t^2+\frac{5}{2^3\cdot 3^2}t^3\right)\epsilon^2+\cdots, \quad (31)$$

$$V_3=U_3\log\frac{(1+t)^{\frac{1}{2}}-1}{(1+t)^{\frac{1}{2}}+1}+(1+t)^{\frac{1}{2}}\left[\frac{1}{3}+\frac{5}{2}t+\left(\frac{2}{3^3\cdot 5}-\frac{11}{2\cdot 3^3\cdot 5}t-\frac{5}{2^2\cdot 3^2}t^2\right)\epsilon^2+\cdots\right], \quad (32)$$

and the solution satisfying the boundary condition at infinity is

$$Y_3=A_3U_3+B_3V_3, \quad (33)$$

where

$$A_3=\frac{4}{3\cdot 5^2\cdot 7}\epsilon^4 a_3, \quad B_3=-\frac{1575}{2^3}\frac{i}{\epsilon^3}b_3, \quad (34)$$

and

$$a_3\equiv 1+\cdots, \quad (35)$$

$$b_3\equiv 1-\frac{37}{2\cdot 3^3\cdot 5^2}\epsilon^2+\cdots. \quad (36)$$

In the case of the fifth harmonic $(\alpha=\alpha_5)$

$$U_5=t+\frac{7}{2}t^2+\frac{21}{2^3}t^3+\cdots, \quad (37)$$

$$V_5=U_5\log\frac{(1+t)^{\frac{1}{2}}-1}{(1+t)^{\frac{1}{2}}+1}+(1+t)^{\frac{1}{2}}\left[\frac{2}{3\cdot 5}+\frac{7}{2}t+\frac{21}{2^2}t^2+\cdots\right], \quad (38)$$

and the solution satisfying the boundary condition at infinity is

$$Y_5=A_5U_5+B_5V_5, \quad (39)$$

where

$$A_5=\text{negligible}, \quad B_5=\frac{3{,}274{,}425}{2^5}\frac{i}{\epsilon^5}b_5, \quad (40)$$

and

$$b_5\equiv 1+\cdots. \quad (41)$$

Fundamental Free Oscillations

The boundary condition at the surface of the conductor for free oscillation is that $E_\xi = 0$ for all ξ's. Therefore it follows from (4) that

$$dY/d\eta = 0, \qquad \eta = \eta_0, \tag{42}$$

where $1/\eta_0$ is the eccentricity of the conducting surface.

For large values of η_0 we use (21) for the fundamental. Performing the differentiation and solving for $i\omega/v$ by successive approximations we find

$$i\frac{\omega}{v} = \frac{1}{2a}\left\{1 - \frac{53}{875}\frac{1}{\eta_0^4} - \frac{3764}{65{,}625}\frac{1}{\eta_0^6} - \frac{8{,}242{,}450}{176{,}859{,}375}\frac{1}{\eta_0^8} + \cdots\right\} + i\frac{(3)^{\frac{1}{2}}}{2a}\left\{1 + \frac{4}{15}\frac{1}{\eta_0^2} + \frac{971}{7875}\frac{1}{\eta_0^4} + \frac{40{,}412}{590{,}625}\frac{1}{\eta_0^6} + \frac{203{,}351{,}122}{4{,}775{,}203{,}125}\frac{1}{\eta_0^8} + \cdots\right\}$$

$$= \frac{1}{2a}\left\{1 - 0.0606\frac{1}{\eta_0^4} - 0.0573\frac{1}{\eta_0^6} - 0.0466\frac{1}{\eta_0^8} + \cdots\right\} + i\frac{(3)^{\frac{1}{2}}}{2a}\left\{1 + 0.2667\frac{1}{\eta_0^2} + 0.1233\frac{1}{\eta_0^4} + 0.0684\frac{1}{\eta_0^6} + 0.0426\frac{1}{\eta_0^8} + \cdots\right\}, \tag{43}$$

where a is the semi-major axis of the spheroid.

Table I. *Wave-lengths and logarithmic decrements for fundamental free oscillation of spheroids of eccentricities between 0 and 0.8.*

$1/\eta_0$	$\lambda/4a$	δ
0.0	1.814	3.628
0.1	1.809	3.618
0.2	1.794	3.588
0.3	1.770	3.537
0.4	1.734	3.461
0.5	1.686	3.356
0.6	1.625	3.214
0.7	1.549	3.024
0.8	1.455	2.772

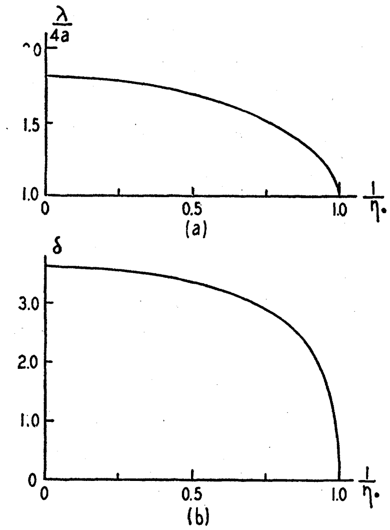

Fig. 2. (a) Wave-length λ and (b) logarithmic decrement δ of fundamental free oscillation of spheroid of eccentricity $1/\eta_0$.

