Excited States of the Alpha-Particle

Because of recent improved expressions for forces between nuclear constituents a re-examination of the excited states of the alpha-particle, whose existence was first predicted by Feenberg,[1] seems to be in order. A very direct approach is afforded by a variational method of calculation, such as the one recently described and applied to the ground states of H^3 and He^4 by Warren and the author.[2]

In this paper an approximation to the ground state was obtained with the use of individual particle coordinates. These are, of course, unsuitable in the problem of excited states. It is therefore necessary to carry out the variational calculation in normal coordinates, similar to those employed by Feenberg:

$$\boldsymbol{\rho}_1 = c(\mathbf{r}_1+\mathbf{r}_2-\mathbf{r}_3-\mathbf{r}_4), \quad \boldsymbol{\rho}_2 = c(\mathbf{r}_1-\mathbf{r}_2), \quad \boldsymbol{\rho}_3 = c(\mathbf{r}_3-\mathbf{r}_4), \quad c = 2^{-\frac{1}{2}}. \tag{1}$$

The notation regarding functions used in this note will otherwise be the same as explained in Eqs. (3), (4), (5) of reference 2. The interaction is taken to be formally equal between all particles,

$$U_{ij} = -Ae^{-r^2_{ij}/a^2}[(1-g)P^M{}_{ij}+gP^H{}_{ij}]. \tag{2}$$

For the numerical work, the constants

$$A = 35.5 \text{ Mev}, \quad a = 2.25\times 10^{-13} \text{ cm}, \quad g = 0.194,$$

have been chosen.

I. *Singlet states.* For the singlet states of the alpha-particle the energy operator has the form

$$T-(1-g/2)\Sigma^{(1)}J_{ij}P_{ij}-(1-2g)\Sigma^{(2)}J_{ij}P_{ij}, \tag{3}$$

where T is the kinetic energy operator, $J_{ij} = Ae^{-r^2_{ij}/a^2}$, $\Sigma^{(1)}$ is extended over unlike, and $\Sigma^{(2)}$ over like-particle pairs. The Coulomb energy is neglected throughout this discussion. If expression (3) is transformed by (1) the P_{ij}'s become simple linear substitutions between the ρ's.

To investigate the stability of the $2P$ state I have chosen, after discarding several functions whose effect is small, a linear combination of the following *ortho*-normal set:

$$\psi_1 = \phi_{100}{}^{(1)}\phi_{000}{}^{(2)}\phi_{000}{}^{(3)},$$
$$\psi_2 = (\tfrac{1}{2})^{\frac{1}{2}}\phi_{100}{}^{(1)}[\phi_{200}{}^{(2)}\phi_{000}{}^{(3)}+\phi_{000}{}^{(2)}\phi_{200}{}^{(3)}],$$
$$\psi_3 = (\tfrac{1}{2})^{\frac{1}{2}}\phi_{100}{}^{(1)}[\phi_{110}{}^{(2)}\phi_{000}{}^{(3)}+\phi_{000}{}^{(2)}\phi_{110}{}^{(3)}].$$

One thus obtains, for example,

$$H_{11} = (10+\lambda)\sigma\left(\frac{\hbar^2}{2Ma^2}\right) - (4-2g)A\left(1-\frac{\lambda}{2}\right)\Big/(\sigma+1) \times[\lambda(2-\lambda)]^{\frac{3}{2}}\left(\frac{\sigma}{\sigma+1}\right)^{\frac{3}{2}} - (2-4g)A\left(\frac{\lambda\sigma}{\lambda\sigma+1}\right)^{\frac{3}{2}},$$

