term in $[\mathbf{r}'\times\mathbf{P}]\sigma_1$ occurring in Eq. (18.9). In a similar way the term in $[\mathbf{r}'\times\mathbf{P}]\sigma_2$ is canceled. It is thus seen that the transformation of Eq. (18.6) together with the wave equation transformation of Eq. (19.2) transforms the wave equation in the frame K into the wave equation in the frame K'. Since at a large distance the values of $\mathbf{p}'$ used in the transformed Eq. (17.3) and the center of mass Eq. (18) are the same the numbers of collisions taking place into corresponding solid angles are equal. No special consideration of the geometrical factors is necessary here anyway since they are the same for the Born first approximation as for the general case and since they have been found to be satisfactory in a previous section.

Note added in proof: Considerations similar to those made for Eq. (17) have been carried out also for the Majorana and Heisenberg exchange equations.

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On Multiple Scattering of Neutrons

I. Theory of the Albedo of a Plane Boundary

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The paper contains the rigorous solution of the following problem in multiple scattering: a beam of particles impinges with arbitrary velocity distribution upon the plane boundary surface of a body which extends towards infinity on the other side of the boundary. In this body the particles have a finite probability of being either captured or scattered without loss of energy. The probability of scattering shall be spherically symmetrical in the laboratory frame of reference. Number and velocity distribution of the returning particles are given explicitly; density as well as velocity distribution of the particles inside the body are determined by the formulae but not worked out in detail since they lack direct physical interest. The result is found to depend on the ratio of the capture to the scattering cross section and on the velocity distribution of the incident particles. Applying the theory to the diffuse reflection of slow neutrons at paraffin surfaces it is found that agreement with observations and previous determinations of the capture cross section can exclusively be obtained, if the active level of the "deuteron with spin zero" is virtual. The connection of these results with some other experiments on the velocity and magnetic moment of the neutrons is discussed.

Introduction

Neutrons before reaching the point of observation usually have to travel through various layers of different materials in which they undergo collisions depending on the nature of the material penetrated. These collisions can be elastic, inelastic, or capture collisions. Quantum mechanics has supplied us with a large amount of information concerning the *single* processes while the problem of the effect of consecutive collisions on the beam has not yet in our opinion been solved satisfactorily. A large number of authors (Fermi,[1] etc., Yost and Dickinson,[2] Wick,[3] Ornstein,[4] etc.) have treated special cases like the stationary state of neutrons that are losing their energy through collisions in hydrogenated substances, elastic diffusion of neutrons accompanied by capture, albedo, etc. Without attempting to enter into any detailed discussions of these papers we think that the

[1] E. Fermi, Ricerca Scienta. VII–II, 13 (1936).
[2] Yost and Dickinson, Phys. Rev. **50**, 128 (1936).
[3] G. C. Wick, Atti del Acad. Reale dei Lincei, **23**, 775 (1936).
[4] L. S. Ornstein, Kon. Akad. van Wet. te Amsterdam, Proc. XXXIX No. 9 (1936).

difference in treatment and results will become apparent from our own presentation below. The use of the diffusion equation and similar methods will appear unjustified on account of the finite mean free path of the neutrons; it furthermore turns out that the influence of the boundary is of far greater importance than would be anticipated.

In the present paper we treat the problem of neutrons, incident with an arbitrary velocity distribution upon the plane surface of an infinitely extended body which fills the half-space $x>0$. About this body we make the following three assumptions:

1. Collisions between the neutrons and the particles of the body occur without loss of kinetic energy of the neutrons.
2. The scattering of neutrons is spherically symmetrical in the laboratory frame of reference.
3. There exists a finite probability for capture.

These three assumptions are fairly well satisfied in nature by thermal neutrons incident upon a hydrogenated substance and by neutrons of higher velocity passing through matter of sufficiently high atomic number. The question of greatest practical importance is at present the penetration through hydrogenated substances since here experimental data are already present which permit comparison between theory and observation. (Confer Section 4.)

Section 1. The Fundamental Integro-Differential Equation

In this section the theory of the motion of particles suffering multiple collisions in matter will be developed, and adapted to the case of neutrons incident upon a plane boundary. The motions of the particles are described by a general equation of conservation of particles. In order to specify the particle, six coordinates besides the time are required, three to determine its position, and three to determine its velocity. We denote by $w(x, y, z, v_x, v_y, v_z, t)dxdydzdv_xdv_ydv_z$ the probability that at the time t the particle be found in the six-dimensional volume element $d\tau = dxdydzdv_xdv_ydv_z$ located at the point (x, y, z, v_x, v_y, v_z). The general equation of conservation states that the change in the number of particles in the volume element $d\tau$ in a small time dt, given by $(\partial w/\partial t)dtd\tau$ is equal to the difference between the number of particles entering and leaving $d\tau$ due to their motion, plus the difference between the number entering and leaving $d\tau$ due to collisions. In the absence of collisions the equation is simply

$$(\partial w/dt)dtd\tau = -\mathbf{v}\cdot\text{grad}\, wdtd\tau. \qquad (1)$$

The effect of a scattering collision is to leave unchanged the particle's position in space, but to alter its velocity; while the result of a capture collision is to remove the particle from the element $d\tau$. We denote by

$$\phi(v_x, v_y, v_z, v_x'-v_x, v_y'-v_y, v_z'-v_z)dv_xdv_ydv_z$$

the probability (per unit time) that a particle suffer a collision removing it from the point $(x, y, z, v_x', v_y', v_z')$ into the range $dv_xdv_ydv_z$ at (x, y, z, v_x, v_y, v_z). Then the total number of particles entering $dxdydzdv_xdv_ydv_z$ in the time dt due to scattering is given by

$$dtdxdydzdv_xdv_ydv_z\int w(x, y, z, v_x', v_y', v_z', t)$$
$$\phi(v_x, v_y, v_z, v_x'-v_x, v_y'-v_y, v_z'-v_z)dv_x'dv_y'dv_z',$$

while the number leaving is simply

$$dtdxdydzdv_xdv_ydv_zw(x, y, z, v_x, v_y, v_z, t)$$
$$\int\phi(v_x', v_y', v_z', v_x-v_x', v_y-v_y', v_z-v_z')dv_x'dv_y'dv_z'.$$

