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Note added in proof: In the above argument for the disintegration of Ta^{180} no stable W^{180} is assumed to exist. Professor A. J. Dempster kindly informed me of his recent discovery of this isotope (Phys. Rev. **52**, 1074 (1937)). This removes the objection to the negative electron emission from Ta^{180}. This process must be considered as possible in addition to K electron capture, either due to a branching reaction or an isomer of Ta^{180}, both isomers disintegrating with periods of little difference.

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The Oppenheimer-Phillips Process

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The mechanism proposed by Oppenheimer and Phillips for the disintegration of nuclei by deuterons with proton emission (*d-p* reaction) is examined. A formula is derived which expresses the probability of this process in terms of the sticking probability of the neutron (§2) and the penetrability of the potential barrier. The importance of the finite (rather than zero) nuclear radius for the penetrability is pointed out and the penetrability is calculated for various values of the radius (§3). The energy distribution of the emitted protons is found to be given directly by the sticking probability of the neutron (§5). Therefore it may differ considerably from the distribution in "ordinary" nuclear reactions by containing relatively more high energy protons. A measurement of the energy distribution would allow direct conclusions about the width of low nuclear levels which is of importance for the theory of the α-decay and therefore of the nuclear radius (§5). The probability of the O-P reaction is compared with that of ordinary nuclear reactions. The O-P mechanism is found to prevail in the *d-p* reactions for nuclear charges of about 25 and higher; if the reaction leads to a nucleus which emits fast β-rays, the O-P mechanism will be valid at still lower charges. The relative probability of *d-p* as compared to *d-n* reactions is found to be (on the average) unity for very light nuclei, to decrease with increasing atomic number until the O-P process becomes prevalent, and to increase from there on. The excitation function of reactions with nuclei up to $Z\sim 30$ is found to be an inadequate test for the O-P mechanism (§6). The question of secondary (cascade) disintegration following the *d-p* reaction is discussed and it is found that such disintegrations (e.g. *d-pn* or *d-pα*) should be rare with deuteron energies below the top of the potential barrier (§7).

§1. General

In any ordinary nuclear reaction the first step is the entry of the incident particle as a whole into the initial nucleus. In the "compound nucleus" thus formed, the nuclear particles rearrange themselves until the compound nucleus breaks up into final nucleus and produced particle. However, according to Oppenheimer and Phillips,[1] this general scheme does not apply to reactions of the *d-p* type:[2] Here the incident deuteron does not enter the nucleus as a whole but splits up outside the nucleus into a proton which leaves as the "produced particle," and a neutron which is absorbed by the nucleus.

The Oppenheimer Phillips (O-P) process is, therefore, in principle *simpler* than the ordinary type of nuclear reactions. While the ordinary reactions are double processes, consisting of the formation and disintegration of the compound nucleus, the O-P process is a simple absorption, i.e. formation of the compound nucleus which is in this case identical with the final nucleus. The proton does not enter the reaction at all except by carrying away the surplus energy and momentum, this latter function being, of course, quite essential because otherwise a simple absorption process could never occur. Since the O-P process differs essentially from all other

[1] Oppenheimer and Phillips, Phys. Rev. **48**, 500 (1935).
[2] I.e., reactions produced by deuterons (d) with emission of a proton (p). For the nomenclature, see Livingston and Bethe, Rev. Mod. Phys. **9**, 245 (1937).

nuclear reactions, the usual "dispersion theory"[3–5] does not apply to it but a special theory must be developed which will be done in §2. In this theory, the cross section of the O-P process will be expressed in terms of the neutron width of the states of the final nucleus[4, 5] or of the "sticking probability" of the neutron.[5]

It is, of course, a matter of primary importance whether a *d-p* disintegration will occur more easily according to the O-P scheme or to the ordinary scheme, i.e., with the deuteron entering as a whole. It was first believed[1] that the O-P process was favored mainly because the neutron, being uncharged, penetrates more easily into the nucleus than the whole deuteron. However, while this effect is certainly important for heavy nuclei, it appears negligible for light ones ($Z\sim 20$). The essential point in favor of the O-P process seems to be that in the ordinary scheme the *d-p* reaction would have to compete with other processes, notably the *d-n* reaction, while in the O-P scheme there would be no such competition (reference 5, §80).

Even so, the O-P process will be restricted to fairly heavy nuclei. For the lightest nuclei, the potential barrier is so low that it presents no obstacle to the entering deuteron nor to the outgoing proton, the latter fact being important for the question of the competition of the proton emission with the neutron emission. (cf. §6).

§2. Oppenheimer-Phillips Process and Compound Nucleus

Since the O-P process is a simple quantum process, its cross section is given by

$$\sigma=\frac{2\pi}{\hbar}\left|\int V_{AN}\,\chi_A\,\chi_D\,\chi_B{}^*\,\psi_D{}^c\,\psi_P{}^{e*}\,d\tau\right|^2, \quad (1)$$

where A denotes the initial, B the final nucleus, D the deuteron, P the proton, N the neutron. V_{AN} is the interaction between initial nucleus and neutron, the χ's are the internal wave functions (discrete states) and the ψ's the wave functions describing the motion in space (continuous spectrum). The upperscript c denotes normalization per unit current, e per unit energy and a per unit amplitude. The relation between these normalizations is

$$\psi^c=v^{-\frac{1}{2}}\psi^a, \quad (2a)$$

$$\psi^e=(M^2v/2\pi^2\hbar^3)^{\frac{1}{2}}\psi^a, \quad (2b)$$

where v is the velocity, M the mass of the particle in question.

The expression (1) is related to the neutron width of nucleus B which is defined by[6]

$$\Gamma_N=2\pi\left|\int V_{AN}\,\chi_A\,\chi_B{}^*\,\psi_N{}^e\,d\tau'\right|^2 \times(2i+1)(2s+1)/(2j+1), \quad (3)$$

where the integration goes over the same coordinates as in (1) except those of the proton, and i, s, j are the spins of nucleus A, neutron N and nucleus B. In order to correlate (1) and (3), we shall assume that the nuclear wave functions χ_A and χ_B are rapidly varying in comparison with the particle wave functions ψ_N etc. Then, e.g. the integral in (3) may be written approximately

$$\left|\int V_{AN}\,\chi_A\,\chi_B{}^*\,d\tau'\right|^2|\psi_N{}^e|^2{}_{\text{Av}}, \quad (3a)$$

the average being taken over the positions of the neutron important in the integral (3). According to reference 4, §4, the main contribution to (3) comes from positions of the neutron at the surface of the nucleus; therefore the average should be taken over these positions.

Combining now (1) with (3), we find

$$\sigma=\frac{\Gamma_N}{|\psi_N{}^e|^2{}_{\text{Av}}}\frac{2j+1}{(2i+1)(2s+1)}\frac{1}{\hbar} \times\left|\int\psi_D{}^c\,\psi_P{}^{e*}\,\chi_D\,d\tau_P\right|^2{}_{\text{Av}}. \quad (4)$$

In the last expression, the integration goes over the positions of the proton; its result depends on the coordinates of the neutron and should be averaged over the surface of the nucleus. (Moreover, an average should be taken with respect to the directions of the outgoing proton, cf. §4.)

[3] Breit and Wigner, Phys. Rev. **49**, 519 (1936).
[4] Bethe and Placzek, Phys. Rev. **51**, 450 (1937).
[5] Bethe, Rev. Mod. Phys. **9**, 69 (1937).

