On the Rotation of the Atomic Nucleus

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(Received March 23, 1938)

The spacing of the levels in the fine structure of alpha- and beta-ray processes and the existence of metastable nuclear states (isomers, isobars) are in contradiction with the existence of low lying levels corresponding to the rotation of the nucleus as a whole. The exchange of the nuclear constituents effected through rotation, together with the fact that the particles are not rigidly bound to equilibrium positions in the nucleus, will in some cases forbid, in other cases perturb the lowest levels, and cause the first state of excitation to lie considerably higher. Simple models illustrating these effects are discussed in II. In III an estimate is made for the position of the lowest excited level for heavy nuclei. It is found to vary inversely with the mass.

I. INTRODUCTION

ACCORDING to Bohr and Kalckar[1] the lower levels of the heavier nuclei can be systematized if the nucleus is compared to a droplet. The low lying levels which are of importance in nuclear reactions will then be the vibrations of this droplet. Still lower levels are obtained, however, if the rotations of the droplet are taken into consideration. If we take as an example the nucleus Pb^{208}, assume a uniform distribution of mass throughout the nucleus and take its radius to be 12×10^{-13} cm, the moment of inertia 2.00×10^{-46} g cm^2 will be obtained. If we furthermore assume that the lowest rotational state of this nucleus corresponds to $J=0$, then the pure rotational energies corresponding to $J=1, 2, 3, \cdots$ will be 3.3, 10, 20, $\cdots$ kilovolts, approximately.

According to a proposition first put forward by Thibaud[2] the rotations of the nucleus might be used to explain fine structure observed in the emission of alpha-particles. The energy differences of rotation, however, are smaller than those observed in the fine structure. It is indeed reasonable to assume that, in general, nuclei will not have excited states which lie only a few thousand volts above the ground level.

Such extremely low levels would give rise to difficulties in the following ways:

(1) If in the process of alpha-particle emission the fundamental states of both the emitting and the resulting nuclei have $J=0$, it should be possible to observe not only the process $J=0$ to $J=0$ but also the processes $J=0$ to $J=1$, $J=0$ to $J=2$, $J=0$ to $J=3$. In the latter processes the alpha-particle would carry away the angular momentum $\hbar$, $2\hbar$, and $3\hbar$, respectively. According to Gamow's[3] formula the relative probabilities of these processes for a nuclear charge $Z=82$ would be approximately as given in Table I. Instead of finding such closely spaced levels as given in Table I no fine structure has been observed in transformations in which there is reason to assume that the fundamental states of initial and final nuclei have $J=0$.

(2) The explanation for the fine structure in the cases in which it is observed[4] is based on the idea that fine structure occurs if a transition to the fundamental state of the final nucleus would require the alpha-particle to carry away a very high angular momentum. Since such a process is improbable, the disintegration will lead in most cases to excited states which possess angular momenta differing from that of the original

TABLE I. *Relative probabilities of transitions calculated from Gamow's formula for a nuclear charge $Z=82$.*

TRANSITION	RELATIVE PROBABILITY	ENERGY CHANGE
$J=0$ $J=0$	1.00	0
$J=0$ $J=1$	0.84	3.3 Kev
$J=0$ $J=2$	0.60	10 Kev
$J=0$ $J=3$	0.36	20 Kev

[1] N. Bohr and F. Kalckar, Kgl. Danske Videnskab. Selskab. **14**, No. 10 (1937).

[2] J. Thibaud, Comptes rendus **191**, 656 (1930).

[3] G. Gamow, *Structure of Atomic Nuclei* (Clarendon Press, 1937), Eq. (56), p. 103.

[4] Reference 3, p. 108.

nucleus by fewer units. These excited states frequently lie several hundred thousand volts, or even more than a million volts, above the fundamental state of the final nucleus. If the final nucleus had rotational states lying close to its fundamental level there would be no reason why the alpha-process should not lead to these low lying levels rather than to those observed.

(3) The strong gamma-radiation accompanying some beta-disintegrations has been given an explanation similar to that of the fine structure of the alpha-rays.[5] Also in this case the hard gamma-rays would be difficult to understand if low lying energy levels were available.

(4) Metastable states of nuclei which do not emit gamma-rays but rather eject electrons from the atomic shell (coefficient of internal conversion equal to unity) are found to be lying more than a million volts over the fundamental state. If low lying rotational levels exist these strong, discrete beta-rays should always be accompanied by at least equally strong gamma-rays of only slightly smaller energy leading from the metastable state to the low lying rotational states.

(5) Metastable states of much longer life have been postulated by Weizsäcker[6] in order to explain isomeric nuclei. These long lives he explained by assuming that all transitions from the metastable state to states of lower energy would be connected with a great change of angular momentum. Evidently if Weizsäcker's hypothesis is to be upheld, one must assume the existence of a special level with a high angular momentum lying lower than all but a few of the rotational states of the droplet.

(6) The existence of the low lying levels of the droplet would not be consistent with the explanation given for the long periods of the potassium and rubidium radioactivities. According to Klemperer[7] a large change in the angular momentum should account for the small transition probability. Klemperer's suggestion has been confirmed for the case of rubidium by Kopfermann and Konopinski.[8] Low rotational levels of the product nucleus would make possible less strongly forbidden transitions. Still smaller transition probabilities have been assumed to account for the stability of the neighboring isobars. The low lying levels would invalidate this explanation too.

We conclude therefore that no such low lying levels are present in the nucleus.

It is our purpose to show that the application of the exclusion principle to the constituents of nuclei gives the possibility of reconciling the droplet model with the absence of low lying levels such as would follow from the rotation of the droplet. As the simplest model we consider N alpha-particles on a circle and spaced at equal distances. The rotation of the ring into itself (i.e., around an axis passing through the center of the circle and perpendicular to the plane of the circle) will interchange the alpha-particles. Since the first exchange takes place after a rotation by the angle $2\pi/N$, the Bose statistics of the alpha-particles will exclude all rotational states except those having the angular momentum $J=0$, $J=N$, $J=2N$, $\cdots$.