If we write

$$i\omega/v = \chi + i\psi, \tag{44}$$

the ratio of the half-wave-length $\lambda/2$ to the major axis $2a$ of the spheroid is

$$\lambda/4a = \pi/2a\psi \tag{45}$$

and the logarithmic decrement δ is

$$\delta = 2\pi\chi/\psi. \tag{46}$$

Unfortunately the series (43) converge too slowly to permit accurate calculations for eccentricities much greater than 0.7. In Table I the quantities $\lambda/4a$ and δ are given for eccentricities from 0 to 0.8. The calculated values are plotted on the graphs of Fig. 2, the portions of the curves corresponding to values of the eccentricity near to unity being plotted from the data in Table II, which will be computed later. We note that both the wave-length and the decrement decrease with increasing eccentricity.

Since (21) does not converge rapidly enough to describe the fundamental free oscillations for eccentricities close to unity, we must resort to the solution (27). We shall consider only eccentricities so close to unity that $t_0 = \eta_0^2 - 1$ and its positive powers are negligible. If then we put

$$l \equiv \left[\log\frac{\eta_0+1}{\eta_0-1} - 2\right]^{-1} \tag{47}$$

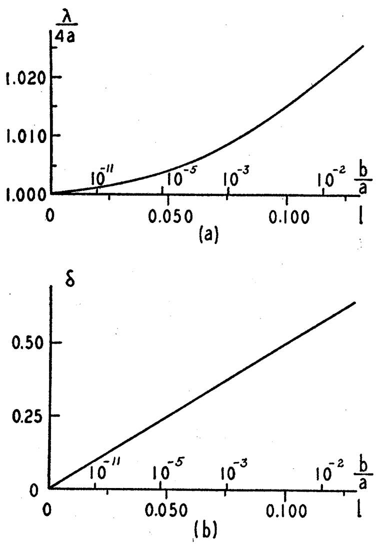

FIG. 3. (a) Wave-length λ and (b) logarithmic decrement δ of fundamental free oscillation of spheroid of major axis a and minor axis b.

and write

$$m_1 \equiv 1 - \frac{1}{5}\epsilon^2 l + \frac{9}{5^3\cdot 7}\epsilon^4 l - \frac{886}{3^4\cdot 5^5\cdot 7}\epsilon^6 l + \cdots$$
$$= 1 - 0.200{,}000\epsilon^2 l + 0.010{,}286\epsilon^4 l - 0.000{,}500\epsilon^6 l + \cdots, \quad (48)$$

the boundary condition (42) gives

$$b_1 = (4/9)i\epsilon^3 l a_1/m_1. \quad (49)$$

We shall compute the wave-length and decrement for six values of l corresponding to values of ratios of the minor axis $2b$ to the major axis $2a$ of the spheroid extending from 0 to about 1/74.

First consider the limiting case ($l=0$) of eccentricity unity, the ratio of b to a being zero. The boundary condition (49) reduces to $b_1=0$. Therefore ϵ is entirely real and the decrement is zero. This does not mean that no energy is radiated from the spheroid, but rather that the energy stored in the electromagnetic field is infinite compared with the energy radiated during a period of oscillation. Solving for ϵ we find

$$\epsilon = 1.5708 \quad (50)$$

correct to five significant figures. This is exactly $\pi/2$. Hence, as $a=f$ in this case, the half-wave-length is exactly equal to the major axis of the spheroid. These values of both wave-length and decrement agree with those obtained by Abraham for the limiting case.

To discuss the remaining cases we multiply (49) by m_1/a_1. Combining the series represented by a_1, b_1 and m_1, we get the formula

$$1 - \{0.360{,}000 + 0.200{,}000l\}\epsilon^2 - \{0.018{,}612 - 0.082{,}286l\}\epsilon^4 + \{0.000{,}093 - 0.000{,}481l\}\epsilon^6 = 0.444{,}444il\epsilon^3. \quad (51)$$

We know that ϵ cannot differ greatly from $\pi/2$ over the range considered. Hence we put

$$\epsilon = (\pi/2)(1 - g - ih). \quad (52)$$

A few trials indicate that an accuracy of one-tenth of one percent or better in the wave-length and better than one percent in the decrement is obtained if we retain terms in g, h, gh, h^2 and neglect all higher powers. Then (51) leads to the simultaneous equations

$$(2.2214 - 0.9735l)g + (1.5472 - 2.4039l)h^2 = 5.1677l(h - 2gh), \quad (53)$$

$$(2.2214 - 0.9735l)h - (3.0944 - 4.8078l)gh = 1.7226l(1 - 3g - 3h^2), \quad (54)$$

which we may solve for g and h for any given value of l in the range considered. In this range it is unnecessary to distinguish between a and f.

In Table II are entered the results of the computation. The second column contains the ratio of b to a for the assumed value of l. These results are plotted in Fig. 3.