[6] For the statistical weight factors, cf. reference 4, appendix. An elementary derivation is this: In the capture cross section (1), we are interested in the transition from a state with given orientation of the spins of neutron and initial nucleus A, to a compound state with any spin orientation. The neutron width (3), on the other hand, gives the transition probability from a compound state with given spin orientation to any separated state. Compared to the capture cross section, the neutron width must therefore contain a factor which gives the number of orientations of the neutron spin, $2s+1$, times the number of spin orientations for the initial nucleus, $2i+1$, divided by that number for the final nucleus, $2j+1$.

The quantity $\Gamma_N/|\psi_N|^2$ which is just the first factor in (3a), no longer contains any reference to the neutron wave function. It may be expressed in terms of the sticking probability of neutron and initial nucleus if the definition of a sticking probability is extended to negative kinetic energies of the neutron. Such an extension is necessary because most final states B reached in the d-p process have less energy than the initial nucleus A plus a free neutron (cf. §7). For fast neutrons, the connection between neutron width and sticking probability is[7]

$$\frac{\Gamma_N}{|\psi_N{}^e|^2}=(2s+1)(2i+1)\hbar\, D'\, \xi_N\, \pi R^2\, v_N, \quad (5)$$

where v_N is the velocity of the neutron, ξ_N its sticking probability, R the radius of the nucleus and D' the spacing of the levels[8] of nucleus B. For our neutrons, the velocity v_N would become imaginary. To avoid this difficulty, we arbitrarily replace the neutron energy $E_N=\frac{1}{2}M_N v_N{}^2$ by the deuteron energy $E_D=\frac{1}{2}M_D v_D{}^2$ and obtain

$$G_N{}^*=\Gamma_N/|\psi_N{}^e|^2=(2s+1)(2i+1)\hbar\, D' \times \xi_N{}'\, \pi R^2\, v_D\sqrt{2}. \quad (6)$$

The modified sticking probability $\xi_N{}'$ will be identical with ξ_N for high energies and will be somewhat smaller than ξ_N for low energies.[9]

Inserting (6) into (4) and introducing wave functions normalized per unit amplitude for deuteron and proton with the help of (2a, b), the cross section (4) reduces to

$$\sigma=\frac{(2j+1)}{\sqrt{2}\pi}\frac{M_P{}^2\, v_P\, R^2}{\hbar^3}D'\, \xi_N{}'\,|\textstyle\int\psi_D{}^a\,\psi_P{}^{a*}\chi_d\, d\tau_P|^2_{\text{Av.}} \quad (7)$$

The integral over the proton coordinates must be calculated with the neutron kept fixed.

[7] Konopinski and Bethe, to appear shortly, §2. See also reference 5, §54D.

[8] More accurately, $1/D'$ is the total number of levels per unit energy, counting each level according to its statistical weight $2j+1$.

[9] When applied to slow neutrons, (6) gives

$$\xi_N{}'=\frac{\pi}{2}\frac{\Gamma'}{D}\frac{\hbar^2}{MR^2}Q^{-\frac{1}{2}}, \quad (6a)$$

where $D=\frac{1}{2}D'(2i+1)(2s+1)$ is the actual spacing of neutron resonance levels, $\Gamma'=\Gamma_N E_N{}^{-\frac{1}{2}}$ and Q is the energy evolution in a $d-p$ reaction. Taking $R=10^{-12}$ for radioactive nuclei (§5, 6), we have for atomic weights around 100, approximately $\hbar^2/MR^2=0.7$ MV. Q is about 5 MV, D may be estimated to be about 5 volts (reference 5, Chapter X) and Γ' about 10^{-3} volts$^{\frac{1}{2}}$ (reference 5, §62). This would make $\xi_N{}'$ about 0.1 from slow neutron experiments.

§3. The Penetrability of the Potential Barriers

The integral in (7) which determines the cross section contains three factors:

1. The proton wave function ψ_P. This function will have almost constant amplitude provided the proton is energetic enough not to be appreciably affected by the potential barrier. This will always be true for the fastest protons emitted, i.e. those corresponding to low states of the final nucleus B (cf. also §5, 7).

2. The wave function of the center of gravity of the deuteron, ψ_D, will fall off in an exponential fashion as the deuteron approaches the nucleus, due to the Coulomb potential barrier.

3. The internal wave function of the deuteron, χ_D, will fall off exponentially with the distance between proton and neutron. Therefore, if the neutron is kept fixed at the surface of the nucleus, this function will decrease with increasing distance of the center of the deuteron from the nucleus. This factor varies therefore in the opposite direction than the second; thus there will be an optimum distance r_0 (of the deuteron from the nucleus) at which the product $\psi_D\,\chi_D$ will be a maximum. We may say that at this distance the deuteron will ordinarily "break up" into its constituents.

The wave functions ψ_D and χ_D may be calculated using the WKB (Wentzel-Kramers-Brillouin) method.[10] This has been done by Oppenheimer and Phillips,[1] and their results may be taken over directly. The only point in which we differ from their treatment is that it does not seem legitimate to us to set the radius R of the nucleus equal to zero. Especially if the deuteron energy approaches the height of the potential

[10] To the wave function ψ_D of the center of the deuteron, the WKB is directly applicable, since the corresponding wave equation is separable in polar coordinates and thus the problem reduces to a one-dimensional one. The wave equation for χ_D, however, is not separable; if we introduce polar coordinates s, ϑ, φ for the position of the proton relative to the center of the deuteron, the interaction between neutron and proton will depend only on s while the Coulomb interaction between proton and nucleus depends also on ϑ. In order to apply the WKB to such a nonseparable problem, the direction of grad ψ must first be known. This direction is given by symmetry considerations if nucleus, neutron and proton all lie on a straight line. Then grad ψ will obviously also be along that line. Thus we can find ψ for all these "straight-line positions." This is sufficient since, for given distances R and r of neutron and deuteron center from the nucleus, χ will be largest when the neutron is closest to the deuteron, i.e. when all particles lie on a straight line.

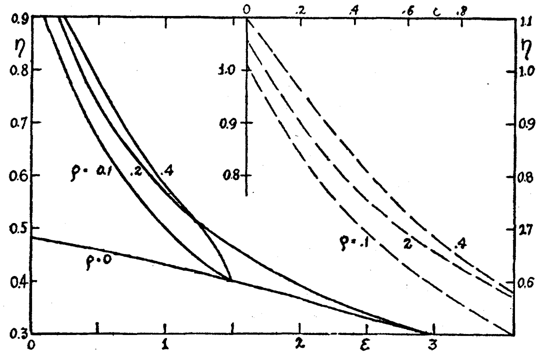

FIG. 1. The break-up distance r_0 of the deuteron as a function of its kinetic energy E for various values of the nuclear radius R. Abscissa $\epsilon=E/I$ (I=deuteron binding energy). Ordinate $\eta=Ir_0/Ze^2$. Curves for various values of $\rho=IR/Ze^2$.

barrier, it is necessary to consider the finite nuclear radius. This is the case for practically all experiments carried out with light nuclei such as Na, Mg, Al;[11] and even for Cu bombarded by deuterons of 3–5 MV, the finite nuclear radius is important.