If we now permit rotations of a group of alpha-particles around any axis in space such simple exclusions of rotational states will be obtained only for nuclei for which particularly symmetrical configurations can be assumed. Thus, Wheeler[9] has found that for four alpha-particles arranged in a tetrahedron the first rotational excitation has $J=3$. If we arrange six alpha-particles on the vertices of an octahedron (this structure has been proposed by Wefelmeier[10] for Mg^{24}) the lowest excitation will lie at $J=4$. For twelve alpha-particles at the vertices of an icosahedron the lowest rotational excitation will be at $J=6$ (this icosahedron was assumed by Wefelmeier for Fe^{54} with the thirteenth alpha-particle in the center of the icosahedron and with two additional neutrons about the localization of which no definite statements are made). Apart from such symmetrical cases, $J=1$ or $J=2$ should be expected as the lowest allowed rotational state since, in most cases, there will be either no rotation which effects an interchange of the constituents of the nucleus, or else only a few rotations, mostly through the angle π, will effect such an interchange.

[5] Reference 3, p. 150.
[6] C. F. v. Weizsäcker, Naturwiss. **24**, 813 (1936).
[7] O. Klemperer, Proc. Roy. Soc. **A148**, 638 (1935).
[8] Kopfermann, Physik. Zeits. **38**, 960 (1937); Konopinski, N. Y. Meeting Am. Phys. Soc., Feb. 25, 1938.
[9] J. A. Wheeler, Phys. Rev. **52**, 1089 (1937).
[10] W. Wefelmeier, Naturwiss. **25**, 525 (1937).

We must remember, however, that the constituents of a nucleus cannot be represented as firmly bound to equilibrium positions. Indeed, the dimensions of the nucleus alone show that the kinetic energies of the nuclear constituents are larger than the nuclear binding energies, that is, the kinetic and potential energy in the nucleus are comparable and the binding energy is obtained as a relatively small difference between the two. Therefore it will be important to consider the following exchange between the constituents of the nucleus: we first perform a rotation which, although it does not actually produce an interchange of the particles, does bring them into positions which are close to positions obtainable by simple exchange; then we have to move the constituents of the nucleus only through comparatively short distances in order to complete the exchange. It will be necessary to investigate which of these two motions will correspond to a higher frequency: the rotation through a finite angle or the displacement through short distances. The first will be the case if the constituents are bound to equilibrium distances with potentials large compared with their kinetic energies and, if thus, no appreciable displacements can be obtained without passing over potential barriers. Then the nucleus may possess all rotational states which are allowed according to the symmetry of the equilibrium position. If, however, the rotation through a finite angle takes a longer time and corresponds to a smaller frequency than the subsequent displacements through short distances, then the process of rotation and the rotational levels will be strongly disturbed and we may expect the nucleus to act like a body of higher symmetry than would correspond to any "equilibrium position" of the constituent particles. The effect may be that the predicted low rotational levels are absent.

In the next section we shall consider the effect of exchanging the particles for simple models consisting of constituents satisfying Bose statistics (alpha-particles). In the last section we shall discuss the angular momentum and the excitation energy which we may expect for the lowest levels in heavy nuclei according to this modified droplet model.

II. Examples

In the following we select a few simple examples which illustrate how the possibility of small displacements from equilibrium increases the effective symmetry of a system and how this increase in symmetry reacts on the structure of the allowed rotational energy levels.

Two particles on a circle

Let us picture two identical particles, obeying Einstein-Bose statistics, free to rotate in a plane about the same fixed point to which both are coupled by inextensible rods of the same length. Between the two particles acts a potential which depends upon the angle θ between their two radii vectors. The potential has the same value for θ and $2\pi-\theta$. The minimum occurs for a value of $\theta=\theta_0$ close to, but definitely less than, 180°. In the extreme case where the potential valley is very deep and very narrow (rigid coupling) the energy levels of the system are simply those of a rotator with one degree of freedom and a moment of inertia I:

$$E=E_0+\hbar^2m^2/2I. \tag{1}$$

All values of m are allowed because the system has no symmetry with respect to its rotational axis. If, now, the rigidity of the binding is decreased, it must be remembered that the system actually possesses two equilibrium positions, at θ_0 and $2\pi-\theta_0$, and that the potential barrier between them, while high, allows the possibility of resonance between the two equilibrium positions. Thus each of the levels considered in the case of rigid coupling is really a double level with a separation $2\Delta E$ proportional to the frequency with which penetration through the barrier occurs. One easily finds from the W–K–B approximation that

$$\Delta E=\pm h\nu_{\text{class}}(n) \times \exp -\int\{2I[V(\theta)-E_{\text{vib}}(n)]\}^{\frac{1}{2}}d\theta/\hbar. \tag{2}$$

$\nu_{\text{class}}(n)$ represents the classical frequency of vibration in the nth vibrational level, and the integral in the exponent depends upon the area under the potential barrier. The positive or negative sign in the equation has to be applied

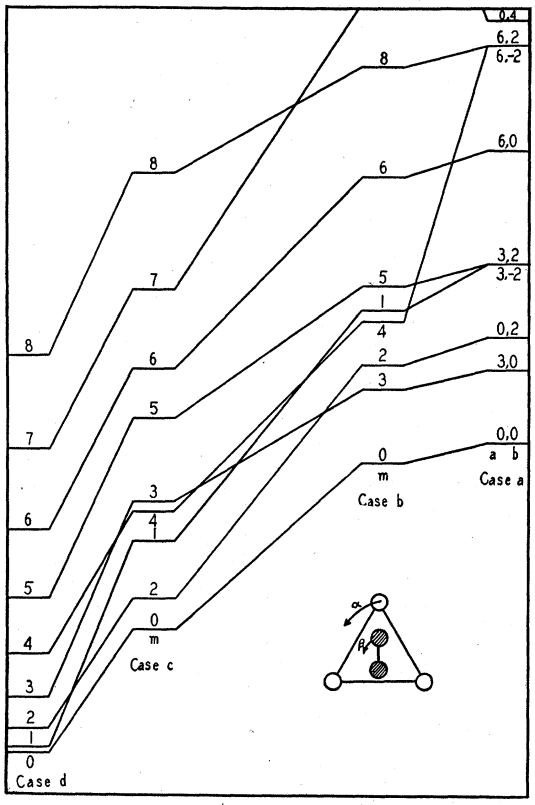

FIG. 1. The diagram in the lower right corner represents the triangle-dumbbell model. The curves represent the energy levels for different cases of coupling. In case (*d*) the coupling is rigid and the levels correspond to different values of the rotational quantum number m. In case (*c*) the coupling is loosened and we see the effect of the dumbbell penetrating the potential barrier to turn into another symmetrical position. In case (*a*) the dumbbell and triangle are allowed to rotate freely with the angular momenta $b\hbar$ and $a\hbar$. The total angular momentum $m\hbar=a\hbar+b\hbar$. In case (*b*) a small coupling is introduced causing repulsion between levels of the same m value. (See levels for $m=4$.)