FORCED OSCILLATIONS

We now turn our attention to oscillations forced by an electric field $E = E_0 e^{-i\omega t}$ parallel to the x axis. As the assumption of a uniform field

TABLE II. *Wave-lengths and logarithmic decrements for fundamental free oscillation of spheroids of eccentricities near to unity.*

l	b/a	$\lambda/4a$	δ
0.000	0.00	1.000	0.000
0.020	1.02 $(10)^{-11}$	1.001	0.098
0.050	3.34 $(10)^{-5}$	1.004	0.247
0.080	1.42 $(10)^{-3}$	1.009	0.396
0.100	4.96 $(10)^{-3}$	1.015	0.494
0.125	1.35 $(10)^{-2}$	1.023	0.613

over the region occupied by the conducting spheroid is a valid approximation only when the wave-length is long compared with the shorter axis of the spheroid, we are limited in our discussion of oscillations in the neighborhood of resonance to eccentricities close to unity. This, however, is the range of most interest.

If γ is the angle which the unit vector $\boldsymbol{\xi}_1$ (Fig. 1) makes with the x axis, we find

$$\cos\gamma=\eta\left(\frac{1-\xi^2}{\eta^2-\xi^2}\right)^{\frac{1}{2}}. \quad (55)$$

Therefore the boundary condition at the surface of the conductor is

$$E_\xi+E\eta_0\left(\frac{1-\xi^2}{\eta_0{}^2-\xi^2}\right)^{\frac{1}{2}}=0. \quad (56)$$

With the use of (4) for E_ξ this condition becomes

$$(\partial A/\partial\eta)_0=i(\kappa/\mu)^{\frac{1}{2}}\epsilon a(1-\xi^2)E_0e^{-i\omega t}, \quad (57)$$

where we have replaced $f\eta_0$ by its equal a.

Evidently A must be a linear combination of the solutions representing the odd harmonics, that is,

$$A=\{C_1u_1(\xi)Y_1(\eta)+C_3u_3(\xi)Y_3(\eta)+C_5u_5(\xi)Y_5(\eta)+\cdots\}(1-\xi^2)^{\frac{1}{2}}e^{-i\omega t}, \quad (58)$$

where $u_1(\xi)$, $u_3(\xi)$, $u_5(\xi)$ are given by (14), (16), (18), respectively, and $Y_1(\eta)$, $Y_3(\eta)$, $Y_5(\eta)$ by (27), (33), (39).

Determining the coefficients C_1, C_3, C_5 from the boundary condition (57) we have

$$C_1=i\left(\frac{\kappa}{\mu}\right)^{\frac{1}{2}}\epsilon aE_0\left\{1-\frac{2}{5^4\cdot 7}\epsilon^4+\frac{8}{3\cdot 5^6\cdot 7}\epsilon^6+\cdots\right\}\Big/\left(\frac{dY_1}{d\eta}\right)_0, \quad (59)$$

$$C_3=i\left(\frac{\kappa}{\mu}\right)^{\frac{1}{2}}\epsilon aE_0\left\{\frac{1}{3\cdot 5^2}\epsilon^2-\frac{2}{3^2\cdot 5^4}\epsilon^4+\cdots\right\}\Big/\left(\frac{dY_3}{d\eta}\right)_0, \quad (60)$$

$$C_5=i\left(\frac{\kappa}{\mu}\right)^{\frac{1}{2}}\epsilon aE_0\left\{\frac{1}{3^4\cdot 5\cdot 7}\epsilon^4+\cdots\right\}\Big/\left(\frac{dY_5}{d\eta}\right)_0. \quad (61)$$

To save writing we shall put

$$k\equiv 2\pi c(\kappa/\mu)^{\frac{1}{2}}aE_0. \quad (62)$$

Then the current I_0 at the center ($\xi=0$) of the antenna is

$$I_0=2\pi c(A)_{\xi=0,\ \eta=\eta_0,}$$
$$=ik\epsilon\Bigg[\left\{1-\frac{2}{5^4\cdot 7}\epsilon^4+\frac{8}{3\cdot 5^6\cdot 7}\epsilon^6+\cdots\right\}u_1(0)\frac{Y_1(\eta_0)}{(dY_1/d\eta)_0}+\frac{1}{3\cdot 5^2}\epsilon^2\left\{1-\frac{2}{3\cdot 5^2}\epsilon^2+\cdots\right\}u_3(0)\frac{Y_3(\eta_0)}{(dY_3/d\eta)_0}+\frac{1}{3^4\cdot 5\cdot 7}\epsilon^4\{1+\cdots\}u_5(0)\frac{Y_5(\eta_0)}{(dY_5/d\eta)_0}+\cdots\Bigg]e^{-i\omega t}, \quad (63)$$

where

$$u_1(0)=1+\frac{1}{2\cdot 5^2}\epsilon^2-\frac{89}{2^3\cdot 5^4\cdot 7^2}\epsilon^4+\frac{733}{2^4\cdot 3^4\cdot 5^5\cdot 7^2}\epsilon^6+\cdots, \quad (64)$$

$$u_3(0)=-\frac{3}{2}\left\{1-\frac{13}{2\cdot 3^3\cdot 5^2}\epsilon^2+\cdots\right\}, \quad (65)$$

$$u_5(0)=\frac{15}{2^3}\{1+\cdots\}, \quad (66)$$

from (14), (16) and (18). We note that $u_1(0)$ is the reciprocal of the series a_1 given by (29).