Using the WKB method, we obtain for the wave functions

$$\psi_D(r)\sim\exp\left\{-\frac{(2M_D)^{\frac{1}{2}}}{\hbar}\int_r^{r_E}\left(\frac{Ze^2}{x}-E\right)^{\frac{1}{2}}dx\right\},\quad(8)$$

$$\chi_D(r,R)\sim\exp\left\{-\frac{(2\mu_D)^{\frac{1}{2}}}{\hbar}\times\int_0^{2(r-R)}\left(I-\frac{Ze^2}{r}+\frac{Ze^2}{r+\frac{1}{2}\zeta}\right)d\zeta\right\}.\quad(9)$$

Here $M_D=2M$ is the mass of the deuteron, $\mu_D=\frac{1}{2}M$ the reduced mass of proton and neutron in the deuteron, M the neutron (= proton) mass. I is the binding energy of the deuteron, ζ the distance between proton and neutron in the deuteron, and

$$r_E=Ze^2/E.\quad(8a)$$

The equations (10), (11) may be re-written:

$$\psi_D\,\chi_D\sim\exp\left\{-\frac{2M^{\frac{1}{2}}}{\hbar}\frac{Ze^2}{I^{\frac{1}{2}}}F(\epsilon,\eta,\rho)\right\},\quad(10)$$

where

$$F(\epsilon,\eta,\rho)=\epsilon^{-\frac{1}{2}}f(\epsilon\eta)+(\eta/1-\eta)^{\frac{1}{2}}\times[f(1-\eta)-f(\{1-\eta\}\{2-\rho\eta^{-1}\})],\quad(11)$$

$$f(x)=\arccos x^{\frac{1}{2}}-x^{\frac{1}{2}}(1-x)^{\frac{1}{2}},\quad(12)$$

$$\eta=Ir/Ze^2,\quad(11a)$$

$$\rho=IR/Ze^2,\quad(11b)$$

$$\epsilon=E/I.\quad(11c)$$

Of the two terms in (11), the first represents the penetration of the deuteron through the Coulomb potential barrier (ψ_D), the second the penetration of the neutron through the "potential barrier" due to its binding to the proton.

As already mentioned, the product $\psi\chi$ will have a sharp maximum for some value r_0 of the deuteron distance r, i.e., F will have a minimum for a certain value η_0 of η. The contributions to the integral in (7) will come mainly from the neighborhood of r_0; therefore, to the accuracy attempted in this paper, it will suffice to calculate F at its minimum η_0. The condition $\partial F/\partial\eta=0$ gives the following equation for η_0:

$$(1-\epsilon\eta_0)^{\frac{1}{2}}=\eta_0(1-\eta_0)^{-\frac{1}{2}}[-f'(1-\eta_0)+(2-\rho\eta_0^{-2})\times f'(\{1-\eta_0\}\{2-\rho\eta_0^{-1}\})]+\tfrac{1}{2}(1-\eta_0)^{-\frac{3}{2}}\times[f(1-\eta_0)-f(\{1-\eta_0\}\{2-\rho\eta_0^{-1}\})].\quad(13)$$

From this equation, ϵ can easily be obtained as a function of η_0 for given ρ.

The result is given in Fig. 1 for some values of ρ which are important experimentally. According to (11b), ρ is the ratio of the deuteron binding energy $I=2.2$ MV to the height of the potential barrier Ze^2/R. With the large nuclear radii proposed by the author,[12] the potential barrier for deuterons varies from about 3 MV for Na to 10 MV for U (reference 5, Table XXXIII) corresponding to a variation of ρ from 0.7 for Na to 0.2 for U. With the small radii of Gamow, ρ would be about three-quarters of these values (0.5 to 0.15).

As is seen from Fig. 1, the "break-up distance" η_0 is not very sensitive to the nuclear radius which is represented by ρ. For zero deuteron energy, η_0 is slightly larger than unity which means that the deuteron breaks up at a point at

[11] Lawrence, McMillan and Thornton, Phys. Rev. **48**, 493 (1935); Henderson, Phys. Rev. **48**, 480 (1935).

[12] Bethe, Phys. Rev. **50**, 977 (1936).

which the Coulomb potential is slightly less than the binding energy I of the deuteron. With increasing deuteron energy, the break-up position shifts to points of higher Coulomb potential, i.e., to smaller values of η_0. This continues until the break-up occurs at the surface of the nucleus itself, i.e., until $\eta_0=\rho$. From then on, the deuteron penetrates as a whole into the nucleus and the penetration probability is given by the ordinary Gamow theory (reference 5, §68). The deuteron energy for which the deuteron breaks up just at the surface of the nucleus (i.e., for which $\eta_0=\rho$), is equal to the height of the potential barrier minus the deuteron binding energy

$$E_0=Ze^2/R-I. \tag{14}$$

In addition to the experimentally important values $\rho=0.2$ and 0.4, there is also given in Fig. 1 a curve for $\rho=0$. This curve which corresponds to the assumption of Oppenheimer and Phillips behaves rather differently from the others. The reason is connected with the behavior of the "potential barrier for the neutron"

$$V_N=I+\frac{Ze^2}{r}-\frac{Ze^2}{2r-x} \tag{15}$$

(cf. (9); x = distance between neutron and nucleus, $2r-x$ distance proton-nucleus). For small nuclear radii ($\rho<\frac{1}{6}$), this potential may become negative for positions of the neutron just outside the nucleus so that the barrier does not quite extend to the surface of the nucleus; for larger nuclear radii, V_N remains always positive right down to $x=R$. This constitutes a qualitative difference between the assumption of zero radius and the actual situation.

After the break-up distance η_0 has been determined, the value of $F(\epsilon, \rho, \eta_0)$ may be found by direct insertion into (11) (12). The result is given in Fig. 2 where this minimum value of F is plotted against the deuteron energy for various values of ρ. It is seen that the various curves are approximately parallel so that we may write approximately

$$F(\epsilon, \rho, \eta_0)\approx F(\epsilon, 0)-g(\rho). \tag{16}$$

Inserting this into (10), we find that the energy dependence of the penetration function $\psi_D\chi_D$ is almost the same for all values of the nuclear radius and that this radius is only important for the absolute value of $\psi\chi$, and therefore for the absolute yield. (The yield will, of course, increase with increasing radius.) The approximate validity of (16) is the reason why the penetration function with zero radius, as calculated by Oppenheimer and Phillips, agrees with the observed excitation functions of reactions of the d-p type.[11]

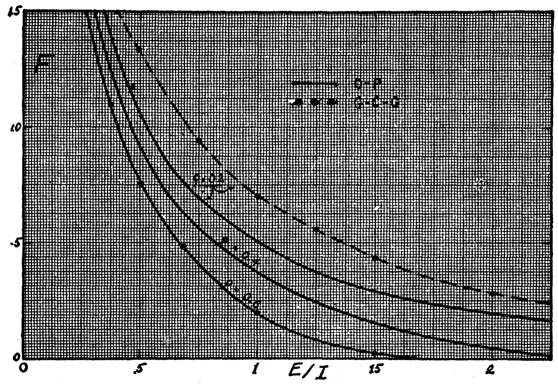

FIG. 2. The Oppenheimer-Phillips penetration function F. The penetrability of the potential barrier is $P=\exp.(-0.6\,ZF)$ where Z is the nuclear charge. Abscissa E/I = kinetic energy of deuteron/deuteron binding energy. Curves for different values of $\rho=IR/Ze^2$ = deuteron binding energy/nuclear potential barrier. $\rho=0.6$, 0.4, 0.2 correspond to $Z=13$, 25 and 70, respectively. The broken curve gives the penetration function F in the ordinary Gamow-Condon-Gurney theory for $\rho=0.2$; for $\rho=0.6$ the G-C-G function (represented by the dots) practically coincides with the O-P function.

