On the Saturation of Nuclear Forces

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The forces between heavy elementary particles, i.e., neutrons and protons, are investigated on the hypothesis that they emit electron-positron pairs. It is also assumed that the interaction of the heavy particles with the field of light particles is large compared with the kinetic energy of the light particles. It is shown that potentials result which are of the same order of magnitude as the kinetic energy of the light particles. When many heavy particles interact the total potential energy is found to be proportional to the number of heavy particles.

INTRODUCTION

THE mass defect of most atomic nuclei is roughly proportional to the number of heavy elementary particles contained in the nucleus. This shows that the binding energy per elementary particle is, in first approximation, independent of the number of particles constituting the nucleus.

Since the heavy particles obey the Fermi-Dirac statistics the average kinetic energy of a heavy particle will vary with the two-thirds power of the density whenever the density is sufficiently high and the velocities of the heavy particles are small compared with the velocity of light. Thus, under these conditions, the kinetic energy in a given volume element of a heavy nucleus will be proportional to the five-thirds power of the density. If, on the other hand, the total potential energy is a sum of potential energies acting between pairs of heavy particles and if the range of the potential energy is small compared with the nuclear radius the total potential energy in a given volume within the nucleus will be proportional to the square of the density. If attraction is assumed between all heavy particles, the binding energy would then increase with increasing density and stability could not be reached before the nuclear radius becomes as small as the range of the attractive forces. In this case, however, the mass defect for heavy nuclei would increase more rapidly than the number of heavy particles contained in the nucleus.[1]

In order to escape this difficulty Heisenberg[2] and Majorana have assumed an interaction between heavy particles which is of the exchange type. For such interactions it follows that a neutron or proton can only interact with a limited number of other neutrons or protons. The following mechanism for this exchange interaction has been proposed:[3] a neutron emits an electron and a neutrino and turns into a proton; a proton absorbs the emitted pair and becomes a neutron. Thus an exchange of charge between a neutron and proton takes place.

This picture, however, cannot explain in a simple way why the forces between two protons (between which no such simple exchange of charge can take place) is of the same order of magnitude as the force between a neutron and proton.[4, 5] Moreover, the probability of electron-neutrino pair emission can be estimated by extrapolating the probabilities of beta-decay processes. Performing this extrapolation either in accordance with the theory proposed by Fermi[6] or with that proposed by Uhlenbeck and Konopinski[7] one obtains a too small value for the interaction between heavy particles.

Gamow and Teller[8] and Wentzel[9] have therefore suggested that an emission and reabsorption of electron-positron pairs should be responsible for the nuclear forces. The equality of proton-proton and proton-neutron forces is then more easily understood and also sufficient freedom is obtained to explain the magnitude of the nuclear

[1] These arguments were first put forward by Heisenberg at the Solvay Congress, 1933.
[2] W. Heisenberg, "Rapports de Congres Solvay, 1933."
[3] D. Iwanenko and I. Tamm, Nature **133**, 981 (1934).
[4] M. A. Tuve, L. R. Hafstad and N. P. Heydenburg, Phys. Rev. **50**, 806 (1936).
[5] G. Breit, E. Condon and R. Present, Phys. Rev. **50**, 825 (1936).
[6] E. Fermi, Zeits. f. Physik **88**, 161 (1934).
[7] K. Konopinski and G. Uhlenbeck, Phys. Rev. **48**, 7 (1935).
[8] G. Gamow and E. Teller, Phys. Rev. **51**, 289 (1937).
[9] G. Wentzel, Helv. Phys. Acta **10**, 107 (1936).

forces. Since, however, according to this hypothesis the charge is not exchanged and since a study of the neutron-proton interaction forces has eliminated the possibility that the major part of the neutron-proton interaction[10] is due to spin exchange we must conclude that if nuclear forces are due to electron-positron pair emission the main part of the forces between heavy elementary particles is not of the exchange type.[11] The purpose of the present paper is to show that the interaction between heavy particles resulting from the pair-emission hypothesis is compatible with the proportionality between mass defect and number of heavy particles.