according as the vibrational functions in the two minima are superposed with the opposite sign or with the same sign. If rotations of the system are taken into account, the symmetry of the wave function with regard to the exchange of two alpha-particles requires that it should remain unchanged if a rotation through 180° is performed and the oscillator is moved from one minimum to the other. Since a half-revolution multiplies by $(-1)^m$ (m is the rotational quantum number), only the energy shift which corresponds to superposition of the vibrational functions with the same sign is allowed for $m=0, 2, \cdots$, and with the opposite sign for $m=1, 3, \cdots$. Thus putting $|\Delta E|=Q$, we obtain for the total energy

$$E=E_{\text{vib}}(n)+\hbar^2m^2/2I-(-1)^mQ. \tag{3}$$

As the rigidity of the binding is decreased, Q in general increases. When $2Q$ becomes as great as $(\hbar^2/2I)\{m_0^2-(m_0-1)^2\}$, already the odd level $m=m_0-1$ is raised higher than the even level $m=m_0$. This corresponds to the frequency of interchange $2Q/h$ becoming greater than the frequency of rotation $\hbar m_0/2\pi I$. Finally we shall find that a number of even m levels will lie below even the lowest odd level ($m=1$).

This approximates then the result we find for the case of equilibrium position of alpha-particles opposite each other (model for Be^8), where only even values of J occur.

Triangle-dumbbell model

A second instructive example is provided by what we may call the triangle-dumbbell model, illustrated in the lower right-hand portion of Fig. 1. The two rigidly coupled particles in the dumbbell are identical, obey Einstein-Bose statistics, and have only a single degree of freedom, that of rotation about their common center of gravity, which is also the center of rotation for the triangle. The latter is composed of three identical particles, again rigidly coupled and following the symmetric statistics. As extreme cases we may consider (*a*) no interaction between the two rotational degrees of freedom and (*d*) rigid coupling. In the first case the energy levels are determined by the sum of the quantum expressions for two free rotators of moments of inertia A and B:

$$E=\hbar^2a^2/2A+\hbar^2b^2/2B. \tag{4}$$

Here the rotational quantum number b may take on only the values 0, ±2, ±4 $\cdots$ and a the values 0, ±3, ±6 $\cdots$, because of the symmetry of the system. In case (*d*) the coupling is so strong that we obtain effectively the problem of a single rotator having no symmetry against rotation, and possessing a moment of inertia equal to the sum $A+B$ of the two separate moments. The energy of the system is now

$$E=E_0+\hbar^2m^2/2(A+B), \tag{5}$$

where m may take on *all* integral values.

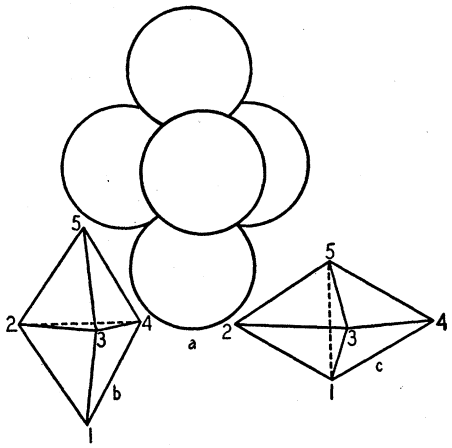

FIG. 2. (*a*) Bipyramid alpha-particle model for the nucleus Ne^{20}. (*b*, *c*), a distortion separating alpha-particles 2,4 and bringing particles 5,1 into contact. The result is an exchange plus rotation through 90 degrees.

The question as to how the transition occurs between the two preceding cases now presents itself in an interesting form, for it is clear that the order of the various rotational levels must go through some sort of mixing process.

First, let us commence with the case of a very weak field, say

$$V(\beta-\alpha)=-W\cos 6(\beta-\alpha),$$

which possesses a sixfold periodicity because of the symmetry of the two parts of the system. It is easy to calculate from the unperturbed wave functions

$$\Psi=(2\pi)^{-1}\exp i(a\alpha+b\beta), \tag{6}$$

that the first-order perturbation energy vanishes. In the second order we have

$$E=\hbar^2a^2/2A+\hbar^2b^2/2B - (W^2/36\hbar^2)(A^{-1}+B^{-1})^{-1}\left\{1-\left(\frac{Ba-Ab}{3A+3B}\right)^2\right\}^{-1}, \tag{7}$$

which represents a repulsion between energy levels possessing the same values of $m=a+b$. (See case (b) in Fig. 1.) On the other hand, proceeding from the side of strong interactions toward the intermediate case, we find it simplest to separate the wave equation

$$(\hbar^2/2A)d^2\psi/d\alpha^2+(\hbar^2/2B)d^2\psi/d\beta^2+\{E-V(\beta-\alpha)\}\psi=0 \tag{8}$$

by writing the wave function as the product

$$\psi=f(\varphi)\exp(im\theta) \tag{9}$$

of a part representing rotation of the system as a whole by the angle $\theta=(A\alpha+B\beta)/(A+B)$ times a part which depends only upon the relative orientation $\varphi=\beta-\alpha$ of the triangle and dumbbell. The wave function of the internal torsional vibration, $f(\varphi)$, must satisfy the equation

$$(\hbar^2/2)(A^{-1}+B^{-1})d^2f/d\varphi^2+\{E_\varphi-V(\varphi)\}f=0, \tag{10}$$

and it is also clear that it must obey a certain periodicity condition. From the picture in the lower right-hand portion of Fig. 1, one sees that a rigid rotation of the whole system by an angle $\theta=(2\pi A/3+2\pi B/2)/(A+B)$ followed by a rotation of the triangle relative to the dumbbell by an angle $\varphi=2\pi/6$ will carry the system from its original position to one where the rod has been given a half-turn, the triangle a third of a revolution. The requirement that the wave function have the same value for the two equivalent configurations tells us at once (see Eq. (9)) that

$$f(\varphi+2\pi/6)=f(\varphi)\exp -im(2\pi A/3+2\pi B/2)/(A+B). \tag{11}$$

The solutions of Eqs. (10) and (8), subject to this condition, may be obtained with the help of the W-K-B approximation and give us the energy values

$$E=E_{\text{vib}}(n)-(-1)^n2h\nu_{\text{class}}(n) \times\{\cos 2\pi m(A/3+B/2)/(A+B)\} \times\exp\{-\textstyle\int|p(\varphi)|d\varphi/\hbar\}+\hbar^2m^2/2(A+B), \tag{12}$$

where n is the quantum number of the torsional vibration.