For comparison, we have also given in Fig. 2 the penetrability of the deuteron as a whole according to the ordinary Gamow-Condon-Gurney (G-C-G) theory which is

$$\psi_D(R)\sim\exp\left\{-\frac{2M^{\frac{1}{2}}}{\hbar}\frac{Ze^2}{I^{\frac{1}{2}}}\epsilon^{-\frac{1}{2}}f(\epsilon\rho)\right\}. \tag{17}$$

This penetrability coincides, of course, with the O-P penetrability above the critical energy (14). The figure shows that for low potential barrier ($\rho=0.6$) the G-C-G curve is practically identical with the O-P curve even at low energies while for high barrier ($\rho=0.2$) there is a considerable difference, the G-C-G curve being considerably steeper. For $\rho=0.4$, the result would be nearer to that for $\rho=0.6$ than to that for $\rho=0.2$. Thus it is necessary to go to fairly heavy nuclei in order to find an appreciable difference between the O-P and the G-C-G excitation functions (cf. Fig. 3). This fact does not preclude the possibility that the O-P *mechanism* is valid for much lighter nuclei (cf. §6).

§4. Integration and Normalization

Compared to the exponential in (10), all other factors in the cross section (7) are relatively unimportant. We shall therefore use crude methods for their determination so that our final result can only give the order of magnitude of the cross section. Any attempt at greater accuracy seems unwarranted in view of the use of the WKB method for the calculation of deuteron wave function, and of the approximations made in §2.

The normalization condition on the internal wave function χ_D of the deuteron is $\int \chi_D^2 d\tau = 1$, the integration being over the position of the neutron relative to the proton.[13] For this normalization, it is sufficient to replace χ_D by the wave function of a free deuteron, which amounts to neglecting the polarization of the deuteron by the Coulomb field of the nucleus. The normalized wave function of a free deuteron is

$$\chi_D{}^0 = (\alpha/2\pi)^{\frac{1}{2}} e^{-\alpha s}/s \tag{18}$$

with

$$\alpha = (MI)^{\frac{1}{2}}/\hbar \tag{18a}$$

and s the distance between neutron and proton. Thus (9) must be multiplied by $\alpha^{\frac{1}{2}}/(2\pi)^{\frac{1}{2}} s$.

The required integration over the proton coordinates, especially the angles, is facilitated by the fact that the square of the integral must be averaged over all directions of the motion of the proton, and all positions of the neutron at the surface of the nucleus. We shall introduce polar coordinates with the direction from the center of the nucleus to the neutron as axis; the angular coordinates of deuteron and proton in this system may be $\vartheta_D \varphi_D$, $\vartheta_P \varphi_P$ whereas the directions of motion of the two particles (at infinity) may be specified by $\Theta_D \Phi_D$, $\Theta_P \Phi_P$. If the distance of the proton from the nucleus is large compared to that of the neutron (nuclear radius) which will be true whenever the deuteron energy is small compared to the potential barrier, we may put $\vartheta_D = \vartheta_P = \vartheta$, $\varphi_D = \varphi_P = \varphi$. We have then to integrate (7) over $\vartheta\varphi$ and to average the square of the integral over $\Theta_D \Phi_D \Theta_P \Phi_P$.

We expand, in the usual way, the wave functions of deuteron and proton in polar coordinates. If ϑ' is the angle between $\mathbf{r}_P$ and the direction of motion of the proton, we have

$$\begin{aligned} \psi_P &= (k_P r_P)^{-1} \sum_{l'} (2l'+1) f_{l'}(r_P) P_{l'}(\cos \vartheta') \\ &= 4\pi (\lambda\!\!\!^{-}_P / r_P) \sum_{l'm'} f_{l'}(r_P) Y_{l'm'}(\vartheta_P \varphi_P) Y_{l'm'}{}^*(\Theta_P \Phi_P) \end{aligned} \tag{19}$$

and a similar expression for the deuteron. Here $f_{l'}$ is a radial wave function which, at large distances from the nucleus, has the form

$$f_{l'} = e^{i\delta_{l'}} \sin (k_P r_P - \tfrac{1}{2} l' \pi + \delta_{l'}). \tag{19a}$$

The last factor in (7) becomes then

$$\begin{aligned} |\int \chi_D \psi_D{}^a \psi_P{}^{a*} d\tau_P|^2{}_{\mathrm{Av}} = (4\pi)^{-2} \sum_{lml'm'} \int \sin \Theta_P d\Theta_P \\ \times \sin \Theta_D d\Theta_D d\Phi_P d\Phi_D |Y_{l'm'}(\Theta_P \Phi_P)|^2 |Y_{lm}(\Theta_D \Phi_D)|^2 \\ \times |(4\pi)^2 \lambda\!\!\!^{-}_P \lambda\!\!\!^{-}_D \int dr_P (r_P{}^2/r_P r_D) f_l(r_D) f_{l'}(r_P) \chi_D(r_P, 0) \\ \times \int \sin \vartheta d\vartheta d\varphi [\chi_D(r_P, \vartheta)/\chi_D(r_P, 0)] \\ \times Y_{lm}(\vartheta\varphi) Y_{l'm'}{}^*(\vartheta\varphi)|^2. \end{aligned} \tag{20}$$

Here $\chi_D(r_P, 0)$ is the value of the internal wave function of the deuteron if nucleus, neutron and proton lie on a straight line, as calculated in §3. The last (angular) integral will be a slowly varying function of r_P, therefore we may write

$$\begin{aligned} |\int dr_P \cdots \int \sin \vartheta d\vartheta d\varphi \cdots|^2 \\ \approx |\int dr_P \cdots|^2 |\int \sin \vartheta d\vartheta d\varphi \cdots|^2{}_{\max}, \end{aligned}$$

the value of the last integral being taken at that value of r_P which makes the integrand of the first integral a maximum, i.e. at $r_P = 2r_0 - R$ (cf. §3).

The integrations over $\Theta_D \Phi_D \Theta_P \Phi_P$ can be carried out immediately, they give unity for any $lml'm'$. To evaluate the remaining integrals, we observe that $f_{l'}$ has about the same form for all important values of l'; therefore we re-write (20)

$$\begin{aligned} (4\pi)^2 \lambda\!\!\!^{-}_P{}^2 \lambda\!\!\!^{-}_D{}^2 \sum_l |\int dr_P (r_P/r_D) f_l(r_D) f_{l'}(r_P) \\ \times \chi_D(r_D, 0)|^2{}_{\mathrm{Av}(l')} \times \sum_{l'm'm} |\int \sin \vartheta d\vartheta d\varphi \\ \times [\chi_D(\vartheta)/\chi_D(0)] Y_{lm}(\vartheta\varphi) Y_{l'm'}{}^*(\vartheta\varphi)|^2. \end{aligned} \tag{21}$$

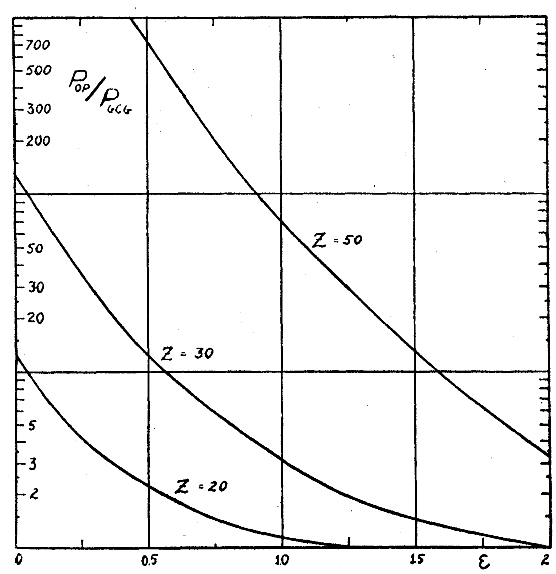

Fig. 3. The ratio of the penetrabilities of the potential barrier in the Oppenheimer-Phillips and the Gamow-Condon-Gurney theories for various nuclear charges as a function of E/I = kinetic energy of deuteron/binding energy.