In order to do this it will be necessary to drop the assumption that the total potential energy within the nucleus can be represented as a sum of interactions between pairs of heavy particles and that the potential energy is proportional to the square of their number. Indeed, this is correct only if the virtual states into which the light particles may be emitted are occupied only with a very small probability. If, however, the term in the Hamiltonian, H', which gives rise to the creation and annihilation of the light particles is large compared with the kinetic energy, E_{kin}, of the light particles in the virtual state a saturation phenomenon may result and the potential energy may increase less strongly than with the square of the number of particles. It then follows from the argument given in the beginning of the introduction that if the potential energy increase less strongly than the five-thirds power of the density the density of heavy nuclei will not depend strongly on the number of particles within the nucleus, and the mass defect divided by the number of particles in the nucleus will remain roughly constant.

We shall assume therefore, in the following discussion, that $H' \gg E_{\text{kin}}$. This assumption is just the opposite of the one usually made, namely $H' \ll E_{\text{kin}}$, which served to justify the treatment of the emission and absorption of light particles by a perturbation method.

In order to simplify the treatment we assume that not only the charge but also the spin of a heavy particle remains unchanged if light particles are emitted. Furthermore, the Coulomb interactions will be neglected. Thus, finer features such as spin dependence of nuclear potentials and the difference between proton and neutron mass will not be obtained from the present development.

Part I. One Heavy Particle

We consider at first one heavy particle and discuss the effect of the operator H' giving rise to emission and absorption of electron-positron pairs. The kinetic energy of the light particles, E_{kin}, will be neglected in accordance with the hypothesis $E_{\text{kin}} \ll H'$. The effect of E_{kin} will be taken into consideration as a perturbation in Part III.

It is convenient to adopt the "hole theory" of positrons in developing the formalism for the emission and absorption of pairs. Thus, emission of a pair will correspond to the transition from a state of negative to a state of positive energy, and an annihilation of a pair, a transition from a positive to a negative state.

Since we assume that a heavy particle after emission is in the same state in which it was before emission of a pair there is a possibility that a second pair be emitted before the first is reabsorbed. In general, a single heavy particle might emit successively in this way any number of pairs without reabsorbing them. If, however, pairs can be emitted only into a limited number of states, the exclusion principle will limit the number of pairs which can be emitted.

In particular we shall suppose the interaction H' to be such that emission (or absorption) can take place only into four states $A\uparrow$, $A\downarrow$, $B\uparrow$, $B\downarrow$. $A\uparrow$ and $A\downarrow$ are two spherically symmetrical electron states with positive kinetic energy, differing only in spin direction. $B\uparrow$ and $B\downarrow$ on the other hand, are states of negative kinetic energy. The absence of an electron from $B\uparrow$ is equivalent

[10] If the neutron-proton interaction be due to pure spin exchange there could not be an attraction between a neutron and a proton for both parallel and anti-parallel spins.

[11] If the probabilities of electron-positron, electron-neutrino and positron-neutrino emissions are of the same order of magnitude, an exchange of charge would remain possible and there would be no necessity for abandoning exchange forces. This suggestion is implied in Feenberg's discussion (Phys. Rev. **51**, 777 (1937)) of heavy particle interactions. The order of magnitude of nuclear forces would then still require an explanation (see Kemmer, Phys. Rev. **52**, 906 (1937)) which can, perhaps, be given in terms of heavy electrons and neutrinos (see Bhabha, Nature **141**, 117 (1938) and Kemmer, Nature **141**, 116 (1938)).

to the existence of a positron in a corresponding state with its spin pointing downward. We shall assume that the dependence on space coordinates of the A's and B's is the same. In general all negative states orthogonal to the B's shall be assumed as filled. $B\uparrow$ or $B\downarrow$, however, shall be filled only if the factor $B\uparrow$ or $B\downarrow$ explicitly appears in the proper function.

The formalism used in quantizing the wave equation will be adopted. Associated with the electron wave function $A\uparrow$ is the operator $a\uparrow$ which acts on the functional ψ, i.e., on the wave function of all electrons. $a\uparrow\psi$ is zero if the state $A\uparrow$ is represented in ψ as filled and $a\uparrow\psi=(\psi, A\uparrow)$ if $A\uparrow$ is empty. $(\psi, A\uparrow)$ is the properly antisymmetrized product wave function of ψ and $A\uparrow$. This means that $a\uparrow$ gives rise to the appearance of an electron in the state $A\uparrow$ if $A\uparrow$ be empty and is equivalent to zero if $A\uparrow$ be full. In accordance with the formalism, $a\uparrow^*$ acting on $(\psi, A\uparrow)$ gives ψ and acting on ψ gives zero, so that $a\uparrow^*$ causes the disappearance of an electron from $A\uparrow$ if $A\uparrow$ be full and is equivalent to zero if $A\uparrow$ be empty. In the same way $a\downarrow$, $a\downarrow^*$ are connected with $A\downarrow$; $b\uparrow$, $b\uparrow^*$ with $B\uparrow$; and $b\downarrow$, $b\downarrow^*$ with $B\downarrow$. For normalized functions the commutation laws hold:

$$\begin{aligned} &xx^*+x^*x=1, && xy+yx=0,\\ &xy^*+y^*x=0\ (x\neq y), && x^*y^*+y^*x^*=0, \qquad (1)\\ &\qquad x, y=a\uparrow, a\downarrow, b\uparrow, b\downarrow. \end{aligned}$$

The term giving rise to pair emission and absorption in the Hamiltonian will then be

$$H'=\sigma O, \qquad (2)$$

where σ is a positive constant and the operator O may be written[12]

$$O\equiv a\uparrow b\uparrow^*+b\uparrow a\uparrow^*+a\downarrow b\downarrow^*+b\downarrow a\downarrow^*. \qquad (3)$$

The first and third terms represent the production, the second and fourth terms the absorption of an electron-positron pair.

If we write the pair emission operator, O, as the sum of two commuting operators, $O\uparrow$ and $O\downarrow$:

$$\begin{aligned} &O\uparrow\equiv a\uparrow b\uparrow^*+b\uparrow a\uparrow^*, && O=O\uparrow+O\downarrow,\\ &O\downarrow\equiv a\downarrow b\downarrow^*+b\downarrow a\downarrow^*, && O\uparrow O\downarrow-O\downarrow O\uparrow=\text{zero}, \qquad (4) \end{aligned}$$

iteration of $O\uparrow$ gives, because of (1),

$$\begin{aligned} &O\uparrow O\uparrow=a\uparrow b\uparrow^* b\uparrow a\uparrow^*+b\uparrow a\uparrow^* a\uparrow b\uparrow^*,\\ &(O\uparrow)^3=O\uparrow \text{ and similarly}\\ &(O\downarrow)^3=O\downarrow. \end{aligned}$$

The proper values of $O\uparrow$ and $O\downarrow$ are therefore ±1 and zero.

It is readily seen that the proper functionals of $O\uparrow$ belonging to the proper value -1 will be those for which the state $(A\uparrow-B\uparrow)/\sqrt{2}$ is full and the state $(A\uparrow+B\uparrow)/\sqrt{2}$ is empty. For the functionals belonging to the proper value $+1$ the state $(A\uparrow-B\uparrow)/\sqrt{2}$ is empty and $(A\uparrow+B\uparrow)/\sqrt{2}$ is full; for the proper value zero both states $(A\uparrow-B\uparrow)/\sqrt{2}$ and $(A\uparrow+B\uparrow)/\sqrt{2}$ have to be full (i.e. there is an electron in $A\uparrow$ but there is no corresponding positron present) or both have to be empty (i.e., we have a positron but no electron).. All these proper functionals are degenerate since the presence of electrons in any state orthogonal to $A\uparrow$ and $B\uparrow$ will have no effect on the proper value. Similar statements are obtained for $O\downarrow$ if $A\downarrow$ and $B\downarrow$ are used instead of $A\uparrow$ and $B\uparrow$.

Since $O\uparrow$ and $O\downarrow$ commute the proper values of $O=O\uparrow+O\downarrow$ will be the sums of the proper values of $O\uparrow$ and $O\downarrow$. Thus we obtain ±2, ±1, and zero. The corresponding proper functionals will be products of the proper functionals of $O\uparrow$ and $O\downarrow$. The proper functionals for the lowest proper value, -2, will be characterized by the states' $(A\uparrow-B\uparrow)/\sqrt{2}$ and $(A\downarrow-B\downarrow)/\sqrt{2}$ being filled and the states' $(A\uparrow+B\uparrow)/\sqrt{2}$ and $(A\downarrow+B\downarrow)/\sqrt{2}$ being empty. We again have degeneracy since any state orthogonal to $A\uparrow$, $A\downarrow$, $B\uparrow$, $B\downarrow$ may be filled or empty. This degeneracy means merely that the state of the nucleus will not depend upon extranuclear electrons or positrons. The lowest proper value of H' is therefore -2σ. Since σ is assumed to be large all other states lie high and therefore will not be excited. We shall be concerned only with the lowest state.[13]

We propose to consider the energy -2σ as part of the proper energy (or mass) of the proton or neutron. Because of its negative value it cannot represent, of course, the complete proper energy.