The integral in this last expression is just the total probability per unit time of any scattering collision, which will be called Γ. In similar manner the probability per unit time of capture will be called Ω, so that the number of particles leaving $d\tau$ due to capture is $dtdxdydzdv_xdv_ydv_z\Omega w(x, y, z, v_x, v_y, v_z)$. Collecting the above expressions we obtain the general conservation equation

$$\partial w/\partial t+\mathbf{v}\cdot\text{grad}\, w+(\Gamma+\Omega)w$$
$$=\int w(\mathbf{r}, \mathbf{v}, t)\phi(\mathbf{v}, \mathbf{v}'-\mathbf{v})dv_x'dv_y'dv_z'. \qquad (2)$$

It is desired to apply Eq. (2) to the case of a beam of neutrons incident upon a plane parallel plate of thickness a and infinite area. The axis of the plate is taken as the x axis; the position of the neutron is specified by the single coordinate x. It will be assumed the particle loses no energy in scattering, so that the velocity is completely determined by the angle the direction of the motion makes with $0-x$. As velocity coordinate, we take the cosine of this angle denoted by ζ. Only the steady state will be considered so that $\partial w/\partial t=0$. The magnitude of the velocity will be called v_0. Then Eq. (2) becomes

$$v_0\zeta\frac{\partial w}{\partial x}+(\Gamma+\Omega)w=\int_{-1}^{1} w(x,\,\zeta')\phi(\zeta,\,\zeta'-\zeta)d\zeta'. \quad (3)$$

In the case of spherically symmetric scattering, ϕ is independent of the scattering angle and is equal to $\Gamma/2$ by normalization. Making this substitution, and setting $A=\Gamma+\Omega/v_0$, $B=\Gamma/2v_0$, we obtain

$$\zeta\partial w/\partial x+Aw=B\int_{-1}^{1} w(x,\,\zeta)d\zeta. \quad (4)$$

The boundary conditions defining the desired solution of this equation will now be investigated. It is first observed that the continuity of w alone is required, since only the first derivative with respect to x appears in the equation. This is to be expected from physical considerations, since the distribution function itself cannot have a step at the edges of the plate, but the rate of change will have, due to the abrupt inset of collisions. Two boundary conditions may now be written down, expressing the fact that outside the plate the velocity direction of the particle remains unchanged, since no collisions can occur. Applied to the incident boundary (defined by $x=0$) this requires that there the distribution of particles having a forward motion be simply the assumed original distribution incident upon the plate; which will be called $f(\zeta)$. Applied to the further boundary (defined by $x=a$) it requires that there the distribution of particles having a backward motion be identically zero, since none can be scattered back after leaving the plate. Summarizing these conditions we obtain:

$$w(0,\,\zeta)=f(\zeta) \quad \text{for} \quad \zeta>0, \quad \text{(a)}$$

$$w(a,\,\zeta)=0 \quad \text{for} \quad \zeta<0. \quad \text{(b)}$$

It is instructive to observe that Eq. (2) is of the type to which the method of Fokker[5] has been applied. In this method the right member is expanded in powers of the *displacement* to which the particle is subject in the assumed time interval dt. This expansion is valid only subject to certain restrictions on the collision function which are not generally satisfied. Therefore an exact solution of Eq. (4) has been obtained, and the result is different from that given by the diffusion theory, which is identical with Fokker's expansion carried to the second order.

Section 2. Solution of the Integro-Differential Equation

In this section the solution of Eq. (4) under the given boundary conditions will be derived. Since the distribution inside the plate cannot be observed, a complete solution is not required; only the values of w at $x=0$ and $x=a$ are physically significant, giving the reflection and transmission coefficients respectively. In this paper only the reflection coefficient, or albedo, will be obtained.

It is first observed that (4) is a linear, inhomogeneous, first-order differential equation in x, whose formal solution is given by the integral equation

$$w(x,\,\zeta)=\eta(\zeta)e^{-Ax/\zeta}+B/\zeta\int_0^x \bar{w}(y)e^{-A(x-y)/\zeta}dy, \quad (5)$$

with $\bar{w}(y)$ defined by

$$\bar{w}(y)\equiv\int_{-1}^{1} w(y,\,\zeta)d\zeta \quad (6)$$

and with $\eta(\zeta)$ an arbitrary function of ζ. Setting $x=0$ in (5) we obtain at once

$$\eta(\zeta)=w(0,\,\zeta) \quad (7)$$

which shows that $\eta(\zeta)$ is the distribution at the incident boundary, and is equal to $f(\zeta)$ for positive ζ. Setting $x=a$ in (5) and using (7) we obtain, for negative ζ:

$$\zeta w(0,\,\zeta)=-B\int_0^a \bar{w}(y)e^{Ay/\zeta}dy, \quad \zeta<0. \quad (8)$$

[5] Cf. Max Planck, Akad. der Wiss., Berlin, Sitzungsberichte, Erster Halbband (1917).