From (27) we find

$$Y_1(\eta_0)=-\frac{3}{2}\frac{i}{\epsilon}c_1, \quad (67)$$

$$\left(\frac{dY_1}{d\eta}\right)_0=-\frac{3}{2}\frac{i}{\epsilon}\frac{b_1m_1}{l}-\frac{2}{3}\epsilon^2a_1, \quad (68)$$

to a sufficient degree of approximation for eccentricities so near to unity as those under consideration, where

$$c_1\equiv 1+\frac{1}{2\cdot 5^2}\epsilon^2-\frac{89}{2^3\cdot 5^4\cdot 7^2}\epsilon^4+\frac{733}{2^4\cdot 3^4\cdot 5^5\cdot 7^2}\epsilon^6+\cdots$$
$$=u_1(0)=1+0.020{,}000\epsilon^2-0.000{,}363\epsilon^4+0.000{,}004\epsilon^6+\cdots, \quad (69)$$

and from (33)

$$Y_3(\eta_0) = -\frac{525}{2^3}\frac{i}{\epsilon^3}c_3, \quad (70)$$

$$\left(\frac{dY_3}{d\eta}\right)_0 = \frac{1575}{2^2}\frac{i}{\epsilon^3}\frac{b_3 m_3}{l} + \frac{8}{3\cdot 5^2\cdot 7}\epsilon^4, \quad (71)$$

where

$$c_3 \equiv 1 + \frac{23}{2\cdot 3^3\cdot 5^2}\epsilon^2 + \cdots, \quad (72)$$

$$m_3 \equiv 1 - \frac{2}{3}l + \frac{1}{2\cdot 3\cdot 5}\epsilon^2 l + \cdots. \quad (73)$$

As the term in the fifth harmonic is quite negligible we shall not trouble to write it down.

The second term in (71) is negligible compared with the first, and we need retain only the first term in the series for b_3 and c_3 and the first two terms in the series for m_3. Let us put

$$s_1 \equiv 1 - \frac{2}{5^4\cdot 7}\epsilon^4 + \frac{8}{3\cdot 5^6\cdot 7}\epsilon^6 + \cdots$$
$$= 1 - 0.000{,}457\epsilon^4 + 0.000{,}024\epsilon^6 + \cdots, \quad (74)$$

$$s_3 \equiv 1 - \frac{2}{3\cdot 5^2}\epsilon^2 + \cdots. \quad (75)$$

In the series s_3 we need retain only the first term. Thus we obtain for the current I_0 at the center of the antenna

$$I_0 = k\left[\frac{(l\epsilon s_1 c_1^2)(4l\epsilon^3 a_1/9)}{(4l\epsilon^3 a_1/9)^2 + (b_1 m_1)^2} - i\left\{\frac{(l\epsilon s_1 c_1^2)(b_1 m_1)}{(4l\epsilon^3 a_1/9)^2 + (b_1 m_1)^2} - \frac{l\epsilon^3}{300}(1 + \tfrac{2}{3}l)\right\}\right]e^{-i\omega t}. \quad (76)$$

The third harmonic contributes only a very small term to the imaginary part of the current.

The series a_1, b_1, m_1 and s_1 are specified by (29), (30), (48) and (74), respectively. The series for c_1^2, obtained by squaring (69), is

$$c_1^2 = 1 + 0.040{,}000\epsilon^2 - 0.000{,}327\epsilon^4 - 0.000{,}007\epsilon^6 + \cdots. \quad (77)$$

The square of the current amplitude I_{00} at the center is

$$I_{00}^2 = k^2\left[\frac{(l\epsilon s_1 c_1^2)^2}{(4l\epsilon^3 a_1/9)^2 + (b_1 m_1)^2}\right] \times \left[1 - \frac{\epsilon^2}{150}b_1(1 + \tfrac{2}{3}l)\right] \quad (78)$$

to a sufficient degree of approximation. To find the frequency of resonance we must equate to zero the derivative of this with respect to ϵ, or, more conveniently, with respect to ϵ^2. This leads to the formula

$$b_1 = 0.14750 l^2\epsilon^6 \frac{1 - 0.00560l}{(1 - 0.43837l)^2}, \quad (79)$$

where, in the very small right-hand member, we have replaced ϵ by $\pi/2$ in the numerator and denominator of the fractional factor. The justification for this lies in the fact that the values of ϵ computed from (79) are found to differ very little from $\pi/2$ in the range under consideration.

We have computed the wave-length λ and the current I_0 at resonance, and also the current amplitude I_{00} and the angle ϕ by which the current leads the electromotive force, for six values of l. These are contained in Table III and are plotted in Fig. 4. In the second column of the table is given the ratio of the minor to the major axis of the spheroid corresponding to the assumed value of l.