[12] A somewhat more general form for H' may be obtained by adding $\lambda(a\uparrow a\uparrow^*+a\downarrow a\downarrow^*+b\uparrow b\uparrow^*+b\downarrow b\downarrow^*)$ where λ is a new constant. However, as long as $\sigma-\lambda\gg E_{\text{kin}}$ our main argument remains unchanged.

[13] The exclusion of the proper value zero means that the wave functions of atomic electrons must be orthogonal to $[A\uparrow]$ and $[A\downarrow]$. This amounts to a boundary condition which all atomic wave functions must satisfy near the nucleus. If the spatial extension of $[A\uparrow]$ and $[A\downarrow]$ is very small compared with that of an atom the influence of this boundary condition will be small. It might account for an essential part of the isotope effects in atomic spectra.

This energy, -2σ, can be, however, incorporated into the proper energy only if it can be shown that the corresponding part of the energy for N particles (neutrons or protons) will be $-2N\sigma$ independently of their configuration. This proof is given in the next section.

Part II. Several Heavy Particles

The operator $H'(N)$ corresponding to the emission and absorption of pairs by N heavy particles will be the sum of operators H_k'

$$H'(N)=\sum_{k=1}^{N}H_k'=\sigma\sum_{k=1}^{N}O_k, \tag{5}$$

where H_k' corresponds to the emission and absorption of pairs by the kth particle. We may thus write:

$$O_k\equiv a_k\uparrow b_k\uparrow^*+b_k\uparrow a_k\uparrow^*+a_k\downarrow b_k\downarrow^*+b_k\downarrow a_k\downarrow^*, \tag{3k}$$

where $a_k\uparrow$ and $a_k\uparrow^*$ are operators associated with the electron state $A_k\uparrow$ which is the same spherical function as $A\uparrow$ with its center being located at the kth particle. Similar statements hold for $a_k\downarrow$, $a_k\downarrow^*$, $b_k\uparrow$, $b_k\uparrow^*$ and $b_k\downarrow$, $b_k\downarrow^*$. The proper functions of (5) are not, in general, simple because $A_k\uparrow$ is not orthogonal to $A_m\uparrow$, and so on for $A\downarrow$ and the B's.

However, two particular cases can be investigated at once: the first is that in which the N particles are far enough apart so that all A's and all B's can be considered orthogonal. Then the lowest proper value of $H'(N)$ is the sum of the lowest values of H_k' or just $-2N\sigma$. The proper functionals belonging to this value are those functionals in which all $(A_k\uparrow-B_k\uparrow)/\sqrt{2}$ and $(A_k\downarrow-B_k\downarrow)/\sqrt{2}$ are filled and all $(A_k\uparrow+B_k\uparrow)/\sqrt{2}$ and $(A_k\downarrow+B_k\downarrow)/\sqrt{2}$ are empty.

The second simple case is obtained when the positions of all heavy particles coincide[14] and all O_k's become identical with O_1. Then (5) becomes

$$H'(N)=N\sigma O_1$$

and the lowest proper value is again $-2N\sigma$. In the corresponding proper functionals $(A_1\uparrow-B_1\uparrow)/\sqrt{2}$ and $(A_1\downarrow-B_1\downarrow)/\sqrt{2}$ are filled and $(A_1\uparrow+B_1\uparrow)/\sqrt{2}$ and $(A_1\downarrow+B_1\downarrow)$ are empty.

[14] The fact that the Pauli principle would prohibit more than four heavy particles' occupying the same point is important in the determination of the behavior of particles in nuclei but not of essential interest in this consideration of the potential in which they move.

To show that the lowest proper value of $H'(N)$ is always $-2N\sigma$ we define the quantities

$$\epsilon_{km}=\int A_k\uparrow^*A_m\uparrow dv=\int A_k\downarrow^*A_m\downarrow dv$$
$$=\int B_k\uparrow^*B_m\uparrow dv=\int B_k\downarrow^*B_m\downarrow dv \tag{6}$$

with $\epsilon_{kk}=1$.