The figure shows the influence of the possibility of displacements of the triangle and dumbbell with respect to each other. In the case (*d*) of the rigid coupling the low symmetry of our model caused the level of the total angular momentum $m=1$ to be the first excited state. In the intermediate cases (*c*) and (*b*) the level $m=1$ moves up strongly whereas the energy of the levels $m=2$ and $m=3$ are not raised to an

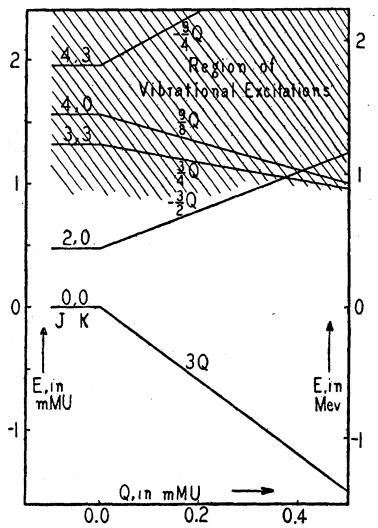

FIG. 3. The energy levels at the left represent the low rotational states allowed for the bipyramid neon model of Fig. 2. $J\hbar$ stands for total angular momentum and $K\hbar$ is the component of this along the axis of figure of the rigid model. The possibility of exchange between different configurations of the same energy (see Fig. 2) perturbs the levels. The shift is shown as a function of the value of the exchange integral Q for which a reasonable estimate is 0.17 mMU. Additional excited levels are found if the vibrations of the model are taken into consideration.

equal extent. Eventually, as we approach case (a), both the energy of the lowest excitation and also its angular momentum ($m=3$) become considerably greater than in the case of rigid coupling.

Bipyramid model

A further example of the qualitative considerations discussed above is furnished by a problem which can be considered as a model for the nucleus Ne^{20}. We may picture the stable state of the system as that in which five alpha-particles are situated at the vertices of a double pyramid (see Fig. 2). If we assume that the "bond distances" are the same between all pairs of alpha-particles, then this configuration gives the maximum possible number of bonds (9). Diagrams (b) and (c) show that a distortion of the nucleus in which the 2–4 bond is stretched and the distance between 1 and 5 is shortened will carry the system over into an equivalent position of the same energy but different orientation. For purposes of identification we may symbolize the original configuration by the expression 15 and the equivalent one by 42. By 15 we indicate that the alpha-particles 1 and 5 are located at the two opposite poles of the bipyramid and the three alpha-particles at the base are numbered clockwise when viewed from 1 towards 5. Thus 51 and 15 differ in the order of the alpha-particles in the base. Corresponding to the 20 ordered pairs, we have 20 different configurations of equal energy. Transitions similar to the change 15 to 42 pictured in the figure are 15 to 23, and 15 to 34. One can see qualitatively that other changes from 15, such as from 15 to 51, will encounter higher potential barriers.

A simple criterion for the most important "channels of intercommunication" is the value, r^2, of the sum of the squares of the displacements of the individual alpha-particles which occur during the distortion. In terms of the distance, c, between neighboring alpha-particles, r^2 is easily computed to be $0.74c^2$ for the change from 15 to 42, $2c^2$ for the change to 51, $1.75c^2$ for the change to 13, $2c^2$ for the change to 31, and $2.07c^2$ for the change to 24. As the penetration factor goes down exponentially with r^2, it is clear that the most important type of exchange is the one represented in the figure.

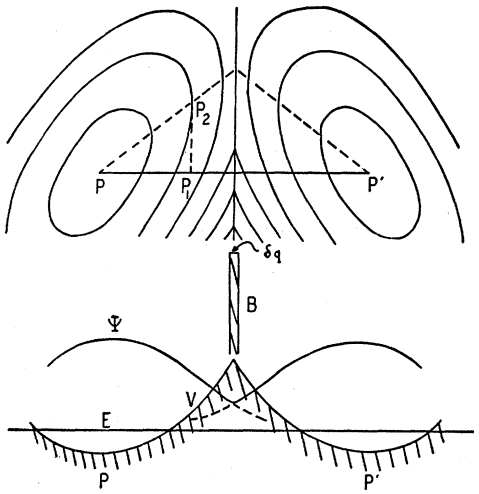

FIG. 4. The lower portion is a schematic diagram for the variation of potential energy which occurs when the system is subjected to a distortion carrying it from one equilibrium position P to an equivalent one P'. The order of magnitude of the potential barrier between P and P' is estimated to be 1.5 mMU for the change from 15 to 24. The fictitious potential barrier B has been introduced in such a way as to obtain a simple form for the zero-order wave function. The contour lines of this wave function are pictured above.

Neglecting at first these exchange phenomena, we may divide the energy of the nucleus into the kinetic energy of rotation of a symmetric top and the zero-point energy of the $3\times5-3-3=9$ vibrational degrees of freedom. In terms of the moments of inertia C and A along and at right angles to the figure axis, and the total angular momentum $J\hbar$ and its component $K\hbar$ along the figure axis, we have

$$E=E_0+J(J+1)\hbar^2/2A+(K^2\hbar^2/2)(C^{-1}-A^{-1}). \quad (13)$$

We shall not be interested in states of the system in which the vibrational degrees of freedom, q_1, q_2, $\cdots$, are excited, and shall therefore treat the system as a rigid rotator.