It will be noted that the real part of the current is substantially the same for all values of l considered, and that the current leads the electromotive force at resonance for all values of l greater than zero by an angle ϕ which increases with increasing l. For $l=0$ the half-wave-length is exactly equal to the major axis of the spheroid as in the case of free oscillation, but for values of l greater than zero the resonant wave-length is slightly shorter than the wave-length of free oscillation. Experiments on an-

TABLE III. *Wave-lengths and currents at resonance.*

l	b/a	$\lambda/4a$	$I_0/ke^{-i\omega t}$	I_{00}/k	ϕ
0.000	0.00	1.0000	1.045	1.045	0°.0
0.020	1.02 $(10)^{-11}$	1.0004	$1.045-0.028i$	1.045	1°.5
0.050	3.34 $(10)^{-5}$	1.0027	$1.045-0.071i$	1.047	3°.9
0.080	1.42 $(10)^{-3}$	1.007	$1.045-0.113i$	1.050	6°.2
0.100	4.96 $(10)^{-3}$	1.011	$1.045-0.141i$	1.055	7°.7
0.125	1.35 $(10)^{-2}$	1.017	$1.046-0.175i$	1.061	9°.5

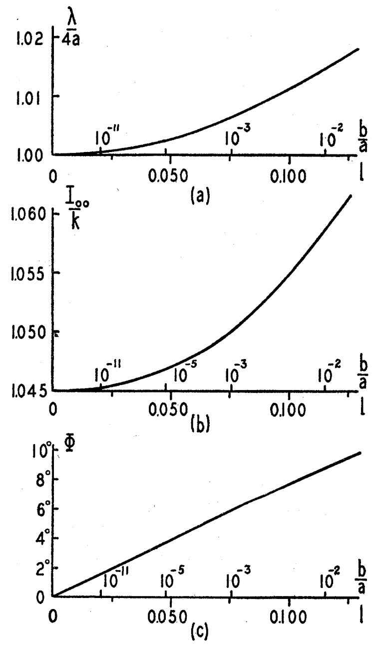

FIG. 4. (a) Wave-length λ, (b) current amplitude I_{00} and (c) lead ϕ of current at resonance.

tennas in the form of right circular cylinders indicate that the half-wave-length at resonance is from five to six percent longer than the antenna.[4] This is in satisfactory qualitative agreement with our calculated ratios of $\lambda/2$ to $2a$ for the prolate spheroid, for the diameter of a prolate spheroid falls to half of its maximum value at a distance $x=0.866a$ from the center whatever the eccentricity may be, and consequently the prolate spheroid which best fits a right circular cylinder is one which is slightly wider at the center and somewhat longer,[5] as illustrated in Fig. 5. We may call this prolate spheroid the *equivalent spheroid* for the given cylinder. Nevertheless it is very desirable that experimental measurements of the ratio $\lambda/4a$ should be made for actual spheroids in order to corroborate the theory.

We have computed resonance curves for $l=0.02$, 0.05 and 0.10. The data for these are given in Tables IV to VI, where ϵ_r represents the value of ϵ at resonance. The current amplitude I_{00} at the center of the antenna and the phase ϕ by which the current leads the electromotive force are depicted in Fig. 6. It will be noted that the resonance is sharper and that the phase shift toward $\pm\pi/2$ as we depart from resonance is more rapid the smaller l.

The distribution of current along the length of a thin antenna at resonance has been measured. Since $\xi=x/f\eta_0=x/a$, the quantity I_ξ/I_0 represents the ratio of the current, at a point whose distance from the center of the antenna is ξa, to the current at the center. The experimental measurements indicate that, at resonance,

$$I_\xi/I_0=\cos\tfrac{1}{2}\pi\xi \qquad (80)$$

within the experimental error.

In developing an expression for this ratio from our theory, we obtain sufficient accuracy if we neglect the third and higher harmonics. Then we get the formula

$$\frac{I_\xi}{I_0}=(1-\xi^2)\left[1-\frac{\epsilon^2}{2\cdot5}\left(1-\frac{4}{5^2\cdot7}\epsilon^2+\frac{8}{3\cdot5^4\cdot7}\epsilon^4\right)\xi^2+\frac{\epsilon^4}{2^3\cdot5\cdot7}\left(1-\frac{8}{3^2\cdot5^2}\epsilon^2\right)\xi^4-\frac{\epsilon^6}{2^4\cdot3^3\cdot5\cdot7}\xi^6+\cdots\right]. \qquad (81)$$

Let p be the ratio of the frequency for which it is desired to evaluate (81) to the frequency

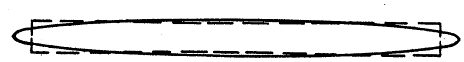

FIG. 5. Prolate spheroid equivalent to cylinder.

for resonance with $l=0$. Then

$$p=\epsilon/1.5708, \qquad (82)$$

and (81) becomes

$$I_\xi/I_0=(1-\xi^2)[1-0.24674p^2\{1-0.05640p^2+0.00371p^4\}\xi^2+0.02174p^4\{1-0.08773p^2\}\xi^4-0.00099p^6\xi^6+\cdots]. \qquad (83)$$