The commutation relations may be written:

$$x_kx_m^*+x_m^*x_k=\epsilon_{mk},$$
$$x_k,\ y_k=a_k\uparrow,\ a_k\downarrow,\ b_k\uparrow,\text{ or }b_k\downarrow, \tag{7}$$
$$x_ky_m^*+y_m^*x_k=\text{zero},\quad x\neq y\text{ etc.}$$

It is possible to choose a set of $4N$ orthogonal states $A_\lambda\uparrow$, $A_\lambda\downarrow$, $B_\lambda\uparrow$ and $B_\lambda\downarrow$ related to $A_k\uparrow$, $A_k\downarrow$, $B_k\uparrow$ and $B_k\downarrow$ respectively by the linear relations:

$$X_k=\sum_\lambda T_{k\lambda}X_\lambda,\quad X=A\uparrow,\ A\downarrow,\ B\uparrow,\text{ or }B\downarrow,$$
$$X_\lambda=\sum_k T_{\lambda k}^{-1}X_k. \tag{8}$$

Operators x_λ and x_λ^* associated with X_λ are defined by:

$$x_\lambda=\sum_k T_{\lambda k}^{-1}x_k,\qquad x_k=\sum_\lambda T_{k\lambda}x_\lambda,$$
$$x=a\uparrow,\ a\downarrow,\ b\uparrow,\text{ or }b\downarrow, \tag{9}$$
$$x_\lambda^*=\sum_k(T_{\lambda k}^{-1})^*x_k^*,\quad x_k^*=\sum_\lambda T_{k\lambda}^*x_\lambda^*.$$

It will be convenient to take for $T_{k\lambda}$ an orthogonal transformation,[15] so that

$$T_{k\lambda}^*=T_{\lambda k}^{-1}. \tag{10}$$

In fact, under such a transformation, $H'(N)$ retains the simple form that it has in (5).

$$\begin{aligned}H'(N)&=\sigma\sum_k\sum_{\lambda,\rho}T_{k\lambda}T_{k\rho}^*(a_\lambda\uparrow b_\rho\uparrow^*\\&\qquad+b_\lambda\uparrow a_\rho\uparrow^*+a_\lambda\downarrow b_\rho\downarrow^*+b_\lambda\downarrow a_\rho\downarrow^*)\\&=\sigma\sum_{\lambda,\rho}\delta_\lambda^{\rho}(a_\lambda\uparrow b_\rho\uparrow^*\\&\qquad+b_\lambda\uparrow a_\rho\uparrow^*+a_\lambda\downarrow b_\rho\downarrow^*+b_\lambda\downarrow a_\rho\downarrow^*)\\&=\sigma\sum_\lambda O_\lambda,\end{aligned} \tag{11}$$

where $O_\lambda\equiv(a_\lambda\uparrow b_\lambda\uparrow^*+b_\lambda\uparrow a_\lambda\uparrow^*+a_\lambda\downarrow b_\lambda\downarrow^*+b_\lambda\downarrow a_\lambda\downarrow^*)$.

[15] It will be seen that $T_{k\lambda}$ is the transformation which brings the Hermitian matrix $\{\epsilon_{mk}\}$ to principal axes.

On the other hand, if the transformation T is orthogonal it follows that the orthogonal set of functions X_λ cannot be normalized. This is so because, if we assume the set X_λ both orthogonal and normalized, then any orthogonal transformation (8) would yield a set X_k which would be both orthogonal and normalized which is in contradiction to the above statements. We therefore define the scalars c_λ by the relations[16]

$$\int X_\lambda^* X_\mu dv = c_\lambda \delta_\lambda^{\mu}, \quad X = A\uparrow, A\downarrow, B\uparrow, \text{or } B\downarrow, \tag{12}$$

leading to the commutation relations

$$\begin{aligned} &x_\lambda x_\mu^* + x_\mu^* x_\lambda = \sum_{k,m} T_{k\lambda}^* T_{m\mu} \epsilon_{km} = c_\lambda \delta_\lambda^{\mu}, \\ &x_\lambda y_\mu^* + y_\mu^* x_\lambda = \text{zero}, \quad x = a\uparrow,\ a\downarrow,\ b\uparrow, \text{ or } b\downarrow \\ &\quad (x \neq y) \qquad\qquad \text{etc.} \end{aligned} \tag{13}$$