Only for certain values of J and K is it possible to obtain a rotational wave function whose value is unaltered by those rigid rotations of the model which effect an interchange of the alpha-particles, namely,

$$K=0\,;\ \text{even } J:\ \Psi\sim P_J(\cos\theta)f_{\text{vib}}(q)\,;$$

$$K=3n\neq0;\ \text{any } J:\ \Psi\sim P_J^{(K)}(\cos\theta)\times\{\exp(iK\chi)+(-1)^J\exp(-iK\chi)\}f_{\text{vib}}(q). \quad (14)$$

Here χ measures rotation about the axis of figure of the double pyramid, and θ is the angle between this axis and some fixed direction in space. The left-hand portion of Fig. 3 shows the position of the low rotational levels allowed according to our analysis. The absolute values of the rotational energies have been obtained by using $A=4.7\times10^{-48}$ g cm^2, $C=3.0\times10^{-48}$ g cm^2.

The occurrence of exchange processes by penetration through potential barriers means now that the total wave function will be a superposition of functions of the type (14):

$$\Psi=(20)^{-\frac{1}{2}}\sum_{c=1}^{20} f_K{}^{(J)}(\theta_c, \chi_c)f_{\text{vib}}(q_{1c}, q_{2c}, \cdots). \quad (15)$$

Here the summation goes over all 20 stable configurations of the five alpha-particles. The Eulerian angles θ_c and χ_c and the vibrational coordinates q_c are defined in the same way for the various configurations, so that Ψ is symmetric with respect to all interchanges of the alpha-particles. If we write V_c for the potential for which the cth term in Ψ would represent an accurate solution of the wave equation, and V for the actual potential, then by making a first-order perturbation calculation for the change in energy due to leakage through the barriers, we obtain

$$E=E_0+E_{\text{rot}}(J, K)$$
$$+3\int f_K{}^{(J)*}(\theta_{42}, \chi_{42})f_{\text{vib}}{}^*(q_{42})(V-V_{15})$$
$$\times f_K{}^{(J)}(\theta_{15}, \chi_{15})f_{\text{vib}}(q_{15})d\tau. \quad (16)$$

In the derivation we have made use of the symmetry of Ψ and the fact that from a given configuration, 15, the most important exchange processes lead only to the three equivalent equilibrium positions 42, 23, and 34.

To carry out the integration over angles, we note from Fig. 2 that the orientation of the configuration 15 differs from that of 42 by a rotation, R, of 90°. Hence according to group theory, the integral of $f_K{}^{(J)*}(\theta_{42}, \chi_{42})f_K{}^{(J)}(\theta_{15}, \chi_{15})$ over all space orientations may be expressed in a simple manner in terms of the elements $D^{(J)}{}_{K,K}(R)$ and $D^{(J)}{}_{-K,K}(R)$ of the standard rotational transformation matrix[11] for $R=90°$, whose values are given by

$$D^{(J)}{}_{KK}(0, 90°, 0)$$
$$=(-1)^{J-K}D^{(J)}{}_{-K,K}(0, 90°, 0)=c_{JK};$$
$$c_{JK}=2^{-J}\sum_{f=0}^{J-K}(-1)^f\frac{(J+K)!}{f!(J+K-f)!}$$
$$\times\frac{(J-K)!}{f!(J-K-f)!}. \quad (17)$$

After the integration over angles, we have left the vibrational part of the exchange integral, defined by

$$Q=\int f^*{}_{\text{vib}}(q_{42})(V-V_{15})f_{\text{vib}}(q_{15})dq. \quad (18)$$

Our result for the position of the low rotational levels, taking account of exchange, is expressed in the equation

$$E=E_0+E_{\text{rot}}(J, K)$$
$$+3\begin{Bmatrix}c_{JK}\\2c_{JK}\end{Bmatrix}Q\cdot\begin{Bmatrix}K=0\text{; even } J\\K=3n\neq0\text{; any } J\end{Bmatrix}. \quad (19)$$

The energy values are shown in Fig. 3 as a function of the magnitude of the vibrational exchange integral, Q. A reasonable estimate gives $Q=-0.17$ mMU, so that the first excited rotational level, $J=2$, is raised up 0.75 mMU with respect to the ground state. However, already if Q be as great as -0.38 mMU, the effect is so pronounced that the levels $J=3$ and $J=2$ become interchanged, with the level of greater angular momentum lying lower. The example shows in a striking manner the disturbance of the normal order of rotational levels caused by the existence of resonance between geometrically similar configurations of a system of five identical particles.

To obtain the estimate we have just used for the value of the vibrational exchange integral Q, we express the potential and kinetic energies for small oscillations from equilibrium in terms of suitable normal coordinates $q_1, q_2, \cdots, q_n$:

$$V=(m_\alpha/2)\Sigma\omega_i{}^2q_i{}^2, \quad (20)$$
$$T=(m_\alpha/2)\Sigma\dot{q}_i{}^2. \quad (21)$$

[11] E. Wigner, *Gruppentheorie und Quantenmechanik* (Braunschweig, 1931), p. 180.

Here $\omega_i/2\pi$ is the classical frequency of vibration of the corresponding normal coordinate. The wave function for the normal state (zero-point vibration) has the value

$$\Psi=(m_\alpha/\pi\hbar)^{n/4}(\omega_1\omega_2\cdots\omega_n)^{\frac{1}{4}}\exp\{-\Sigma m_\alpha\omega_i q_i/2\hbar\}, \quad (22)$$

if we suppose the vibration to occur in the neighborhood of one equilibrium position P. Actually the representative point of the system can penetrate through the potential barrier represented schematically in the lower portion of Fig. 4 and vibrate about the second minimum P'. If, as shown in the figure, we join together in the center of the barrier the unperturbed wave functions for the two minima, we obtain a solution of the wave equation

$$(\hbar^2/2m_\alpha)\Sigma\partial^2\Psi/\partial q_i^2+(E-V_1)\Psi=0 \quad (23)$$

for the unperturbed value of E. Here the potential V_1 is not the same as the actual potential V, however, but has been obtained from it by superposing a fictitious potential wall, very narrow and extremely high, in such a way as to bring about continuity in the normal derivatives of the wave functions on the two sides. Using q_{norm} to denote the coordinate measured parallel to the direction PP', and calling the height and width of the barrier B and δq, we easily obtain in fact the relation

$$-(\hbar^2/m_\alpha)(\partial\Psi_{\text{left}}/\partial q_{\text{norm}})=B\delta q\Psi \quad (24)$$

by integrating (23).