TABLE IV. *Current for $l=0.02$, $b/a=1.02\ (10)^{-11}$, $\epsilon_r=1.5701$.*

ϵ/ϵ_r	$I_0/ke^{-i\omega t}$	I_{00}/k	ϕ
0.875	$0.011-0.120i$	0.120	84°.8
0.950	$0.081-0.292i$	0.303	74°.6
0.975	$0.275-0.474i$	0.548	59°.9
0.990	$0.726-0.495i$	0.879	34°.3
1.000	$1.045-0.028i$	1.045	1°.5
1.010	$0.757+0.453i$	0.882	−30°.9
1.025	$0.313+0.464i$	0.560	−56°.0
1.050	$0.107+0.302i$	0.320	−70°.5
1.125	$0.022+0.137i$	0.139	−80°.7

[4] C. R. Englund, Bell System Tech. J. 7, 404 (1928).
[5] Rayleigh, Phil. Mag. 8, 105 (1904).

When $p=1$ this reduces to

$$I_\xi/I_0=(1-\xi^2)[1-0.2337\xi^2+0.0198\xi^4-0.0010\xi^6+\cdots], \quad (84)$$

which is exactly $\cos \pi\xi/2$ to the fourth decimal place. When $l=0$ it should be remembered that the term in the third harmonic disappears from (76) and therefore, in that case, we make no error in neglecting it.

In Table VII are listed the values of I_ξ/I_0 over the range of ξ for various values of p near to resonance.

Next we shall calculate the rate of absorption of energy and the radiation resistance of the antenna. Since the element of distance ds at the surface of the spheroid in the direction of increasing ξ is

$$ds=f\left(\frac{\eta_0^2-\xi^2}{1-\xi^2}\right)^{\frac{1}{2}}d\xi, \quad (85)$$

the electromotive force $d\mathcal{E}$ corresponding to the increment $d\xi$ is

$$d\mathcal{E}=E_0e^{-i\omega t}\cos\gamma ds=E_0e^{-i\omega t}ad\xi \quad (86)$$

from (55).

At any ξ the ratio of the part I_ξ' of the current in phase with the electromotive force to the part I_0' in phase with the electromotive force at the center of the antenna is

$$\frac{I_\xi'}{I_0'}=\frac{(1-\xi^2)^{\frac{1}{2}}u_1(\xi)}{u_1(0)}=\frac{(1-\xi^2)^{\frac{1}{2}}u_1(\xi)}{c_1} \quad (87)$$

from (58).

Hence we obtain from (76) for the mean rate of absorption of energy.

TABLE V. *Current for $l=0.05$, $b/a=3.34\ (10)^{-5}$, $\epsilon_r=1.5666$.*

ϵ/ϵ_r	$I_0/ke^{-i\omega t}$	I_{00}/k	ϕ
0.850	$0.042-0.238i$	0.242	80°.0
0.900	$0.104-0.347i$	0.363	73°.3
0.950	$0.352-0.529i$	0.635	56°.4
0.975	$0.707-0.525i$	0.880	36°.6
1.000	$1.045-0.071i$	1.047	3°.9
1.025	$0.783+0.418i$	0.888	−28°.1
1.050	$0.450+0.482i$	0.660	−46°.9
1.100	$0.182+0.360i$	0.403	−63°.2
1.150	$0.100+0.270i$	0.288	−69°.7

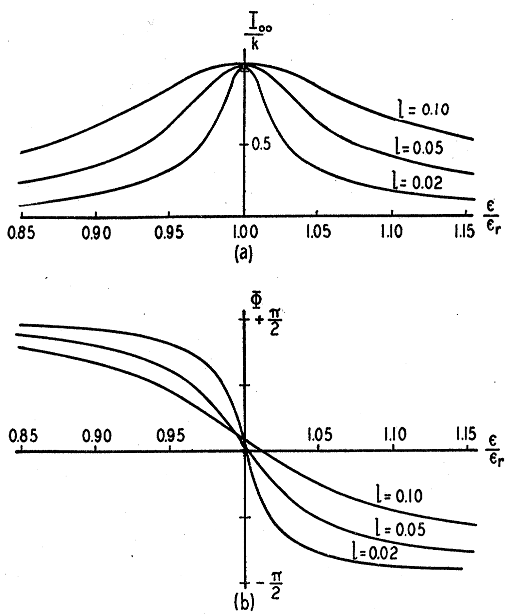

FIG. 6. (a) Current amplitude I_{00} and (b) lead ϕ of current plotted against ratio ϵ/ϵ_r of frequency to frequency at resonance.

$$\left\langle\frac{dU}{dt}\right\rangle_{Av}=\tfrac{1}{2}kE_0\frac{(l\epsilon s_1c_1^2)(4l\epsilon^3a_1/9)}{(4l\epsilon^3a_1/9)^2+(b_1m_1)^2}\frac{a}{c_1}\times\int_{-1}^{1}(1-\xi^2)^{\frac{1}{2}}u_1(\xi)d\xi. \quad (88)$$