The operator O_λ may be again considered as the sum of the commuting operators $O_\lambda\uparrow$ and $O_\lambda\downarrow$:

$$\begin{aligned} &O_\lambda\uparrow \equiv a_\lambda\uparrow b_\lambda\uparrow^* + b_\lambda\uparrow a_\lambda\uparrow^*, \quad O_\lambda = O_\lambda\uparrow + O_\lambda\downarrow, \\ &O_\lambda\downarrow \equiv a_\lambda\downarrow b_\lambda\downarrow^* + b_\lambda\downarrow a_\lambda\downarrow^*, \\ &\qquad O_\lambda\uparrow O_\lambda\downarrow - O_\lambda\uparrow O_\lambda\downarrow = \text{zero}. \end{aligned} \tag{14}$$

The proper values of O_λ will then be the sums of those for $O_\lambda\uparrow$ and $O_\lambda\downarrow$ and the proper functionals will be product functions of those for $O_\lambda\uparrow$ and $O_\lambda\downarrow$. From (13), iteration of $O_\lambda\uparrow$ shows that

$$\begin{aligned} (O_\lambda\uparrow)^3 &= c_\lambda^2 O_\lambda\uparrow, \\ (O_\lambda\downarrow)^3 &= c_\lambda^2 O_\lambda\downarrow, \end{aligned} \tag{15}$$

so that the lowest proper value of each is $-c_\lambda$ and the lowest proper value of O_λ is $-2c_\lambda$; this value belongs to those functionals in which the states $(A_\lambda\uparrow - B_\lambda\uparrow)/(2c_\lambda)^{\frac{1}{2}}$ and $(A_\lambda\downarrow - B_\lambda\downarrow)/(2c_\lambda)^{\frac{1}{2}}$ are filled and the states $(A_\lambda\uparrow + B_\lambda\uparrow)/(2c_\lambda)^{\frac{1}{2}}$ and $(A_\lambda\downarrow + B_\lambda\downarrow)/(2c_\lambda)^{\frac{1}{2}}$ are empty.

According to the commutation relations (13) the operators O_λ commute and

$$O_\lambda O_\mu - O_\mu O_\lambda = \text{zero}$$

and therefore the proper values of $H'(N)$ will be given by σ times the sums of the proper values of the O_λ. In particular, the lowest proper value will be $-2\sigma\sum_\lambda c_\lambda$ and the corresponding functionals will be those in which all $(A_\lambda\uparrow - B_\lambda\uparrow)/(2c_\lambda)^{\frac{1}{2}}$ and $(A_\lambda\downarrow - B_\lambda\downarrow)/(2c_\lambda)^{\frac{1}{2}}$ are filled and all $(A_\lambda\uparrow + B_\lambda\uparrow)/(2c_\lambda)^{\frac{1}{2}}$ and $(A_\lambda\downarrow + B_\lambda\downarrow)/(2c_\lambda)^{\frac{1}{2}}$ are empty. From (13)

$$\sum_\lambda c_\lambda = \sum_\lambda \sum_{k,m} T_{k\lambda} T_{m\lambda}^* \epsilon_{km} = \sum_k \epsilon_{kk} = N. \tag{16}$$

The lowest proper value of $H'(N)$ is then $-2N\sigma$ and independent of the relative position of the heavy particles.

[16] The numbers c_λ are the proper values of the matrix $\{\epsilon_{mk}\}$.

Part III. Kinetic Energy of Light Particle Field

We now introduce the kinetic energy of the light particles, K, as a perturbation in the calculation of the proper values of

$$H = H' + K. \tag{17}$$

We are interested in the lowest value of H. The lowest level of H' has been seen to be an infinitely degenerate one of energy -2σ. The introduction of K as a perturbation splits these levels and the problem of interest is to find the lowest of the resulting states.

We shall adopt the usual convention of putting the kinetic energy of the state in which no electrons or positrons are present equal to zero. The average value of K in the state ψ is therefore calculated by finding $\int \psi^* K \psi dv$ and subtracting from this value the average value of K in all negative energy levels. The average value for the lowest proper functions of H' is then

$$\begin{aligned} &\tfrac{1}{2}\int (A\uparrow - B\uparrow)^* K (A\uparrow - B\uparrow) dv \\ &\quad + \tfrac{1}{2}\int (A\downarrow - B\downarrow)^* K (A\downarrow - B\downarrow) dv + \bar{\kappa} \\ &\quad - \int B\uparrow^* K B\uparrow dv - \int B\downarrow^* K B\downarrow dv - \bar{\kappa}_0, \end{aligned} \tag{18}$$

where $\bar{\kappa}$ is the average value of K in all states orthogonal to A's and B's and $\bar{\kappa}_0$ is the average value of K in all negative states orthogonal to the B's.