We now introduce in (4) the Laplace adjoint function to w, defined by

$$v(\zeta,\gamma)=\int_0^a w(x,\zeta)e^{-\gamma x}dx \tag{9}$$

and for brevity we set

$$\bar{v}(\gamma)=\int_{-1}^1 v(\zeta,\gamma)d\zeta=\int_0^a \bar{w}(x)e^{-\gamma x}dx. \tag{10}$$

When $v(\zeta,\gamma)$ is known, w may be obtained from the inversion formula:

$$w(x,\zeta)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} v(\zeta,\gamma)e^{\gamma x}d\gamma, \tag{11}$$

which is always valid if there exists a strip $\alpha<\gamma_1<\beta$ in the $\gamma=\gamma_1+i\gamma_2$ plane within which the integral (9) is absolutely convergent, the path of integration being defined by taking σ inside that strip. In our case this strip is the whole γ plane. Now multiplying (4) by $e^{-\gamma x}$ and integrating we obtain

$$\zeta\int_0^a e^{-\gamma x}(\partial w/\partial x)dx+A\int_0^a e^{-\gamma x}wdx = B\int_0^a \bar{w}(x)e^{-\gamma x}dx, \tag{12}$$

and on integrating the first term by parts:

$$\zeta e^{-\gamma a}w(a,\zeta)-\zeta w(0,\zeta)+\gamma\zeta v(\zeta,\gamma) +Av(\zeta,\gamma)=B\bar{v}(\gamma), \tag{13}$$

or

$$v(\zeta,\gamma)=\frac{\zeta w(0,\zeta)-\zeta e^{-\gamma a}w(a,\zeta)}{A+\gamma\zeta}+\frac{B\bar{v}(\gamma)}{A+\gamma\zeta}. \tag{14}$$

Multiplying by $d\zeta$ and integrating:

$$\bar{v}(\gamma)=\int_{-1}^1\frac{\zeta[w(0,\zeta)-e^{-\gamma a}w(a,\zeta)]d\zeta}{A+\gamma\zeta} +B\bar{v}(\gamma)\int_{-1}^1\frac{d\zeta}{A+\gamma\zeta}. \tag{15}$$

The treatment will now be confined to the special case $a=\infty$. The significance of this restriction is seen if we set $e^{-\gamma a}=0$ in Eq. (13) and assign to γ the value $-A/\zeta$. We then obtain

$$w(0,\zeta)=-B/\zeta\bar{v}(-A/\zeta). \tag{16}$$

This relation shows that $w(0,\zeta)$, in which alone we are interested, may be obtained directly from $\bar{v}(\gamma)$ by setting $\gamma=-A/\zeta$, without resorting to the inversion formula (11). The restriction $e^{-\gamma a}=0$ is therefore equivalent to $a(\Gamma+\Omega)/\zeta v_0\gg 1$. For thermal neutrons in paraffin $(\Gamma+\Omega)/v_0=1/\lambda$ is approximately 3.3 cm^{-1}, so that the condition is well satisfied by plates of thickness greater than 1.5 cm.

With this specialization Eq. (15) becomes

$$\bar{v}(\gamma)\left[1-B\int_{-1}^1\frac{d\zeta}{A+\gamma\zeta}\right]=\int_{-1}^1\frac{\zeta w(0,\zeta)}{A+\gamma\zeta}d\zeta. \tag{17}$$

The integral on the right may be split into two; for negative ζ we replace $\zeta w(0,\zeta)$ by the expression given by Eq. (8). Hence:

$$\bar{v}(\gamma)\left[1-B\int_{-1}^1\frac{d\zeta}{A+\gamma\zeta}\right]=\int_0^1\frac{\zeta w(0,\zeta)}{A+\gamma\zeta}d\zeta -B\int_{-1}^0\frac{d\zeta}{A+\gamma\zeta}\int_0^\infty \bar{w}(y)e^{Ay/\zeta}dy \tag{18}$$

or from (10):

$$\bar{v}(\gamma)\left[1-B\int_{-1}^1\frac{d\zeta}{A+\gamma\zeta}\right]=\int_0^1\frac{\zeta w(0,\zeta)}{A+\gamma\zeta}d\zeta -B\int_{-1}^0\bar{v}(-A/\zeta)\frac{d\zeta}{A+\gamma\zeta}. \tag{19}$$

To simplify the form of this equation we introduce a new variable $z=-A/\gamma$, and define $u(z)=-B/z\bar{v}(-A/z)$, so that $u(\zeta)=w(0,\zeta)$ from (16). Finally we let $\sigma=B/A=\Gamma/2(\Gamma+\Omega)$. Then (19) becomes:

$$u(z)\left[1+\sigma z\int_{-1}^1\frac{d\zeta}{\zeta-z}\right]=\sigma\int_0^1\frac{\zeta w(0,\zeta)}{\zeta-z}d\zeta +\sigma\int_{-1}^0\frac{\zeta u(\zeta)}{\zeta-z}d\zeta. \tag{20}$$

This equation contains only one unknown, the function $u(z)$, which is an analytic function of the complex variable z and which is identically the desired velocity distribution $w(0,z)$ on the

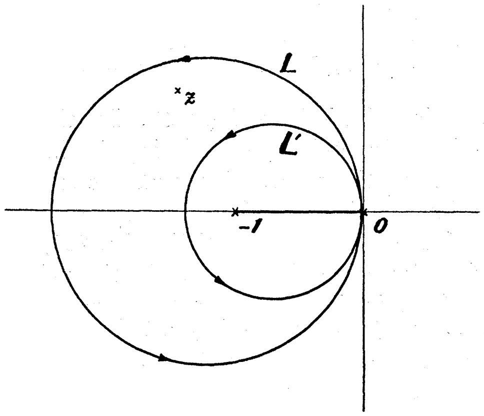

FIG. 1.

segment of the real axis defined by $-1<z<0$. If we consider now the incident velocity distribution given by $f(z)=\delta(z-z_0)$,[6] and introduce the symbol $\rho(z)\equiv 1+\sigma z\int_{-1}^{1} d\zeta/(\zeta-z)$ we may re-write Eq. (20) as

$$\rho(z)u(z)-\frac{\sigma z_0}{z_0-z}-\sigma\int_{-1}^{0}\frac{\zeta u(\zeta)}{\zeta-z}d\zeta=0. \tag{21}$$

In the left half-plane $R(z)<0$ the function $u(z)$ is by definition given by the integral

$$u(z)=-B/z\int_0^\infty \bar{w}(x)e^{Ax/z}dx,$$

from which follows at once:

I. $u(z)$ is regular in the entire left half-plane $R(z)<0$, and vanishes at infinity like $1/z$.

On the other hand we obtain directly from Eq. (21):

II. The difference $\rho(z)u(z)-\sigma z_0/(z_0-z)$ is regular in the right half-plane $R(z)>0$, and (since $\lim_{z=\infty}\rho(z)=1-2\sigma$) also in the right half-plane $u(z)$ vanishes at infinity like $1/z$.