Now

$$\int_{-1}^{1}(1-\xi^2)^{\frac{1}{2}}u_1(\xi)d\xi=\int_{-1}^{1}P_{11}(\xi)u_1(\xi)d\xi=4/3 \quad (89)$$

from (14). Consequently, from the relation $a_1c_1=1$ and from (62),

$$\left\langle\frac{dU}{dt}\right\rangle_{Av}=\frac{64}{27}\pi^3\left(\frac{\kappa}{\mu}\right)^{\frac{1}{2}}cE_0^2\frac{a^2f^2}{\lambda^2}\frac{a_1^4}{s_1}\times\frac{(l\epsilon s_1c_1^2)^2}{(4l\epsilon^3a_1/9)^2+(b_1m_1)^2} \quad (90)$$

TABLE VI. *Current for $l=0.10$, $b/a=4.96\ (10)^{-3}$, $\epsilon_r=1.5539$.*

ϵ/ϵ_r	$I_0/ke^{-i\omega t}$	I_{00}/k	ϕ
0.850	$0.144-0.428i$	0.451	71°.5
0.900	$0.306-0.546i$	0.625	60°.8
0.950	$0.669-0.575i$	0.882	40°.7
1.000	$1.045-0.141i$	1.055	7°.7
1.050	$0.822+0.360i$	0.897	−23°.6
1.100	$0.504+0.450i$	0.676	−41°.8
1.150	$0.326+0.410i$	0.524	−51°.5

in terms of the amplitude E_0 of the exciting field, or

$$\left\langle\frac{dU}{dt}\right\rangle_{\text{Av}}=\frac{16}{27}\pi\left(\frac{\mu}{\kappa}\right)^{\frac{1}{2}}\frac{I_{00}^2}{c}\frac{f^2}{\lambda^2}\frac{a_1^4}{s_1}\times\left[1+\frac{\epsilon^2}{150}b_1(1+\tfrac{2}{3}l)\right] \quad (91)$$

in terms of the amplitude I_{00} of the current at the center of the antenna.

The radiation resistance R, defined as the quotient of the mean rate of absorption of energy by $\frac{1}{2}I_{00}^2$, is

$$R=\frac{32}{27}\pi\left(\frac{\mu}{\kappa}\right)^{\frac{1}{2}}\frac{f^2}{c\lambda^2}\frac{a_1^4}{s_1}\left[1+\frac{\epsilon^2}{150}b_1(1+\tfrac{2}{3}l)\right]. \quad (92)$$

In practical units, the radiation resistance R_p in ohms is

$$R_p=87.670\left(\frac{\mu}{\kappa}\right)^{\frac{1}{2}}\left(\frac{4a}{\lambda}\right)^2\frac{a_1^4}{s_1}\times\left[1+\frac{\epsilon^2}{150}b_1(1+\tfrac{2}{3}l)\right], \quad (93)$$

provided we replace f by a. In Table VIII is given the radiation resistance at resonance for the six values of l considered before.

In Table IX we give the quantity $(\kappa/\mu)^{\frac{1}{2}}R_p$ for various fractions ϵ/ϵ_r of the frequency at resonance, both for $l=0.050$ and for $l=0.100$. The results are plotted in Fig. 7.

Finally we turn our attention to the electromagnetic field of the oscillating antenna. The three nonvanishing field components E_ξ, E_η and H_ϕ are given in terms of A and its derivatives by (4), (5) and (6), respectively. Since the coefficient C_3 in (58) may be expressed in terms of C_1 by the relation

$$C_3=-(2b_1/3^2\cdot 5^4\cdot 7)\epsilon^4 C_1 \quad (94)$$

to a quite sufficient degree of accuracy, we find that

$$A=-\frac{ik}{3\pi c}l\epsilon^2 s_1\frac{4l\epsilon^3 a_1/9-ib_1 m_1}{(4l\epsilon^3 a_1/9)^2+(b_1 m_1)^2}\left\{u_1(\xi)Y_1(\eta)-\frac{2b_1}{3^2\cdot 5^4\cdot 7}\epsilon^4 u_3(\xi)Y_3(\eta)\right\}(1-\xi^2)^{\frac{1}{2}}e^{-i\omega t}. \quad (95)$$

At a distance from the nearest point on the antenna not less than the semi-interfocal distance f the field components are specified to better than one percent by using the functions (21) and (22) for $Y_1(\eta)$ and $Y_3(\eta)$, respectively, although very close to the antenna it is necessary to use (27) and (33).