According to the perturbation theory the lowest energy value into which a degenerate state will split is given by the minimum average value of the perturbation for the set of degenerate proper functions under consideration. It is evi-

dent that the matrix of K, calculated in accordance with the preceding paragraph, is positive definite. Also the matrix of $\kappa-\kappa_0$ is positive definite with the minimum value zero for that state in which no positive levels are occupied and all negative levels are full. Therefore the lowest proper value of H will be, in first approximation, the lowest value of H', i.e., -2σ, plus (18) with $\bar{\kappa}=\bar{\kappa}_0$. Since the space dependence of $A\uparrow$, $A\downarrow$, $B\uparrow$, and $B\downarrow$ is taken to be the same

$$\int A\uparrow^* KA\uparrow dv = \int A\downarrow^* KA\downarrow dv$$

$$= -\int B\uparrow^* KB\uparrow dv = -\int B\downarrow^* KB\downarrow dv$$

and the integrals in (18) may be expressed by

$$E_{\text{kin}} = 2\int A\uparrow^* KA\uparrow dv. \tag{19}$$

and we obtain

$$E_{\min} = -2\sigma + E_{\text{kin}}. \tag{20}$$

In a representation including the A's and B's, K also has nondiagonal elements. The order of magnitude of these elements is the same as that of E_{kin}. For example, the element in K coupling the state in which $(A\uparrow-B\uparrow)/\sqrt{2}$, $(A\downarrow-B\downarrow)/\sqrt{2}$ are full and $(A\uparrow+B\uparrow)/\sqrt{2}$, $(A\downarrow+B\downarrow)/\sqrt{2}$ as well as all other positive energy states are empty with the state in which $(A\uparrow-B\uparrow)/\sqrt{2}$, $(A\downarrow+B\downarrow)/\sqrt{2}$ are full and $(A\uparrow+B\uparrow)/\sqrt{2}$, $(A\downarrow-B\downarrow)/\sqrt{2}$ as well as all other positive states are empty is readily calculated to be $\frac{1}{2}E_{\text{kin}}$. However, the diagonal elements for the two states differ by 2σ. According to second order perturbation formulae the correction to the calculation of the minimum energy associated with this transition will be $E^2_{\text{kin}}/8\sigma$. Similar transitions to electron states orthogonal to the A's and B's involve an energy difference approximately equal to σ. We may then say that expression (20) is correct if contributions of the order of $(E^2_{\text{kin}}/\sigma)$ and higher orders may be neglected.

The order of magnitude of the additional energy, E_{kin}, can be estimated if the spatial extent of $A\uparrow$ is known. If the characteristic linear dimension of $A\uparrow$ (its effective radius for example) be "d" then for sufficiently small values of d: $E_{\text{kin}}\sim cp\sim c\hbar/d$. The significance of this energy is seen in considering many heavy particles.

For N heavy particles the total Hamiltonian is

$$H(N) = \sum_i [H_i'] + \sum_i K_i, \tag{21}$$

where K_i is the operator K for the ith heavy particle alone. If contributions of the order E^2_{kin}/σ are neglected the lowest proper value of (21) is, by the same reasoning which led to (20),

$$E_{\min} = -2N\sigma + \int \psi_0^* \sum_i K_i \psi_0 dv$$

$$- \int \chi^* \sum_i K_i \chi dv, \tag{22}$$

where ψ_0 is that functional belonging to $-2N\sigma$ in which all negative levels orthogonal to the B's are filled and all positive levels orthogonal to the A's are empty. χ is the functional of negative states including the B's. The average value of K indicated in (22) cannot be calculated in general without a specific knowledge of the dependence of $A_\lambda\uparrow$, etc., on space coordinates. Since the form of this dependence is not known and since we make no hypothesis regarding it, we are limited to the evaluation of (22) for those special cases which are independent of the shape of the electron functions.