[13] Then $|\psi_D \chi_D|^2 d\tau_D \, d\tau_{\mathrm{rel}}$ gives the probability of finding the deuteron in the volume element $d\tau_D$ and the relative coordinates between the limits indicated by $d\tau_{\mathrm{rel}}$. In order to obtain the probability for given neutron and proton coordinates, we must multiply $|\psi_D \chi_D|^2$ by the Jacobian

$$\frac{\partial(\mathbf{r}_D, \mathbf{r}_{\mathrm{rel}})}{\partial(\mathbf{r}_N, \mathbf{r}_P)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 1 \end{vmatrix}^3 = 1.$$

The sum over $l'm'$ can be carried out immediately using the completeness relation for the spherical harmonics; then the sum over m is found using the addition theorem

$$\sum_m |Y_{lm}|^2 = (2l+1)/4\pi,$$

so that the last sum in (21) reduces to

$$\int \sin \vartheta d\vartheta d\varphi [\chi_D(\vartheta)/\chi_D(0)]^2. \quad (22)$$

Here we use again the wave function (18) of the free deuteron, and the geometrical relation

$$\begin{aligned} s^2 &= R^2 + r_P^2 - 2Rr_P \cos\vartheta; \\ sds &= Rr_P \sin\vartheta d\vartheta; \quad s(\vartheta=0)=s_0=r_P-R. \end{aligned} \quad (23)$$

Then (22) becomes

$$2\pi(s_0^2/Rr_P)\int (ds/s)e^{-2\alpha(s-s_0)} \approx (2\pi s_0^2/2\alpha R r_P^2)(1-e^{-4\alpha R}) \quad (24)$$

since r_P is the average value of s. Since $1/\alpha = 4.5\cdot 10^{-13}$ cm, $e^{-4\alpha R}$ is negligible even for fairly light nuclei.

The integral over r_P contains the factor $f_l(r_D)\chi_D$ which has a sharp maximum for $r_P = 2r_0 - R$ (§3), and the factor $f_{l'}(r_P)$. The latter function oscillates rapidly with a period of $2\pi\lambda_P$ and has the amplitude unity. It can be shown that the change of $f_{l'}$ with r_P is more rapid near $r_P = 2r_0 - R$ than that of the product $f_l\chi_D$, except for very large proton wave-length λ_P. Therefore we assume that the main contribution to the integral comes from a region of extension λ on either side of the maximum of $f_l\chi_D$, and replace the integral by 2λ times the value of the integrand at $r_P = 2r_0 - R$. Putting the average of $f_{l'}^2(2r_0 - R)$, averaged over l', equal to one-half, (21) becomes finally

$$(4\pi)^2 2\lambda_P{}^4\lambda_D{}^2(r_P/r_D)^2 \sum_l P_{\mathrm{OP}l}(\alpha/2\pi s_0^2)\times[2l+1/4\pi] \times(2\pi s_0^2/2\alpha R r_P^2) = 4\pi(\lambda_P{}^4\lambda_D{}^2/Rr_0^2)\sum_l (2l+1)P_{\mathrm{OP}l}. \quad (25)$$

Here $P_{\mathrm{OP}l}$ is the square of $f_l\chi_D$, the latter being taken without the factor $\alpha^{\frac{1}{2}}/(2\pi)^{\frac{1}{2}}s$ (cf. 18). Thus, for $l=0$, $P_{\mathrm{OP}l}$ is simply the square of (10), the Oppenheimer-Phillips penetrability

$$\begin{aligned} P_{\mathrm{OP}} &= \exp\left\{-\frac{4M^{\frac{1}{2}}}{\hbar}\frac{Ze^2}{I^{\frac{1}{2}}}F(\epsilon, \eta_0, \rho)\right\} \\ &= \exp(-0.6ZF(\epsilon, \rho)). \end{aligned} \quad (26)$$

For $l \neq 0$, slightly smaller results are obtained because of the additional potential barrier due to the centrifugal force (ref. 5, §72, and ref. 7). As for ordinary disintegrations, we replace the sum over l by $Pl_c'^2$ where l_c' is a "critical orbital momentum."

Inserting these results into (7), we obtain the final formula

$$\sigma_{\mathrm{OP}} = 2\sqrt{2}(2j+1)M\hbar^{-2}R\lambda_P{}^3\lambda_D{}^2 l_c'^2 r_0^{-2}D'\xi_N' P_{\mathrm{OP}}. \quad (27)$$

§5. Energy Distribution of the Emitted Protons

The cross section (27) may be written

$$\sigma_{\mathrm{OP}} = \sigma_0(2j+1)D'\,\xi_N'\,E_P{}^{-\frac{1}{2}}, \quad (28)$$

where σ_0 does not depend on the energy E_P of the proton. The excitation energy of the final nucleus being a constant minus the proton energy, the number of protons with an energy between E and $E+dE$, $\sigma(E)dE$, is obtained by summing (28) over all the nuclear states in the energy interval dE. Since dE/D' is the number of levels counting each level according to its degeneracy $2j+1$, we must simply replace $(2j+1)D'$ by dE:

$$\sigma_{\mathrm{OP}}(E)dE = \sigma_0\,\xi_N'\,E^{-\frac{1}{2}}\,dE. \quad (29)$$

In this formula, the factor $E^{-\frac{1}{2}}$ varies relatively slowly with the energy and is, moreover, theoretically rather uncertain. The number of protons per unit energy is given primarily by the sticking probability ξ_N'. This is in striking contrast to disintegrations of the ordinary type in which the number of outgoing particles per unit energy is determined by the number of energy levels of the final nucleus in the corresponding energy interval (ref. 5, §54). This latter relation leads, as is well known, to the Maxwell-Boltzmann law

$$\sigma_{\mathrm{ord}}(E)dE \sim e^{-E/T}\,dE \quad (30)$$

for the energy distribution of the outgoing particles in an ordinary nuclear reaction such as the *d-n* reaction.

While the energy distribution in an ordinary reaction can thus be predicted from the "evaporation model," not much can be said about the sticking probability ξ_N' which determines the distribution of the Oppenheimer-Phillips protons. Two extreme views may be taken

(a) *The sticking probability may have the same order of magnitude for all levels* of the compound nucleus (in our case ≡ final nucleus), including the lowest ones.[14] Then the protons from the *d-p* reaction should be *uniformly distributed in energy* between two limits [see below, (31), (32)].

(b) *The sticking probability may decrease for low compound levels* in such a way as to leave the *reduced neutron width* Γ^* (cf. 6) approximately *constant*. This would mean that ξ_N' varies inversely as the spacing D' of levels: In this case, therefore, the energy distribution of the protons would be the same as in an ordinary nuclear reaction,

[14] Grave theoretical reasons are against this assumption because it would mean the validity of the one-body model for the low states of nuclei.

showing a very sharp drop in intensity towards higher proton energies.

(c) Any intermediate assumption may be made; e.g., we think it plausible that for relatively high excitation of the compound nucleus, ξ is almost constant, while near the lowest level the reduced width G^* becomes constant, with a transition region between.