The components of the radiation field are

$$E_\xi=\frac{k}{3\pi c}\left(\frac{\mu}{\kappa}\right)^{\frac{1}{2}}\frac{l\epsilon^2 s_1}{f}\frac{1}{(\eta^2-\xi^2)^{\frac{1}{2}}}\times\frac{1}{\{(4l\epsilon^3 a_1/9)^2+(b_1 m_1)^2\}^{\frac{1}{2}}}\times\left\{u_1(\xi)-\frac{2b_1\epsilon^4}{3^2\cdot 5^4\cdot 7}u_3(\xi)\right\}e^{i(\epsilon\eta-\omega t+\pi/2-\gamma)}, \quad (96)$$

$$H_\phi=-\frac{k}{3\pi c}\frac{l\epsilon^2 s_1}{f}\frac{1}{(\eta^2-1)^{\frac{1}{2}}}\times\frac{1}{\{(4l\epsilon^3 a_1/9)^2+(b_1 m_1)^2\}^{\frac{1}{2}}}\times\left\{u_1(\xi)-\frac{2b_1\epsilon^4}{3^2\cdot 5^4\cdot 7}u_3(\xi)\right\}e^{i(\epsilon\eta-\omega t+\pi/2-\gamma)}, \quad (97)$$

where

$$\tan\gamma\equiv\frac{b_1 m_1}{4l\epsilon^3 a_1/9}. \quad (98)$$

TABLE VII. *Current distribution I_ξ/I_0.*

ξ \ p	0.90	0.95	1.00	1.05	1.10
±0.1	0.988	0.988	0.9877	0.987	0.987
±0.2	0.953	0.952	0.9511	0.950	0.949
±0.3	0.894	0.893	0.8910	0.889	0.887
±0.4	0.815	0.812	0.8090	0.806	0.803
±0.5	0.715	0.711	0.7071	0.703	0.699
±0.6	0.597	0.592	0.5878	0.583	0.578
±0.7	0.464	0.459	0.4540	0.449	0.443
±0.8	0.318	0.313	0.3090	0.304	0.300
±0.9	0.162	0.159	0.1564	0.153	0.150

TABLE VIII. *Radiation resistance at resonance.*

l	$(\kappa/\mu)^{\frac{1}{2}}R_p$(ohms)
0.000	73.1
0.020	73.0
0.050	72.8
0.080	72.3
0.100	71.8
0.125	71.1

For small l it follows that γ is nearly zero at resonance, nearly $\pi/2$ below resonance, and nearly $-\pi/2$ above resonance.

As the Poynting flux is $-cE_\xi H_\phi$ in the direction of increasing η and the element of area of a spheroidal surface is $2\pi f^2\{(\eta^2-\xi^2)(\eta^2-1)\}^{\frac{1}{2}}d\xi$, the mean rate of radiation of energy is

$$\left\langle\frac{dR}{dt}\right\rangle_{\text{Av}}=\frac{16}{9}\pi^3\left(\frac{\kappa}{\mu}\right)^{\frac{1}{2}}cE_0^2\frac{a^2f^2}{\lambda^2}a_1^4 \times\frac{(l\epsilon s_1c_1^2)^2}{(4l\epsilon^3a_1/9)^2+(b_1m_1)^2}\int_{-1}^{1}[u_1(\xi)]^2d\xi, \quad (99)$$

provided we make use of the relation $a_1c_1=1$ and neglect, as heretofore, the square of the third harmonic. From (14) we find

$$\int_{-1}^{1}[u_1(\xi)]^2d\xi=\frac{4}{3}\left\{1+\frac{2}{5^4\cdot 7}\epsilon^4-\frac{8}{3\cdot 5^6\cdot 7}\epsilon^6+\cdots\right\}=\frac{4}{3s_1}, \quad (100)$$

TABLE IX. *Radiation resistance $(\kappa/\mu)^{\frac{1}{2}}R_p$ in ohms for frequencies near resonance.*

l \ ϵ/ϵ_r	0.850	0.900	0.950	1.000	1.050	1.100	1.150
0.050	55.3	61.0	66.9	72.8	78.7	84.6	90.6
0.100	54.5	60.2	66.0	71.8	77.7	83.6	89.4

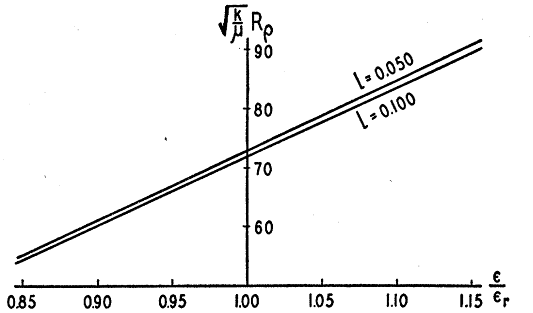

FIG. 7. Radiation resistance R_p in ohms plotted against ratio ϵ/ϵ_r of frequency to frequency at resonance.

where s_1 is the series defined by (74). Therefore

$$\left\langle\frac{dR}{dt}\right\rangle_{\text{Av}}=\frac{64}{27}\pi^3\left(\frac{\kappa}{\mu}\right)^{\frac{1}{2}}cE_0^2\frac{a^2f^2}{\lambda^2}\frac{a_1^4}{s_1} \times\frac{(l\epsilon s_1c_1^2)^2}{(4l\epsilon^3a_1/9)^2+(b_1m_1)^2} \quad (101)$$

in agreement with (90).

In a future communication we hope to discuss the field in more detail.

Note added in proof. Since submitting this article for publication there has come to the attention of the authors a recent paper by L. V. King [Trans. Roy. Soc. London, **236**, 381 (1937)] in which the radiation from a cylindrical antenna is investigated by a very different method from that pursued here.