It can now be seen that these regularity conditions I and II are sufficient to define a function $u(z)$ as the solution of the integral equation (21).

To show this we introduce the operator:

$$Tu(z)=\rho(z)u(z)-\sigma\int_{-1}^{0}\frac{\zeta u(\zeta)}{\zeta-z}d\zeta \tag{22}$$

in which $u(z)$ must be regular for $R(z)<0$. In terms of this operator Eq. (21) reads:

$$Tu(z)-\sigma z_0/(z_0-z)=0.$$

The operatur Tu may be expressed by a simple complex integral:

$$Tu=\frac{1}{2\pi i}\int_L\frac{\rho(s)u(s)}{s-z}ds \tag{23}$$

where L is the path of integration shown in Fig. 1. Thus L includes the point $s=z$ and also the segment $-1<s<0$ of the real axis. To identify (22) and (23) we first observe that the residue from $s=z$ gives:

$$u(z)\left[1+\sigma z\int_{-1}^{1}d\zeta/(\zeta-z)\right].$$

Then we may write:

$$Tu(z)=u(z)\left[1+\sigma z\int_{-1}^{1}\frac{d\zeta}{\zeta-z}\right]+\frac{\sigma}{2\pi i}\int_{L'}\frac{su(s)}{s-z}\int_{-1}^{1}\frac{d\zeta}{\zeta-s}ds \tag{24}$$

where L' does not include $s=z$. Then interchanging the order of integration and using partial fractions we obtain:

$$Tu(z)=u(z)\left[1+\sigma z\int_{-1}^{1}\frac{d\zeta}{\zeta-z}\right]+\frac{\sigma}{2\pi i}\int_{-1}^{1}\frac{d\zeta}{\zeta-z}\int_{L'}su(s)\left[\frac{1}{s-z}-\frac{1}{s-\zeta}\right]ds. \tag{25}$$

Since L' does not include $s=z$ the last term reduces to

$$-\sigma\int_{-1}^{1}\frac{d\zeta}{\zeta-z}\frac{1}{2\pi i}\int_{L'}\frac{su(s)}{s-\zeta}ds$$

and the complex integral is 0 for positive ζ, but is $\zeta u(\zeta)$ for negative ζ. Hence the last term gives

$$-\sigma\int_{-1}^{0}\frac{\zeta u(\zeta)}{\zeta-z}d\zeta$$

and (23) is therefore equivalent to (22). (In this discussion it is assumed $u(z)$ possesses no singularities in the left half-plane $R(z)<0$, a condition

[6] The albedo for any desired incident distribution may then be obtained by integrating the result over that distribution.

which we have seen must be fulfilled in order that Eq. (20) be satisfied.)

Using this representation of Tu and setting

$$\frac{\sigma z_0}{z_0-z}=\frac{1}{2\pi i}\int_L \frac{\sigma z_0}{z_0-s}\frac{ds}{s-z}$$

we obtain from Eq. (21) the equivalent equation

$$\frac{1}{2\pi i}\int_L\left[\rho(s)u(s)-\frac{\sigma z_0}{z_0-s}\right]\frac{ds}{s-z}=0. \qquad (26)$$

Since the expression in the bracket vanishes at infinity like $1/s$ the integration path may be deformed into the imaginary axis, giving for the left side of Eq. (21):

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left(\rho u-\frac{\sigma z_0}{z_0-s}\right)\frac{ds}{s-z};\ R(z)<0.$$

Now according to condition II the expression $(\rho u-\sigma z_0/z_0-s)$ is regular and vanishes at infinity like $1/s$ for $R(s)>0$. Consequently we may take instead of the i axis any path parallel to it which runs in the right half-plane, and obtain therefrom the vanishing of the expression

$$Tu-\frac{\sigma z_0}{z_0-z}=\frac{1}{2\pi i}\int_L\left(\rho u-\frac{\sigma z_0}{z_0-s}\right)\frac{ds}{s-z},$$

which is equivalent to Eq. (21).

We can express the conditions I and II in simpler form by introducing $y(z)\equiv(z_0-z)u(z)/\sigma z_0$, obtaining for $y(z)$ the regularity conditions:

$$R(z)<0, \qquad y(z) \text{ regular}, \qquad (a)$$

$$R(z)>0, \quad y(z)\rho(z) \text{ regular}, \qquad (b)$$

$$\rho(z_0)y(z_0)=1, \qquad (c)$$

$$\lim_{z\to\infty} y(z)=\text{constant}. \qquad (d)$$

It is seen that if $\rho(z)$ can be expressed in the form $\rho(z)=p_1(z)p_2(z)$ where p_2 and $1/p_2$ are regular in the right half-plane $R(z)>0$, while p_1 and $1/p_1$ are regular in the left half-plane $R(z)<0$; then the function $y(z)=1/p_1(z)p_2(z_0)$ satisfies all above conditions. Such a separation is in fact possible. We observe $\rho(\infty)=1-2\sigma$ and define for $\sigma<\frac{1}{2}$:

$$\log\frac{p_1(z)}{(1-2\sigma)^{\frac{1}{2}}}=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\log\frac{\rho(s)}{1-2\sigma}\frac{ds}{s-z} \qquad (27)$$

in the left half-plane $R(z)<0$. In the right half-plane $\log p_1(z)/(1-2\sigma)^{\frac{1}{2}}$ is obtained by analytic continuation. For example the continuation may be carried out by proper deformation of the integration path in Eq. (27); since the function $\log \rho(s)$ is regular in the neighborhood of the i axis, with exception of $s=0$ as may be seen from the relation $\rho(it)=1-2\sigma t$ arctan $(1/t)\geq 1-2\sigma>0$. In similar manner we define for $R(z)>0$:

$$\log\frac{p_2(z)}{(1-2\sigma)^{\frac{1}{2}}}=-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\log\frac{\rho(s)}{1-2\sigma}\frac{ds}{s-z} \qquad (28)$$

and obtain the function $\log p_2(z)/(1-2\sigma)^{\frac{1}{2}}$ for $R(z)<0$ by analytic continuation. The deformed paths L_1 and L_2 for the functions $\log p_1(z)/(1-2\sigma)^{\frac{1}{2}}$ and $\log p_2(z)/(1-2\sigma)^{\frac{1}{2}}$ respectively are taken as shown in Fig. 2, including the small circle K with center on the positive imaginary axis. Inside this circle we have

$$\log p_1/(1-2\sigma)^{\frac{1}{2}}+\log p_2/(1-2\sigma)^{\frac{1}{2}}$$

$$=\frac{1}{2\pi i}\int_{L_1}\log\frac{\rho(s)}{1-2\sigma}\frac{ds}{s-z}-\frac{1}{2\pi i}\int_{L_2}\log\frac{\rho(s)}{1-2\sigma}\frac{ds}{s-z}$$

$$=\frac{1}{2\pi i}\int_K\log\frac{\rho(s)}{1-2\sigma}\frac{ds}{s-z}=\log\frac{\rho(z)}{1-2\sigma}.$$

Hence $\log p_1+\log p_2=\log \rho$ inside K, and therefore the identity holds for all values of z. Since $\log p_1$ for $R(z)<0$ and $\log p_2$ for $R(z)>0$ are regular the desired separation $\rho(z)=p_1(z)p_2(z)$ has been obtained.

Introducing the expression for $y(z)$ in $u(z)$ we obtain:

$$u(z)=\frac{\sigma z_0}{z_0-z}\frac{1}{p_1(z)p_2(z_0)}. \qquad (29)$$

The definitions of $p_1(z)$, $p_2(z)$ may be put in more suitable form. Replacing s by it:

$$\log\frac{p_1(z)}{(1-2\sigma)^{\frac{1}{2}}}=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{z+it}{z^2+t^2}\log\frac{\rho(it)}{1-2\sigma}dt. \qquad (30)$$

Since $\rho(it)=1-2\sigma t$ arctan $1/t$ is an even function

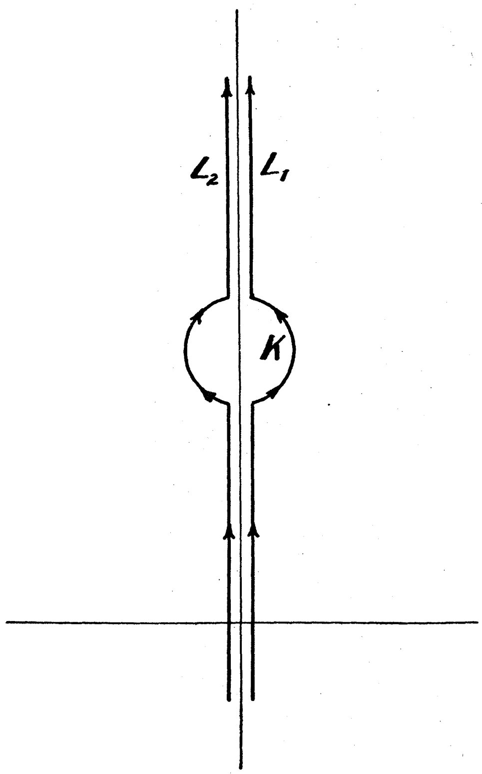

FIG. 2.

there is no contribution from $it \log \rho(it)$ and

$$\log \frac{p_1(z)}{(1-2\sigma)^{\frac{1}{2}}} = -\frac{z}{2\pi}\int_{-\infty}^{\infty} \frac{\log \rho(it)}{z^2+t^2}dt + \frac{z}{2\pi}\log(1-2\sigma)\int_{-\infty}^{\infty}\frac{dt}{z^2+t^2}. \quad (31)$$

Now for $R(z)<0$ we have

$$\int_{-\infty}^{\infty} \frac{dt}{z^2+t^2} = -\frac{\pi}{z}$$

and hence, for $R(z)<0$ and $\sigma \leqq 1/2$:

$$\log p_1(z) = \frac{-z}{2\pi}\int_{-\infty}^{\infty}\frac{\log \rho(it)}{z^2+t^2}dt = z\varphi(z) \quad (32)$$

where $\varphi(z)$ is the integral:

$$\varphi(z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\log(1-2\sigma t \arctan 1/t)}{z^2+t^2}dt. \quad (33)$$

Similarly for $R(z)>0$:

$$\log p_2(z) = -z\varphi(z). \quad (34)$$

Then

$$u(z) = \sigma z_0 e^{z_0\varphi(z_0)-z\varphi(z)}/(z_0-z). \quad (35)$$

Since $u(z)=w(0, z)$ for $z<0$ by definition, the number of particles leaving the surface at an angle defined by z is $u(z)dz$, and the number of these crossing a surface parallel to the plate per unit time is $-zu(z)dz$ since the velocity component normal to the plate is the fraction $-z$ of the total velocity. The total number crossing such a surface per unit time, which is the albedo (for normalized incident intensity), is therefore given by

$$\beta = \int_{-1}^{0} zu(z)dz. \quad (36)$$

SECTION 3. GENERALIZATION FOR VARIOUS VELOCITY DISTRIBUTIONS

In this section there will be derived expressions for the albedo when the distribution of incident particles either is uniform, or obeys a cosine law. As a first step we obtain a relation for

$$\int_{-1}^{0} zu(z)dz$$

from (21), letting $z \to -\infty$:

$$\lim_{-z\to\infty} zu(z) = -\sigma/1-2\sigma\left[\int_{-1}^{0} zu(z)dz + z_0\right]. \quad (37)$$