To obtain the correct energy value, we must remove the above-mentioned fictitious potential. Treating this alteration as a small change, we obtain by a first-order perturbation calculation

$$\Delta E=Q=\int\Psi^2(V-V_1)d\tau=-\int B\delta q\Psi^2 dS$$
$$=(\hbar^2/m_\alpha)\int\Psi_{\text{left}}(\partial\Psi_{\text{left}}/\partial q_{\text{norm}})dS, \quad (25)$$

where the integration goes over the $n-1$ dimensional surface normal to PP'. It would appear at first sight that the result should be divided by 2 because both parts of the total wave function are normalized to 1; but in the one-dimensional case one can readily prove that a compensating factor of 2 enters, just because of the fact that the perturbation is not small. It seems reasonable to expect this result in general, so we use (25) as it stands, without rigorous analysis. The integration may be carried out directly, from (22), and gives

$$Q=-\hbar\omega(m_\alpha\omega/\pi\hbar)^{\frac{1}{2}}(r/2)\exp\{-(m_\alpha\omega/\hbar)(r/2)^2\}. \quad (26)$$

Here r is the distance from P to P' and $\omega/2\pi$ is that frequency with which the system would oscillate if constrained to move along the straight line PP' with a potential energy which at every point P_1 on its path (see upper part of Fig. 4) is given by the value of V, not at P_1, but at that point P_2, equally distant from the surface of join, where the wave function has its maximum value. The contour representation of the wave function in the figure is drawn in such a way as to emphasize that in general no normal mode of vibration will lie in the direction PP'.

Following the methods used in reference 9, we find for the normal vibrations of neon characteristic energies ranging from $\hbar\omega_1=0.8$ mMU to $\hbar\omega_9=2.2$ mMU. An estimate for the value of the vibrational exchange integral is obtained from Eq. (26) by substituting for ω a harmonic mean of the 9 characteristic frequencies, and for r^2 the value $0.74c^2$ mentioned earlier. With $\hbar\omega=1.2$ mMU and $r=7.8\times10^{-13}$ cm we find $Q=-0.17$ mMU, which is the value used above in discussing the energy level perturbations shown in Fig. 3. It is of interest to note that the value of the vibrational exchange integral given by (26) will for no value of ω ever exceed in magnitude the limit $Q=-0.925\hbar^2/m_\alpha r^2$.

III. First Excited States of Heavy Nuclei

The actual behavior of nuclei cannot be represented in any mathematically rigorous form; we must be content to estimate the order of magnitude of the excitation energies as they would follow from the simplest possible assumptions. We must investigate in particular in what way the excitation energies will change if the number of the nuclear constituents increases.

The reason why the first excited rotational levels will be strongly disturbed is that the phase of the wave function will be changed by a rotation. If a further small displacement of the alpha-particles suffices to complete an exchange of the constituents (and restore the sign of the wave function) then it follows that the phase of the wave function must change on a comparatively short path. Such change of phase, however, requires an increase of kinetic energy unless a potential barrier forces the amplitude of the wave function to approach zero irrespective of its phase.

Alpha-particle model

We shall first assume that no potential barriers are present. Let a nucleus composed of N_α alpha-particles[12] be pictured in the $3N_\alpha$-dimensional configuration space. A phase change φ over a distance r will cause the momentum to increase by the amount $\hbar\varphi/r$, where r^2 is the sum of the squares of the displacements of the alpha-particles. If originally the momentum and kinetic energy were zero, the result would be to increase the energy by $\hbar^2\varphi^2/2mr^2$. Even if in the original case a considerable zero-point energy and corresponding momenta p_0 are present, the

[12] The nucleus is assumed to be composed of alpha-particles rather than of protons and neutrons in order to simplify the argument. A detailed consideration of particles which possess spin and obey the Fermi statistics would lead in a more complicated way to results similar to those obtained in the text.

average increase of the kinetic energy is still

$$\{p_0+\hbar\varphi/r\}_{\text{Av}}^2/2m-p_0^2/2m=\hbar^2\varphi^2/2mr^2, \quad (27)$$

since the expression $\{p_0\hbar\varphi/r\}_{\text{Av}}$ vanishes, because of the random direction of p_0 in the unperturbed state.

In order to obtain the perturbation of the rotational levels, the expression $\hbar^2\varphi^2/2mr^2$ should be averaged over all orientations. For an order of magnitude estimate, it will be sufficient to substitute both for φ and for r their maximum possible values. Thus, we set

$$\varphi\sim\pi$$
$$r^2\sim N_\alpha(\delta/2)^2, \quad (28)$$

where N_α again is the number of alpha-particles and δ is their average distance. We thus obtain for the probable increase in excitation energy[13]

$$\Delta E\sim 4\pi^2\hbar^2/2N_\alpha m_\alpha\delta^2, \quad (29)$$

It will be seen that this quantity decreases with the -1 power of the mass and reaches for heavy nuclei a value of about 100 kilovolts. Thus, according to this picture, we shall not expect in general an excitation energy of less than this order of magnitude for nuclei of any atomic number. Exceptions may occur if the nucleus has a spin due to the spin of protons or neutrons; then the lowest excitation energies might be determined by spin-orbit interaction rather than by the factors considered above.

In the model just described, no definite statement can be made about the angular momenta of the levels. Of course, if the nucleus is composed of alpha-particles obeying Bose statistics, the wave function for the lowest level must not have any nodes and must thus belong to the quantum number $J=0$. The first excited level, then, may have any J value for which the rotational energy does not considerably exceed the ΔE given in (29). Therefore we have for the probable upper value for the rotational quantum number J_{exc} of the first excited state the relation

$$\hbar^2 J_{\text{exc}}(J_{\text{exc}}+1)/2I\sim 4\pi^2\hbar^2/2N_\alpha m_\alpha\delta^2. \quad (30)$$

Here I is the moment of inertia of the nucleus, given by

$$I=10^{-1}(18/\pi^2)^{\frac{1}{3}}N_\alpha^{5/3}m_\alpha\delta^2. \quad (31)$$

Thus we obtain

$$J_{\text{exc}}\sim 2.2N_\alpha^{\frac{1}{3}}, \quad (32)$$

which for heavy nuclei gives an angular quantum number $J_{\text{exc}}=8$.