If the N heavy particles are sufficiently widely separated the electron functions ($A_k\uparrow$, etc.) pertaining to each heavy particle may be considered orthogonal. Then the average value of K will be found lowest in that state in which all positive levels orthogonal to the A_k's as well as all $(A_k\uparrow+B_k\uparrow)/\sqrt{2}$, $(A_k\downarrow+B_k\downarrow)$ are empty and all negative levels orthogonal to the B_k's as well as all $(A_k\uparrow-B_k\uparrow)/\sqrt{2}$, $(A_k\downarrow-B_k\downarrow)/\sqrt{2}$ are filled. The average value of $\sum_i K_i$ will be NE_{kin}:

$$E_{\min}(\infty) = -2N\sigma + NE_{\text{kin}}. \tag{23}$$

The other simple case occurs when the N heavy particles may be considered to occupy the same point in space. The proper function of minimum energy in this instance is that in which $(A_1\uparrow-B_1\uparrow)/\sqrt{2}$, $(A_1\downarrow-B_1\downarrow)/\sqrt{2}$ and all negative levels orthogonal to $B_1\uparrow$ and $B_1\downarrow$ are filled and $(A_1\uparrow+B_1\uparrow)/\sqrt{2}$, $(A_1\downarrow+B_1\downarrow)/\sqrt{2}$ and all positive

levels orthogonal to $A_1\uparrow$ and $A_1\downarrow$ are empty. As before these A's and B's are chosen arbitrarily from the N identical possibilities. Then

$$E_{\min}(O) = -2N\sigma + E_{\text{kin}}. \tag{24}$$

Conclusion

The difference between $E_{\min}(\infty)$ in (23) and $E_{\min}(O)$ in (24) is the binding energy obtained if the N particles are brought from infinite distances to the same point. Even if the positions of all heavy particles do not coincide the value of $E_{\min}(O)$ will remain essentially unchanged as long as the mutual distances remain small compared with the linear dimension, d, of the wave functions A and B.

The interaction which we have obtained differs essentially from the ordinary interaction between pairs; if the interaction is the sum of interactions between pairs, then by bringing the heavy particles close together the potential energy will be proportional to the number of pairs, that is, to $\frac{1}{2}N(N-1)$; for the type of nonexchange forces which we have discussed, however, the interaction at close contact will be $E_{\min}(\infty)-E_{\min}(O) = (N-1)E_{\text{kin}}$. Since, for finite separation, no lower potential is to be expected than for coinciding heavy particles, it follows from the discussion given in the introduction that the densities of heavy nuclei will remain essentially constant and that their binding energies will be roughly proportional to the number of heavy particles which they contain. The maximum potential energy per heavy particle will be E_{kin}.

It is evident that the range of the nuclear forces is of the same order of magnitude as the linear dimension, d, of the wave function A or B. From a study of nuclear radii it follows that this range is of the same order of magnitude as the electron radius e^2/mc^2. This means that the wave function A or B will have dimensions comparable to the electron radius and that in the pair emission theory of nuclear binding electron and positron wave-lengths which are small compared with the electron radius do not play an important role. Putting $d=e^2/mc^2$ we obtain $E_{\text{kin}}\sim 137mc^2$. This energy is of the same order of magnitude as the minimum of the potential energy per particle usually assumed in nuclei.

The formalism as given in this paper does not explain the spin dependence of the nuclear forces and does not give the magnetic moments of proton and neutron. An explanation of these phenomena might be forthcoming if it is assumed that the spin of the heavy particle may change during the emission of a pair. A dependence of the spin of the emitted light particles on the spin of the heavy particle has to be introduced. If, moreover, a heavy particle can emit electrons into several orthogonal states simultaneously, sufficient freedom may be gained to describe our empirical knowledge of nuclei.

The form of the wave functions into which emissions can take place remains, in the present state, arbitrary and thus the detailed dependence of the potential on the configuration of heavy particles cannot be calculated. It seems, however, that under rather general assumptions about the functions A and B, the first interaction between two heavy particles approaching each other will be a repulsion. At small distances it has been shown that attraction is obtained. The low binding energies obtained for nuclei like Li^8 or B^{12} in which, owing to the Pauli principle, the average distance between particles will be greater than in Be^8 or C^{12} may be due to this repulsive potential at greater distances.[17]

It is felt that the most serious objection to the theory proposed in this paper is the introduction of the finite distance operator O in Eq. (2). Such an operator may cause creation and annihilation of electrons or positrons at points different from the position of the heavy particle and no way is known for reconciling such processes with the principle of relativity. This means that the present theory will meet with difficulties if one tries to extend it to relativistic velocities of the heavy particles.

In conclusion the authors wish to express their thanks to Professors Bethe, Fermi, Gamow and Oppenheimer for helpful discussions.

[17] The questions mentioned in the two previous paragraphs will be discussed in detail elsewhere.