Now letting $z\to-\infty$ in (33) we obtain $\lim_{-z\to\infty} z\varphi(z) = \log(1-2\sigma)^{\frac{1}{2}}$, and using this in (35):

$$\lim_{-z\to\infty} zu(z) = -\sigma z_0/(1-2\sigma)^{\frac{1}{2}} e^{z_0\varphi(z_0)}. \quad (38)$$

Equating (37) and (38):

$$\int_{-1}^{0} zu(z)dz = -z_0[1-(1-2\sigma)^{\frac{1}{2}}e^{z_0\varphi(z_0)}]. \quad (39)$$

For a general initial distribution $f(z_0)$ this must be averaged, giving the general result:

$$\beta=\frac{\int_0^1[1-(1-2\sigma)^{\frac{1}{2}}e^{s\varphi(s)}]sf(s)ds}{\int_0^1 sf(s)ds}$$

$$=1-(1-2\sigma)^{\frac{1}{2}}\frac{\int_0^1 sf(s)e^{s\varphi(s)}ds}{\int_0^1 sf(s)ds}. \qquad (40)$$

Now β has the form $1-C(1-2\sigma)^{\frac{1}{2}}$, where in general C is of course dependent on σ. It is therefore sufficient in the case of small $(1-2\sigma)$, which corresponds to small capture cross section, to compute the factor C for $\sigma=\frac{1}{2}$, the error in C being of order Ω/Γ (or $1/N$ in Fermi's notation). Now for a uniform incident distribution $f(s)=1$, for a cosine law $f(s)=2s$, so that we require the values of

$$\int_0^1 se^{s\varphi(s)}ds \quad \text{and} \quad \int_0^1 s^2e^{s\varphi(s)}ds \quad \text{for } \sigma=\tfrac{1}{2}.$$

In order to obtain these values we first observe that Eq. (20) possesses the particular solution $u(z)=1/a-z$ in the special case that the incident distribution is given by $w(0,\zeta)=1/a-\zeta$; in which a must satisfy the relation:

$$1+a\sigma\log a-1/(a+1)=0. \qquad (41)$$

This solution has no physical significance, the value of a given by (41) being an eigenvalue of the equation which fits no reasonable boundary conditions. Nevertheless it is of assistance in deriving certain relations concerning $u(z)$. Since Eq. (35) gives the value of $u(z)$ for an arbitrary incident direction, its integral over the incident distribution $1/a-s$ must give $1/a-z$.

$$\frac{1}{a-z}=\sigma e^{-z\varphi(z)}\int_0^1\frac{se^{s\varphi(s)}}{(s-z)(a-s)}ds \qquad (42)$$

using s as integration variable. Replacing z by $-z$ and splitting into partial fractions we obtain:

$$\frac{1}{a+z}=\frac{\sigma e^{z\varphi(z)}}{a+z}\int_0^1\left[\frac{se^{s\varphi(s)}}{s+z}+\frac{se^{s\varphi(s)}}{a-s}\right]ds. \qquad (43)$$

Now inserting in the left member of (39) the value of $u(z)$ given by (35) there results:

$$\sigma z_0e^{z_0\varphi(z_0)}\int_{-1}^0\frac{ze^{-z\varphi(z)}}{z_0-z}dz=-z_0[1-(1-2\sigma)^{\frac{1}{2}}e^{z_0\varphi(z_0)}] \qquad (44)$$

or setting $s=-z$:

$$\sigma e^{z_0\varphi(z_0)}\int_0^1\frac{se^{s\varphi(s)}}{s+z_0}ds=1-(1-2\sigma)^{\frac{1}{2}}e^{z_0\varphi(z_0)}. \qquad (45)$$

This gives an alternative expression for the first term on the right side of (43), the insertion of which yields readily:

$$\int_0^1\frac{se^{s\varphi(s)}}{a-s}ds=\frac{(1-2\sigma)^{\frac{1}{2}}}{\sigma}, \qquad (46)$$

in which σ may be replaced by $\dfrac{1}{a\log(a+1)/(a-1)}$ from (41). This equation gives us immediately the value of

$$\int_0^1 se^{s\varphi(s)}ds$$

for $\sigma=\frac{1}{2}$, when $a\rightarrow\infty$. The result is:

$$\begin{aligned}\int_0^1 se^{s\varphi(s)}ds&=\lim_{a\to\infty}a\int_0^1\frac{se^{s\varphi(s)}}{a-s}ds\\&=\lim_{a\to\infty}a/\sigma(1-2\sigma)^{\frac{1}{2}}\\&=\lim_{a\to\infty}a^2\log\frac{a+1}{a-1}\left[1-\frac{2}{a\log\left(\dfrac{a+1}{a-1}\right)}\right]^{\frac{1}{2}}\\&=\lim_{a\to\infty}a\left[\left(\frac{2}{3a^2}\right)\left(1+\frac{1}{3a^2}\right)\right]^{\frac{1}{2}}\\&=2/\surd 3.\end{aligned} \qquad (47)$$

An expression for

$$\int_0^1 s^2e^{s\varphi(s)}ds$$

will now be derived, subject to the same approximation $\sigma=\frac{1}{2}$. Setting $\sigma=\frac{1}{2}$ in (45),

$$\frac{2}{e^{z\varphi(z)}}=\int_0^1 \frac{se^{s\varphi(s)}}{s+z}ds=\frac{1}{z}\int_0^1 \frac{se^{s\varphi(s)}}{1+s/z}ds. \quad (48)$$

Expansion of the denominator in powers of s/z gives:

$$\frac{2}{e^{z\varphi(z)}}=\frac{1}{z}\left\{\int_0^1 se^{s\varphi(s)}ds-\frac{1}{z}\int_0^1 s^2e^{s\varphi(s)}ds+\cdots\right\} \quad (49)$$

and hence

$$\int_0^1 s^2e^{s\varphi(s)}ds=\lim_{z\to\infty} z\left\{\int_0^1 se^{s\varphi(s)}ds-\frac{2z}{e^{z\varphi(z)}}\right\}. \quad (50)$$