In view of these uncertainties and of the great theoretical importance of the sticking probability, it would be very interesting to measure experimentally the energy distribution of the emitted protons. In order to make the O-P mechanism certainly valid, the experiment should be carried out with heavy nuclei and preferably with an element with only one isotope. The deuteron energy is rather immaterial, but the deuterons should be fairly monochromatic (thin target). Owing to the large density of nuclear levels, the protons will, of course, not show any "group structure" as with light nuclei, except possibly near the high energy limit (corresponding to low states of the residual nucleus). The spectrum would extend to an upper limit

$$E_{\max}=E_D+Q, \qquad (31)$$

where Q is the reaction energy (about 4–6 MV). At the low energy end, the proton spectrum will break off at an energy approximately equal to the Coulomb potential at the distance $2r_0-R$ from the nucleus, i.e. the distance at which the proton is found when the deuteron breaks up. With the notations of §3, we have

$$E_{\min}=\frac{Ze^2}{2r_0-R}=\frac{I}{2\eta_0-\rho}. \qquad (32)$$

For zero deuteron energy, we have about $2\eta_0-\rho=1.9$ so that $E_{\min}=1.2$ MV; for $E_D=I=2.2$ MV, $E_{\min}$ is about I; and if the deuteron energy becomes equal to or greater than the height of the potential barrier, $E_{\min}$ will be equal to the potential barrier. In practically all cases, $E_{\min}$ is large enough so that the corresponding proton range $R_{\min}$ is considerably greater than the range of the incident deuteron. Therefore the whole proton distribution should be easily observable without disturbance by the scattered deuterons.

It is of particular importance that these measurements would give the width of nuclear levels right down to the lowest states of the compound (= final) nucleus, states which could never be examined by ordinary neutron bombardment. The *d-p* reaction supplies neutrons of "negative kinetic energy." The importance of the width of these low levels lies mainly in the theory of the α-decay. It can probably be assumed that the α-width without barrier is of the same order as the neutron width[15] for a given level. Thus the measurement of the proton distribution from the O-P reaction should settle the problem of the nuclear radii. If assumption (a) above (constant sticking probability down to the lowest levels, uniform proton distribution) proves true, the width of the low levels would, according to (6), be very large due to their large spacing. This would make the radii of radioactive nuclei almost as small as in the one-body model of Gamow (about $9\cdot 10^{-13}$ cm). If, on the other hand, assumption (b) proves to be correct (constant G^*, proton distribution showing rapid decrease towards high energies), the large nuclear radii proposed by the author ($12\cdot 10^{-13}$ cm) would result. Assumption (c), which we consider most likely at the moment, would lead to an intermediate value for the radius.

§6. Comparison of Oppenheimer-Phillips and "Ordinary" Processes

Integrating (27) over the proton energy from $E_{\min}$ to $E_{\max}$, we find

$$\sigma_{\mathrm{OP}}=2^{\frac{3}{2}}\lambda\!\!\!^{-}_D{}^2 l_c'^2 r_0^{-2} R(\lambda\!\!\!^{-}_{\min}-\lambda\!\!\!^{-}_{\max})P_{\mathrm{OP}}\xi_{N\mathrm{Av}}', \qquad (33)$$

where $\lambda\!\!\!^{-}_{\min}$ and $\lambda\!\!\!^{-}_{\max}$ are the proton wave-lengths corresponding to the energies $E_{\min}$ and $E_{\max}$ (cf. 31, 32). (Of course, $\lambda\!\!\!^{-}_{\min}>\lambda\!\!\!^{-}_{\max}$.) $\xi_{N\mathrm{Av}}'$ is the average of the sticking probability over the levels of nucleus B between ground state and excitation energy $E_{\max}-E_{\min}$, slightly greater weight being given to the higher states. The taking of such an average is, of course, adequate only if ξ changes slowly with the energy (assumption (a) in §5); if, on the other hand, the reduced neutron width G^* (cf. 6) remains approximately constant (assumption b), we may write

$$\sigma_{\mathrm{OP}}=\sqrt{2}R\,\lambda\!\!\!^{-}_D{}^2\, l_c'^2\, r_0^{-2}\,\lambda\!\!\!^{-}_{\min} \times T'\,E_{\min}^{-1}\,P_{\mathrm{OP}}\,\xi_N', \qquad (33a)$$

[15] The "neutron width" G^* deduced for levels below the neutron dissociation energy is, of course, not an actual width but only a convenient measure of the matrix element.

where T' is the temperature of nucleus B corresponding to the excitation energy $E_{\max}-E_{\min}$ and ξ_N' the corresponding sticking probability of the neutron. T' is in general considerably smaller than $E_{\min}$.

The O-P cross section (33) may be compared with the cross section of an ordinary nuclear reaction produced by a deuteron (reference 5, Eq. (681a))

$$\sigma_{\mathrm{GCG}}=\pi\lambda_D^2\, l_c^2\, P_{\mathrm{GCG}}\, \xi_D\, \Gamma_Q/\Gamma. \qquad (34)$$

Here Γ is the total width of the levels of the compound nucleus formed by adding the deuteron to the initial nucleus, Γ_Q the partial width referring to the outgoing particle, P_{GCG} the penetrability of the potential barrier for the deuteron as a whole, according to the ordinary theory of Gamow, Condon and Gurney, ξ_D the deuteron sticking probability and l_c the critical orbital momentum for the "ordinary" disintegration. The ratio of (34) to (33) is

$$\frac{\sigma_{\mathrm{GCG}}}{\sigma_{\mathrm{OP}}}=\frac{\pi}{2^{\frac{3}{2}}}\frac{l_c^2}{l_c'^2}\frac{r_0^2}{R(\lambda_{\min}-\lambda_{\max})}\frac{\xi_D}{\xi_{N\mathrm{Av}}}\frac{P_{\mathrm{GCG}}}{P_{\mathrm{OP}}}\frac{\Gamma_Q}{\Gamma}. \qquad (35)$$

The critical orbital momentum l_c is discussed in reference 5, §72. For deuteron energies well below the top of the barrier, we have (Eq. 633) $l_c^2=\frac{1}{2}g$ where g is given in (600a) of reference 5. Analogous considerations for the O-P process give $l_c'^2=\frac{1}{2}g'$ where g' differs from g in that the nuclear radius R is replaced by the break-up distance r_0 of the deuteron. Since $g\sim R^{\frac{1}{2}}$, we have $l_c^2/l_c'^2=(R/r_0)^{\frac{1}{2}}$. Then we may write

$$\frac{\sigma_{\mathrm{GCG}}}{\sigma_{\mathrm{OP}}}=k(E_D,Z)\frac{\xi_D}{\xi_{N\mathrm{Av}}}\frac{P_{\mathrm{GCG}}}{P_{\mathrm{OP}}}\frac{\Gamma_Q}{\Gamma}, \qquad (36)$$

where k is a relatively unimportant factor which is given by

$$k=\frac{\pi}{2^{\frac{3}{2}}}\frac{r_0^{\frac{3}{2}}}{R^{\frac{1}{2}}(\lambda_{\min}-\lambda_{\max})}. \qquad (37)$$

Assuming the nuclear radius to be 10^{-12} cm for radioactive nuclei, and to be proportional to $A^{\frac{1}{3}}$ (A = atomic weight) otherwise, we find

$$k=0.14_5\, Z^{\frac{3}{2}}\, A^{-\frac{1}{6}}\, \eta_0^{\frac{3}{2}} \times[(2\eta_0-\rho)^{\frac{1}{2}}-(\epsilon+q)^{-\frac{1}{2}}]^{-1}, \qquad (37a)$$

where $q=Q/I\approx 2$ to 3. The factor k turns out to be almost independent of the deuteron energy, decreasing by 10 to 20 percent when the deuteron energy increases from I to $2I$. For $E_D=I$, the values of k are approximately

$Z=10$	20	30	50	70	92
$k=$ 4.7	8.8	13.5	26	39	57.