Another characteristic J value is the one above which the order of rotational levels will become normal. We shall call this J value the quantum number of collective rotation, J_{col}, since for higher angular momenta the rotational energies approach those of a rigid body. We obtain for J_{col} the relation

$$(\hbar^2/2I)\{(J_{\text{col}}+1)(J_{\text{col}}+2)-J_{\text{col}}(J_{\text{col}}+1)\}$$
$$\sim 4\pi^2\hbar^2/2N_\alpha m_\alpha\delta^2, \quad (33)$$

which leads to

$$J_{\text{col}}\sim 2.4N_\alpha^{\frac{2}{3}}. \quad (34)$$

For heavy nuclei J_{col} is about 36, and the corresponding energy of rotation is $E_{\text{rot}}\sim 2$ Mev. Therefore one cannot expect to determine the moment of inertia of a nucleus described by our present model from a study of energy levels and their angular momenta, since in the region where the rotational levels assume their proper order there will be present a great number of vibrational excitations.

Another interpretation for J_{col} is obtained by comparing the classical frequency of rotation with the frequency of the small displacements which complete the rotational exchange of alpha-particles. This comparison gives

$$\hbar J_{\text{col}}/2\pi I\sim\Delta E/2\pi\hbar, \quad (35)$$

which is equivalent to Eq. (33).

To obtain an interpretation of J_{exc}, we shall compare the time necessary to rotate an alpha-particle on the surface of the nucleus through half the distance between two alpha-particles, with the time $\hbar/\Delta E$ required for an essential disturbance of the phase of the wave function by the displacement of the alpha-particles. We then have

$$\{r/(\delta/2)\}\hbar J_{\text{exc}}/I\sim\Delta E/\hbar, \quad (36)$$

which leads to a value for J_{exc} differing from

[13] This estimate of the first excitation assumes that of all the low lying levels that for $J=0$ is the only one which, due to the favorable phase relations, is not raised.

that given in (32) only by the factor $(\pi^2/10)^{\frac{1}{2}}$. For J values smaller than J_{exc} it is impossible to follow the rotation of the nucleus, even if we restrict ourselves to an angle $\delta/2r$ and if we use the best possible method of following the rotation, that is, a coordinate system fixed by a statistical study of all alpha-particles.

Thomas-Fermi model

In the above estimates, we used a model in which alpha-particles could exchange without crossing potential barriers. It is of interest to see how these results compare with estimates which one can obtain from the Thomas-Fermi model of the nucleus.[14] We consider a nucleus which consists of N particles (protons and neutrons). The number of protons and neutrons shall be even and equal, so that each cell in phase space will be occupied by four particles. If the radius of the nucleus is r and the maximum momentum P_m, we have for the volume in phase space

$$(4\pi/3)P_m{}^3(4\pi/3)r^3=(N/4)(2\pi\hbar)^3. \quad (37)$$

For the maximum angular momentum of a particle we obtain

$$P_m r/\hbar=2.4(N/4)^{\frac{1}{3}}=2.4N_\alpha{}^{\frac{1}{3}}. \quad (38)$$

The close resemblance between this value and J_{exc} may be explained in the following way: if the neutrons and protons are supposed to move in spherically symmetrical fields and if their angular moments are composed according to the rules of vector addition, the lowest state will become an S level in consequence of the attraction between constituent particles. If, then, one neutron or proton is lifted to a level of the same or a neighboring shell a group of excited levels will result the average angular momenta of which will be comparable with the value given in (38). Thus the J values for the lowest excited states will be similar for the α-particle picture and the Thomas-Fermi model.

An estimate of the energy of the first excited state in the Thomas-Fermi model reveals no immediate agreement with the alpha-particle picture. If the excitation energy ΔE is due to the transition from one shell into the next one we may write

$$\Delta E=P_m\Delta P_m/m_H, \quad (39)$$

where ΔP_m, the change of momentum of the excited neutron or proton, is given by

$$(4\pi/3)3P_m{}^2\Delta P_m(4\pi/3)r^3=g(2\pi\hbar)^3. \quad (40)$$

g is the degree of degeneracy in the shell into which the neutron or proton is lifted. Dividing (40) by (37) we obtain

$$3\Delta P_m/P_m=4g/N. \quad (41)$$

Since the density of nuclei is roughly constant P_m will not vary with N in a systematic way. However, $g\sim 2P_m r/\hbar$ will be proportional (according to (38)) to $N^{\frac{1}{3}}$. Thus we get

$$\Delta E\sim\frac{8P_m{}^3r}{3m_HN\hbar}=\frac{8\cdot 10^7\text{ ev}}{N^{\frac{2}{3}}}. \quad (42)$$

This equation differs from (29) both in its dependence on N and in the high values of $\Delta E\sim 2\cdot 10^6$ ev which are obtained from it for heavy nuclei.

The Thomas-Fermi model leads, however, to closer agreement with the alpha-particle picture if one carries out the discussion in greater detail. We shall discuss two such improved pictures.

First we shall assume that we deal with an incomplete shell and that the first excited state corresponds to the transition of a neutron or proton from one degenerate state of this shell to another. Then the excitation energy will correspond to the diminished coupling of the particles in the incomplete shell. Its order of magnitude will be given by the change of interaction between two particles in the same orbit. For short range exchange forces this interaction will be proportional to N^{-1}.[15] Since for the lightest nuclei the interaction is[16] of the order of 10^7 ev to $4\cdot 10^7$ ev for the heaviest nuclei excitation energies of 10^5 ev are obtained. This, as well as the dependence on N is in agreement with (29).

The above estimate was based on a model in which the nuclear proper function is not simply

[14] Similarity of results may be expected in view of the comparison carried out by Euler, Zeits. f. Physik **105**, 553 (1937).

[15] Wigner has found this dependence from the application of the symmetry properties of nuclear wave functions to the kinks in the mass defect curve. See Phys. Rev. **51**, 954 (1937).

[16] Breit, Condon and Present, Phys. Rev. **50**, 825 (1936).

a product function of the individual particles but rather a linear combination of those products which would have the same energy if each particle is assumed to move in a spherically symmetrical field. A second kind of improvement on the individual particle approximation would be to retain the product form of the proper function but to drop the assumption that the particles move in a field of spherical symmetry. Then in estimating the excitation energy ΔE the equations (39), (40) and (41) remain valid. Only g must be set equal to one since the absence of symmetry in the field will remove the degeneracy of the individual particle proper function. Thus we obtain

$$3\Delta P_m/P_m=4/N \qquad (41a)$$

and $$\Delta E=4P_m^2/3m_HN=(2.7\cdot 10^7/N)\text{ ev}, \qquad (43)$$

which has the same dependence on N and the same order of magnitude as (29).