It is apparent from Eq. (49) that the terms in the bracket become equal as $z\to\infty$, and hence the limit as written is indeterminate. We therefore write the factor z as $1/z$ in the denominator, and differentiate numerator and denominator independently to obtain the limit:

$$\int_0^1 s^2e^{s\varphi(s)}ds=\lim_{z\to\infty}\left[\frac{d}{dz}\left(\frac{-2z}{e^{z\varphi(z)}}\right)\right]\Big/\left[\frac{d}{dz}\left(\frac{1}{z}\right)\right]$$
$$=\lim_{z\to\infty} 2z^2\frac{d}{dz}\left[\frac{z}{e^{z\varphi(z)}}\right]. \quad (51)$$

Now it is convenient to modify the form of Eq. (33). Changing the integration variable in Eq. (33) from t to zt we obtain:

$$z\varphi(z)=-\frac{1}{\pi}\int_0^\infty \frac{\log(1-2\sigma tz \operatorname{arccot} zt)}{1+t^2}dt$$
$$=-\frac{1}{\pi}\int_0^\infty \frac{\log 3t^2z^2(1-zt\operatorname{arccot} zt)}{1+t^2}dt \quad (52)$$
$$+\frac{1}{\pi}\int_0^\infty \frac{\log 3t^2z^2}{1+t^2}dt.$$

For brevity we call the first term $-g(z)$, the last is equal to

$$\frac{\log 3z^2}{\pi}\int_0^\infty \frac{dt}{1+t^2}+\frac{1}{\pi}\int_0^\infty \frac{\log t^2}{1+t^2}dt.$$

But we have:

$$\int_0^\infty \frac{\log t^2}{1+t^2}dt=\int_0^1 \frac{\log t^2}{1+t^2}dt+\int_1^\infty \frac{\log t^2}{1+t^2}dt$$
$$=-\int_\infty^1 \frac{\log 1/t^2}{1+1/t^2}\frac{dt}{t^2}+\int_1^\infty \frac{\log t^2}{1+t^2}dt$$
$$=-\int_1^\infty \frac{\log t^2}{1+t^2}dt+\int_1^\infty \frac{\log t^2}{1+t^2}dt=0. \quad (53)$$

Thus we obtain, noting $\int_0^\infty \frac{dt}{1+t^2}=\frac{\pi}{2}$:

$$z\varphi(z)=\log(\sqrt{3})z-g(z) \quad (54)$$

and

$$\frac{z}{e^{z\varphi(z)}}=\frac{1}{\sqrt{3}}e^{g(z)} \quad (55)$$

with

$$\frac{d}{dz}\left(\frac{z}{e^{z\varphi(z)}}\right)=\frac{1}{\sqrt{3}}g'(z)e^{g(z)}. \quad (56)$$

It is now observed that

$$\lim_{z\to\infty} 3t^2z^2[1-zt\operatorname{arccot} zt]=1 \quad (57)$$

and therefore $\lim_{z\to\infty} g(z)=0$ (58)

To find $g'(z)$ we write:

$$g(z)=\frac{1}{\pi}\int_0^\infty \frac{h(zt)}{1+t^2}dt \quad (59)$$

where $h(s)=3s^2(1-s\operatorname{arccot} s)$. Let $zt=\tau$:

$$g(z)=\frac{1}{\pi}\int_0^\infty \frac{h(\tau)}{1+\tau^2/z^2}\frac{d\tau}{z}=\frac{z}{\pi}\int_0^\infty \frac{h(\tau)d\tau}{z^2+\tau^2} \quad (60)$$

and hence

$$g'(z)=\frac{1}{\pi}\int_0^\infty \frac{h(\tau)}{z^2+\tau^2}d\tau-2\frac{z^2}{\pi}\int_0^\infty \frac{h(\tau)}{(z^2+\tau^2)^2}d\tau \quad (61)$$

Then

$$-\lim_{z\to\infty} z^2g'(z)=\frac{1}{\pi}\int_0^\infty h(\tau)d\tau \quad (62)$$

and

$$\int_0^1 s^2e^{s\varphi(s)}ds=\lim_{z\to\infty} 2z^2\frac{d}{dz}\left(\frac{z}{e^{z\varphi(z)}}\right)$$
$$=-\frac{2}{\pi\sqrt{3}}\int_0^\infty \log 3t^2(1-t\operatorname{arccot} t)dt. \quad (63)$$

The definite integral may be put in a numerically more tractable form:

$$\int_0^\infty \log 3t^2(1-t\,\mathrm{arccot}\,t)dt$$

$$=\int_0^{\pi/2}\frac{\log 3\cot^2 t(1-t\cot t)}{\sin^2 t}dt$$

$$=\int_0^{\pi/2}\frac{\log\cos^2 t}{\sin^2 t}dt+\int_0^{\pi/2}\frac{\log\dfrac{3(1-t\cot t)}{\sin^2 t}}{\sin^2 t}dt$$

$$=[\tan t\log\sin^2 t-2t]_0^{\pi/2}+I, \tag{64}$$

where $$I=\int_0^{\pi/2}\frac{\log\dfrac{3(1-t\cot t)}{\sin^2 t}}{\sin^2 t}dt$$

$$=-\pi+\frac{\log(1-\cos^2\pi/2)}{\cos\pi/2}+I=-\pi+I.$$

By numerical integration $I=0.90$, and hence

$$\int_0^1 s^2e^{s\varphi(s)}ds=\frac{2}{\sqrt{3}}\left[1-\frac{0.90}{\pi}\right]=0.826. \tag{65}$$

Section 4. Interpretation of the Result; Comparison with Observation

With the aid of the formulas derived above it is now possible to write explicit expressions for the albedo in the cases in which the incident distribution is normal, uniform, or cosine law.