In view of the many approximations made, the numerical values of k should not be trusted; they may easily be wrong by a factor of 5 either way. However, the tendency of k to increase with increasing Z seems to be real. Moreover, k may actually depend more strongly on the deuteron energy.

The second factor in (36), *viz.* the ratio of the sticking probabilities of deuteron and neutron, is very difficult to estimate. It is likely that ξ_D is considerably larger than ξ_N'; in fact, ξ_D will be approximately unity while the "modified sticking probability" ξ_N', may be about 0.1 (footnote 9).

The penetrabilities of the potential barrier can be found from Fig. 2; e.g. for $\rho=0.2$, $Z=70$, $E_D=2I=4.4$ MV, we have from that figure $F_{\mathrm{OP}}=0.19_5$, $F_{\mathrm{GCG}}=0.28$ which gives, according to (26), $P_{\mathrm{OP}}=3.5\cdot10^{-4}$ and $P_{\mathrm{GCG}}=0.8\cdot10^{-5}$, corresponding to a ratio of 45 in favor of the O-P penetration. With $E_D=I=2.2$ MV the ratio would be 5000. Altogether, the factors beside Γ_Q/Γ in (33), will be about unity for moderate energies (~3–4 MV), decreasing with increasing energy.

This means first of all that *the d-p reaction will practically always follow the O-P mechanism for heavy nuclei.* For the proton width of heavy nuclei is known to be extremely small compared to the neutron (and therefore the total) width (ref. 5, §79), so that $\Gamma_Q/\Gamma\ll1$ if Q is a proton. The ratio Γ_Q/Γ can be estimated to be about 10^{-4} for heavy nuclei and medium excitation energies so that only one *d-p* process in 10,000 would follow the "ordinary" instead of the O-P mechanism.

Furthermore, we see that for heavy nuclei and moderate deuteron energies *the O-P process will be about equally as probable as the d-n reaction.* (The *d-n* reaction follows the G-C-G mechanism with Γ_Q practically equal to Γ.) A more accurate experimental determination of the relative probabilities of *d-p* and *d-n* process could be used to determine the relative sticking probabilities of deuteron and neutron according to (33). However, it must be considered that the *d-n* reaction is, in the case of heavy nuclei, often followed by the emission of an α-particle or a second neutron (cascade disintegration, cf. ref. 5, §79 E). The yields of these reactions must, for our purpose, be added to that of the *d-n* reaction itself.

For lighter elements, the probability of the O-P process will decrease as compared to the "ordinary" processes owing to the change in the relative penetrabilities. This decrease will be partly compensated by the decrease of k (cf. (37)). Nevertheless, the "ordinary type" *d-p* reaction will ultimately become as probable as the O-P process so that we reach the limit of validity of the O-P mechanism. As we shall show, this limit occurs for Z about 25, so that $k\approx 10$. Setting, as before, $\xi_N=\frac{1}{10}\xi_D$, the limit of validity of the O-P mechanism is given by

$$\frac{\Gamma}{\Gamma_{\text{proton}}}\frac{P_{\text{OP}}}{P_{\text{GCG}}}\approx 100. \qquad (38)$$

The figure 100 is of course very uncertain, and a possible error of at least a factor 5 either way must be admitted.

To estimate $\Gamma/\Gamma_{\text{proton}}$, we note (1) that even for medium heavy nuclei Γ is practically equal to the neutron width, (2) that the width for each kind of particles is supposed to be proportional to the number of available states of the respective final nuclei (ref. 5, §54, §79). The latter depends on the available energy (ref. 5, §79) which is

$$U=E_D+Q-B, \qquad (39)$$

where Q is the reaction energy and B the potential barrier. The difference of the available energies for the *d-n* and the *d-p* reactions is thus

$$U_N-U_P=Q_N-Q_P+B_P$$
$$=\epsilon_- -0.8\text{ MV}+B_P, \qquad (40)$$

where B_P is the proton potential barrier, 0.8 MV the difference of the masses of neutron and hydrogen atom, and ϵ_- the energy difference between the (isobaric) nuclei formed in the *d-p* and the *d-n* reaction. If the former nucleus is heavier, ϵ_- would be equal to the maximum energy of the β-particles emitted (plus the energy of the subsequent γ-ray if the β-emission leads to an excited state). If the *d-n* nucleus is heavier, ϵ_- is negative; and if in particular the *d-n* nucleus emits positrons, $\epsilon_-=-1.02-\epsilon_+$ where ϵ_+ is the maximum energy of the positrons. The relative number of available states, and therefore the ratio of the widths, is given by the statistical formula

$$\frac{\Gamma_N}{\Gamma_P}=\exp\left(\frac{U_N-U_P}{T}\right), \qquad (41)$$

where T is the nuclear temperature corresponding to some average excitation energy between U_N and U_P. Now for medium heavy nuclei Q_N is about 5 MV (difference of mass excesses of deuteron and neutron), Q_P-B_P will turn out to be about zero (§7), so that $\frac{1}{2}(U_N+U_P)$ will be 5–7 MV, depending on the deuteron energy. The corresponding nuclear temperature would be about 1.3 MV according to reference 5, Table XXI.

The solution of (38) is approximately

$$U_N-U_P\approx 4T. \qquad (42)$$

To verify this, we show that $P_{\text{OP}}/P_{\text{GCG}}$ is not very different from unity under the conditions given by (42). On the average, ϵ_- in (40) will be zero (see below for a more detailed discussion); therefore (42) corresponds to a proton barrier of $4T+0.8\text{ MV}\approx 6\text{ MV}=2.7\ I$, i.e. (cf. (11b)) to $\rho=0.37$. For $\rho=0.4$ and a deuteron energy of 2.2 MV ($=I$), we have $F_{\text{OP}}=0.37$ (cf. Fig. 2) and $F_{\text{GCG}}=0.39_5$ (cf. (19)). With $R=10^{-12}$ cm for the natural radioactive nuclei, a barrier height of 6 MV corresponds about to $Z=25$. This would give (cf. (26))

$$P_{\text{OP}}/P_{\text{GCG}}=e^{0.6\cdot 25\cdot 0.02_5}=1.4.$$

With a higher deuteron energy, we should find an even smaller ratio $P_{\text{OP}}/P_{\text{GCG}}$. Therefore $\Gamma_N/\Gamma_P=\exp(U_N-U_P/T)$ is only slightly less than 100 (cf. (38)), so that (42) follows immediately.

According to (36), the limit of validity of the O-P mechanism is therefore given by

$$B_P=4T+0.8\text{ MV}-\epsilon_-. \qquad (43)$$

Assuming $T=1.3$, this gives

$$B_P=6\text{ MV}-\epsilon_-. \qquad (43a)$$

The limit of validity depends therefore on the relative stability of the nuclei formed in the *d-n* and *d-p* reactions. If the product of the *d-p* reaction is radioactive, the reaction will follow the O-P mechanism even for relatively light nuclei; if it is stable, the "ordinary" type of reaction will persist to somewhat heavier nuclei. This is due to the competition of the ordinary type with the *d-n* reaction; this competition will be easier when the product of the *d-p* reaction is more stable than that of the *d-n* reaction.