where λ and σ are two convenient variation parameters. A contour plot of H_{11} against λ and σ shows that H_{11} cannot be negative with the above choice of A, a, and g. It increases uniformly as both σ and λ grow, having the value zero for $\sigma=\lambda=0$. But the plot exhibits a groove for $\lambda\approx 1$. The next step was to calculate H_{12}, H_{13}, H_{22}, H_{23}, H_{33} and to evaluate them for different values of σ in the groove $\lambda=1$. For each of these values the lowest root of the determinant $|H_{ij}-E\delta_{ij}|$ was determined. This root, considered as a function of σ, showed the same monotone increasing behavior as did H_{11}, although it was lowered by about 35 percent. Higher functions will, to be sure, depress it further, but the effect is expected to be relatively small. This, together with the general behavior of the root, leads to the conclusion that there exists no stable ^1P state in the discrete spectrum of the alpha-particle.

A similar investigation of the $2S$ state using 3 *ortho*-normal functions leads to the conclusion that there may be a singlet $2S$ state near the limit of the discrete spectrum, but this state would certainly lie above -8 Mev and therefore disintegrate into H^3 and a proton, if not even into two deuterons.

II. *Triplet states.* The "spin-free" Hamiltonian for a triplet state has the form $T-(1-g/2)\Sigma^{(1)}J_{ij}P_{ij}-J_{34}P_{34}-(1-2g)J_{12}P_{12}$ if ψ is antisymmetrical in the coordinates of 3 and 4. Let us call this function u. There is then another triplet function $P_{13}P_{24}u$ which, in the absence of spin dependent forces, produces the same variational energy as u. Because of the presence of Heisenberg forces in (2) the two combined give two states:

$$E = H_{uu} \pm g/2(\mathcal{H})_{uv}$$

and the interaction operator turns out to be

$$\mathcal{H} = J_{13}P_{13}+J_{24}P_{24}-J_{23}P_{23}-J_{14}P_{14}.$$

For u I have taken a combination of the following two functions:

$$\psi_1 = \phi_{000}{}^{(1)}\phi_{000}{}^{(2)}\phi_{100}{}^{(3)},$$
$$\psi_2 = \phi_{200}{}^{(1)}\phi_{000}{}^{(2)}\phi_{100}{}^{(3)}.$$

The behavior of H_{11} if plotted against λ and σ is very similar to that of the corresponding $2P$ element for the singlet case. In particular, there is no region in which it is negative. H_{12} lowers it by about 25 percent, but there is no indication of a minimum which might be further depressed by the inclusion of more functions. H_{uv} is in general about twice as large as H_{uu} but the small factor $g/2$ makes it ineffective. To give typical values: For $\lambda=0.8$ ("groove") and $\sigma=1$, $H_{11}=13$ Mev, $H_{22}=54$ Mev, $H_{12}=-12$ Mev, while $H_{uv}=21$ Mev. Thus $H_{uu}=9.8$ Mev and $E=(9.8\pm 2.1)$ Mev, values which lie far in the continuous spectrum and have, of course, no physical meaning.

Energies derived by a variational method are in general too high. But it seems almost certain that the present method will not miss a stable level by more than a few Mev. The conclusion is, therefore, that the alpha-particle possesses no excited P states if the forces are of the type here assumed (Eq. (2)) and if the constants adopted are approximately correct. If the range of the forces were as large as 2.8×10^{-13} cm (with the force constants changed to produce the correct binding energy) the P states would become stable.

The difference between the present results and those of Feenberg[1] is due to the inclusion of exchange forces in the present calculation. Their effect is to *decrease* the stability of the P levels. An explanation for the γ-rays observed by Crane and co-workers,[3] such as that proposed by Bethe and Bacher,[4] appears to be untenable.

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[1] Feenberg, Phys. Rev. **49**, 328 (1936).
[2] Margenau and Warren, Phys. Rev. **52**, 790 (1937).
[3] Crane, Delsasso, Fowler and Lauritsen, Phys. Rev. **48**, 125 (1935).
[4] Bethe and Bacher, Rev. Mod. Phys. **8**, 147 *et seq.* (1936), apparently assumed that exchange forces would render the P states more stable.