Normal incidence: $f(\zeta)=\delta(1-\zeta)$

$$\beta=1-(1-2\sigma)^{\frac{1}{2}}e^{\varphi(1)}=1-2.91(1-2\sigma)^{\frac{1}{2}}$$
$$=1-2.91[1/(N+1)^{\frac{1}{2}}]. \tag{66a}$$

Uniform distribution: $f(\zeta)=1$

$$\beta=1-(1-2\sigma)^{\frac{1}{2}}(4/\sqrt{3})=1-2.31(1-2\sigma)^{\frac{1}{2}}$$
$$=1-2.31[1/(N+1)^{\frac{1}{2}}]. \tag{66b}$$

Cosine law distribution: $f(\zeta)=2\zeta$

$$\beta=1-(1-2\sigma)^{\frac{1}{2}}(0.826/\tfrac{1}{3})=1-2.48(1-2\sigma)^{\frac{1}{2}}$$
$$=1-2.48[1/(N+1)^{\frac{1}{2}}]. \tag{66c}$$

The result obtained can be made physically plausible by considering separately the case of $\sigma=\frac{1}{2}$ which gives the albedo 1. In the absence of capture processes an incident particle if followed in its path through the material can only either escape to infinity or after successive collisions arrive again at the surface and escape towards negative values of x. Consider a particle after its first collision in the distance, say, x_0 from the boundary; then the probability to travel to "infinity" before it again hits the surface $x=0$ is vanishingly small. All incident particles therefore again leave the material. On the other hand for a finite probability of capture the problem reduces to the determination of the probability that a particle will be captured *before* again hitting the surface. This probability is a finite number which increases with increasing capture cross section and therefore leads to values for the albedo smaller than 1.

Amaldi and Fermi[7] have attempted to approach the geometrical conditions underlying our problem and to determine experimentally the albedo for the case of thermal neutrons in paraffin. From his theoretical discussions of the case Fermi obtained for the albedo the expression: $\beta=1-2(1/N)^{\frac{1}{2}}$, which leads to the value 124 for N if Amaldi and Fermi's result $\beta=0.82$ is employed. On the other hand, if Amaldi and Fermi's experimental value for the albedo is taken and our numerical coefficients are used, we obtain for N the following values respectively

(a) Normal incidence: 261
(b) Uniform distribution: 164
(c) Cosine law distribution: 189

From our data given above it becomes apparent that the ratio of capture to scattering cross section is always considerably smaller than that assumed by Fermi, but can be determined by this method with sufficient accuracy only if we have better information about the velocity distribution of the incoming particles. Considerations based on the diffusion equation lead Fermi to the distribution law $f(\theta)=\cos\theta+\sqrt{3}\cos^2\theta$ which would give for N a value slightly below 200.

It is now of importance to compare the possible values obtained for N from our theory with the second theoretical determination for N which follows from the comparison of the calcu-

[7] E. Amaldi and E. Fermi, Phys. Rev. 50, 899 (1936).

lated cross sections of scattering and capture due to interaction of proton and neutron. If it is assumed that the capture of a proton by a neutron and the accompanying emission of gamma-radiation is due to the magnetic dipoles of the proton and neutron, the theory[8] leads to the following expression for N

$$N=\frac{Cv}{(\mu_P-\mu_N)^2(\epsilon^{\frac{1}{2}}\mp\epsilon'^{\frac{1}{2}})^2}, \qquad (67)$$

in which v stands for the velocity of the neutrons; μ_P and μ_N denote respectively the algebraic value of the magnetic moment of the proton and the neutron in nuclear Bohr magnetons; ϵ and ϵ' stand for the absolute value of the energy level of the compound nucleus proton plus neutron, the first referring to the normal deuteron nucleus with spin 1 the second to the deuteron "with spin 0"; C is a numerical constant depending on atomic constants and nuclear energy levels, the exact definition of which is without interest for us at this moment. The upper or lower sign is valid, depending upon whether the "deuteron with spin 0" represents a stable or an unstable state.

Numerical evaluation of (67) leads to the two values 188 and 76 for N depending upon whether the "deuteron with spin 0" has a real or virtual level. It is furthermore assumed that the neutrons have thermal energy and that the values for μ_P and μ_N equal 2.9 and -2 nuclear Bohr magnetons, respectively.

Before comparing these values with the one obtained from the albedo measurements it must be remembered that (67) holds true for the interaction of free protons and neutrons whereas the observations have been carried out with bound protons. The scattering cross section in the latter case is approximately three times that of free protons.[9] We therefore obtain finally for comparison the two values 565 and 230 for N.

It is obvious by inspection that the value of 124 obtained by using Fermi's numerical factor differs almost 100 percent from the least value possible. Agreement can be obtained with our theory by assuming a velocity distribution of the incident neutrons which lies between the cosine law and perpendicular incidence and *only* for the case of a *virtual level* of the "deuteron of spin 0." The existence of a real level seems to be definitely excluded.

As satisfactory as this agreement might appear, there are still certain difficulties involved in the interpretation which do not permit us to say that a complete proof of the capture theory of neutrons has been established. Recent evidence[10] makes it quite possible that the value for the proton moment has to be reduced to 2.5 nuclear Bohr magnetons. A corresponding reduction in the absolute value of the neutron moment to 1.65 nuclear Bohr magnetons so as to maintain the relation that proton moment minus neutron moment equals deuteron moment, increases the value of N according to (67) by the factor 4/3.

Furthermore recent experiments by M. D. Whitaker[11] on the scattering of neutrons by paramagnetic substances[12] have raised doubt whether the neutron velocity is really down to thermal values and whether the neutron has a magnetic moment as large as the additivity rule mentioned above would postulate. In both cases the value of N would increase and difficulties would arise for the presently accepted theory of deuteron formation.

The calculation presented above seems to indicate that the transmission coefficient for a finite thickness of the scattering body would become to a certain extent independent of the velocity distribution of the incident neutrons. It therefore appears possible to obtain an exact solution to the problem for plates of finite thickness. The authors are working on this question and on the extension of the theory to include inelastic scattering processes.

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