The exact value of the critical nuclear charge above which the O-P mechanism is valid, depends also on the assumed nuclear radii. With the large radii, $B_P=6$ corresponds (reference 5, Table XXXIII) to about $Z=36$; with the small (Gamow) radii, the limit would be $Z=20$ (for $\epsilon_-=0$), and with the intermediate radii assumed in this paper, we have $Z=25$. In a case such as Na^{23} *d-p* where the product nucleus (Na^{24}) emits β-rays of as much as 4.6 MV (including γ-ray energy), the O-P mechanism may be valid in spite of the exceedingly low nuclear charge $Z=11$.

These estimates of the limit of validity are very crude, mainly because of the uncertainty in the numerical value 100 in (38), but also because the nuclear temperature T is not very certain (it is probably much higher for light nuclei such as Na which works against the O-P mechanism in such cases). If, e.g., the correct figure in (38) is 10 instead of 100, we should have $2.3T$ in (39) and assuming now $T=1.6$ MV (generally lighter nuclei!) 4.5 MV in (39a). This would shift the limit of validity to $Z=17$ (with the medium radius), if $\epsilon_-=0$, and to still lower Z when the product nucleus of the *d-p* reaction is radioactive ($\epsilon_->0$).

Much more information on these problems could be obtained from experimental investigations of the relative yields of *d-n* and *d-p* reactions. For very light atoms, the probabilities of the two reactions should vary from case to case and should, on the average, be about equal. For higher atomic weight (perhaps $10<Z<25$), the probability of the proton reaction should show, on the average, a slow decrease because the competition between *d-n* and *d-p* process becomes more and more favorable for *d-n*. The minimum is reached at the limit of validity of the O-P mechanism; further increase of the atomic number will increase the relative probability of proton emission because the O-P penetrability increases compared to the G-C-G penetrability. Moreover, the fluctuations from case to case should become much less pronounced in the O-P region because the competition has stopped. For very heavy nuclei, as already mentioned, the *d-p* reaction is presumably about equally probable as the *d-n* reaction for energies of about 4 MV.

While all the problems mentioned thus far have not yet received much attention experimentally, a rather large amount of work has been done on excitation functions of *d-p* as compared to *d-n* and *d-α* reactions. Here the theory gives much less striking results. Large differences between the O-P and the G-C-G type of reaction can be expected only for rather heavy nuclei for which no experiments are available. In Fig. 3 we have plotted the ratio P_{OP}/P_{GCG} as a function of the deuteron energy for various nuclear charges. The nuclear radius was assumed to be 10^{-12} for naturally radioactive nuclei. It is seen that for $Z=20$ the ratio of the excitation functions is practically unity for all energies above 2 MV ($\epsilon=1$), for $Z=30$ it decreases only from about 3 at 2 MV to 1 at and above 4 MV, and only for $Z\geqslant 50$ the ratio changes rapidly above 2 MV. Therefore we believe that the validity of the O-P mechanism can scarcely be proved by comparing the excitation functions of *d-p* and other reactions for nuclei of $Z\leqslant 30$.

§7. Secondary (Cascade) Disintegrations

As any nuclear process, a *d-p* reaction may in principle be followed by a break-up of the residual nucleus provided sufficient energy is available. If the residual nucleus is sufficiently heavy, it will always disintegrate with neutron emission whenever this is energetically possible (cf. reference 5, §79 and §65). In our case, neutron emission leads back to the original nucleus; it is therefore energetically possible if

$$E_D-E_{\min}-I>0, \tag{44}$$

where $E_{\min}$ is the minimum proton energy given in (32). From Fig. 1, it can be shown that the expression in (44) is always negative for deuteron energies below $B+I$ where B is the height of the potential barrier. Thus a *d-pn* reaction can only occur[16] when the deuteron can go over the

[16] Strictly speaking, there are always some protons of smaller energy than $E_{\min}$. When such slow protons are emitted, a *d-pn* reaction is possible. However, the number of these protons is small because they have to penetrate the potential barrier, and therefore the probability of the *d-pn* reaction will also be small.

potential barrier, and in this case the primary *d-p* reaction is relatively improbable compared to a primary *d-n* reaction, since it is no longer favored by a greater penetrability. Therefore the *d-pn* reaction (which would be hard to observe) will be of rather minor importance.

The question of *d-pα* reactions is rather harder to decide because in this case the energy evolution cannot be determined so accurately. However, it can be said with certainty that the *d-pα* reaction can only have an appreciable probability with a given nucleus if slow neutrons give an *n-α* reaction with the same nucleus. For we have shown above that the product nucleus of the *d-p* reaction will, in general, not have sufficient energy to emit a neutron; it has therefore less excitation energy than the compound nucleus formed by adding a slow neutron to the target nucleus. If that latter compound nucleus emits γ-rays rather than α-particles, i.e., if the capture of slow neutrons is more probable than a *n-α* reaction, the same will be *a fortiori* true of the final nucleus formed in the *d-p* reaction, because the probability of α-emission decreases rapidly with decreasing excitation energy. *n-α* reactions with slow neutrons and heavy nuclei have only been observed for Th and U (reference 2, Table LX); therefore we may expect that only these extremely heavy elements give *d-pα* reactions to any appreciable extent. (A *small* yield of the *d-pα* reaction will, of course, always be obtained; it may be calculated from the penetrability of the potential barrier for α-particles if the energy evolution in the reaction is known). This seems to make unlikely the reaction *Au-d-pα* which was reported by Cork and Thornton[17] and was previously considered probable by the author (reference 5, p. 205).

The rarity of *d-pα* reactions may also be understood if we consider that the *d-p* reaction produces a nucleus with too many neutrons which will naturally have no tendency to lose further charge by emission of an α-particle. This is in contrast to the *d-n* reaction which produces a nucleus of too high charge so that a subsequent α-emission seems favorable.

Our considerations show that *d-p* reactions with deuteron energies below the potential barrier should rarely lead to any cascade disintegration.

[17] Cork and Thornton, Phys. Rev. **51**, 59 (1937).

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A New Method for Investigating Atomic Electron Velocities

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When electrons of sufficient speed pass through helium under conditions favorable to single scattering, the electrons scattered through a suitable angle fall into two distinct classes, those scattered elastically and those scattered inelastically. The former have been scattered by nuclei and the latter by atomic electrons. Because the atomic electrons are in random motion, those electrons which have been scattered by them through a definite angle have a distribution of energies, the most probable energy being that corresponding to a collision with an atomic electron at rest. Jauncey has shown that when a fast electron of energy V_0 collides with an atomic electron having a component velocity u in a certain direction, the electron will have energy given by $V=V_0\cos^2\theta+u(2mV_0/e)^{\frac{1}{2}}\sin\theta$, where θ is the angle of scattering. It can be shown to follow from this relation that the *distribution of energy* among the scattered electrons is identical with the *distribution of component velocities* among the atomic electrons. Moreover, since the last mentioned distribution is closely related to, and identical in shape with, the profile of the Compton modified band in x-ray scattering, measurements of the energy distribution of the scattered electrons will give an experimental determination of the profile of the band. Wave mechanical computations lead to a definite shape for this profile which can now be tested by experiments on electron scattering. A beam of electrons, with energies between 1000 and 4000 volts, was directed into helium at a low pressure and the distribution of energies of electrons scattered at 34.2° measured. It was found that the experimental results gave a profile for the Compton modified band in excellent agreement with the profiles calculated by Hicks and in good agreement with those calculated by Kirkpatrick, Ross and Ritland. Values for the probability of the various component velocities of the atomic electrons are tabulated.