Comparison with experience

Unfortunately there is no direct experimental procedure to determine the position of the first excited level of a heavy nucleus. The arguments adduced in the introduction against low lying rotational levels might be used however to test the orders of magnitude we have obtained in Eqs. (29) and (43) for the excitation energy. We shall discuss here only our last argument (6) concerning the stability of isobars and the radioactivity of potassium and rubidium.

The upper limits of the beta-ray spectra of potassium and rubidium are $7\cdot 10^5$ ev and $2.5\cdot 10^5$ ev, respectively. The explanation of the long periods require that there should be no levels in the product nuclei lying much lower than the beta-limit (unless beta-transitions into these levels are strongly forbidden). Thus we see that the lowest excitation energy of suitable angular momentum must be a few hundred thousand volts in both cases; the limit is more strict for the heavier element, rubidium.

The number of neighboring pairs of isobars can be explained statistically if one postulates a mechanism whereby such a pair will be stable whenever the energy difference between initial and final atoms is less than 50,000 ev.[17] Now with a neutrino of zero mass a beta-disintegration of this energy will yield too short a period of decay if the transition is an allowed one. A satisfactory explanation of the isobars can be obtained, however, by assuming that the angular momenta of the two isobars differ by three or more units. If the difference in angular momenta is great enough the energy difference might be 10^5 ev or even more. This explanation will only hold, however, if in general there exist in the product nucleus no excited levels below 10^5 ev with suitable angular momenta to which transitions may occur. We see therefore that general experience points towards a value of $\sim 10^5$ ev for the lowest excitation energy. The calculations on the "thermal properties" of nuclei indicate[18] that the lowest excited states cannot be much higher than this quantity.

The effect of potential barriers

In conclusion we shall discuss on the basis of the alpha-particle model how the dependence of the lowest nuclear excitation on N_α will change if potential barriers must be crossed when the alpha-particles are displaced. For the lightest nuclei the transition from freely moving alpha-particles to strongly bound alpha-particles is a gradual one. As has been shown in the examples discussed in the second section the low lying energy levels approximate gradually the levels of a rigid top as the potential barriers become higher.

For heavy nuclei, however, even low potential barriers which for light nuclei had no strong effect will become important. The reason is that in heavy nuclei many alpha-particles have to cross the barriers which causes an increase both in the effective width and height of the barrier. Thus for heavy nuclei low lying rotational levels can be avoided only if the displacement of the alpha-particles can be performed on such a path in configuration space on which the "kinetic energy" of the system never becomes negative. It is sufficient to have low potential barriers to effect a sudden change to the rotational spectrum of a rigid body. This change may be called a change of phase (or state) since it occurs suddenly for a system of many particles.

[17] Bethe and Bacher, Rev. Mod. Phys. **8**, 199 (1936).

[18] Weisskopf, Phys. Rev. **52**, 295 (1937); Landau, Physik. Zeits. Sowjetunion **11**, 556 (1937).

It is interesting to note in this connection that the low temperature modification of liquid helium[19] has an unmeasurably small viscosity.[20] Since helium does not solidify unless it is compressed it is reasonable to assume that it is kept in the liquid state because of the zero-point vibrations of the atoms[21] which, for helium, are comparable with the heat of evaporation. The experiments on the viscosity of helium indicate that in a "quantum liquid" the particles may rearrange without crossing potential barriers.

The absence of low lying rotational levels in heavy nuclei definitely indicates therefore that if nuclei are to be compared with a phase of matter in macroscopic experiments the correct analog to use is not a crystallite but a droplet of a "quantum liquid" such as the low temperature modification of liquid helium.

[19] W. H. Keesom and M. Wolfke, "Comm. Phys. Lab. Leyden," No. 190b.
[20] P. Kapitza, Nature **141**, 74 (1938); J. F. Allen and A. D. Misener, Nature **141**, 75 (1938).
[21] Simon, Nature **133**, 460 and 529 (1934); F. London, Proc. Roy. Soc. **153**, 576 (1935).

MAY 15, 1938 PHYSICAL REVIEW VOLUME 53

New Experimental Evidence for the Existence of a Neutrino

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(Received March 24, 1938)

A new method is used for determining the energy of recoil of the nucleus in the individual beta-disintegration process. A gaseous compound of radiochlorine is placed in a cloud chamber. The clearing field is removed long enough before the expansion to allow the ions to spread out so that the resulting droplets can be seen individually. A cluster of droplets appears at the beginning of the track, and this is believed to be produced by the recoil nucleus. From the number of droplets an estimate is made of the kinetic energy of the nucleus, and this is compared with that calculated from the observed curvature of the beta-ray track. It is found that the laws of momentum as well as those of energy indicate that a third particle participates in the disintegration.

INTRODUCTION

IN the experiment to be described[1] the momenta of both the electron and the recoil nucleus have been simultaneously measured for the elementary beta-disintegration process. This is the first experiment which has given any information at all regarding the momentum relations in the *individual* disintegration event. Although the results are of limited accuracy, they strongly indicate that momentum is not conserved between the electron and the nucleus alone. Hence the laws of momentum, as well as those of energy, indicate that a third particle participates in the disintegration.

The idea of observing the disintegration of a substance in the form of a gas in a cloud chamber has been suggested many times as a possible way of measuring the momentum or energy of recoil and the direction of recoil of the nucleus emitting the beta-ray. The difficulty with such a scheme is that the length of track made by the nucleus is far too short for observation, even in a cloud chamber operated at the lowest possible pressure. The nucleus will, however, produce a number of ion pairs concentrated in a very small region of space, and the number of ion pairs will be a function of the kinetic energy of the nucleus. It occurred to the authors that if these ions could be allowed to diffuse into a cluster several millimeters in diameter before the condensation were brought about, the individual droplets could be counted, and hence the kinetic energy of the nucleus could be estimated. It was found that this could be accomplished, and by applying a magnetic field to the chamber it was possible to know the momentum of the electron in each case, so that the estimated momentum of the nucleus could be compared with that of the electron.

[1] Halpern and Crane, Abstract 5, New York Meeting of the American Physical Society, Feb. 25–26, 1938.