Approximately Relativistic Equations

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The stationary states of the system of extranuclear electrons in an atom can be treated without making much explicit use of electrodynamics and, as is well known, the present lack of understanding of the electromagnetic field is relatively unimportant in practical calculations of spectroscopic energy terms. It suffices, in such applications, to deal with wave equations which are correct only to the order v^2/c^2 times the term value, where v stands for the velocity of the electron. The difficulties involved in forming a completely satisfactory quantum electrodynamics or any quantum field theory appear at the moment to be so formidable that one may expect their solution along lines rather different from those attempted so far. It has been a lucky circumstance for the development of atomic theory that the v^0/c^0 approximations sufficed for the gross treatment of energy levels while the v^2/c^2 approximation apparently gives a satisfactory account of their fine structure. The development of the theory of nuclear physics has paralleled that of atomic physics inasmuch as the gross structure of nuclear levels has been of primary interest. This has been done with an apparent sacrifice of even approximate agreement with relativity through the introduction of potentials varying in an arbitrary way with the distance. The present paper is a continuation of a previous attempt to improve this state of affairs. While in atomic theories, Maxwell's equations can be used as a guide in the setting up of a wave theory, no field concept of comparable certainty is as yet available for nuclear interactions. Fortunately, however, it turns out that the requirement of relativistic invariance to the order v^2/c^2 together with the known symmetries of the electromagnetic field are sufficient to determine the v^2/c^2 approximations to the wave equations in the electronic case. Even though the retardation of electromagnetic potentials is involved in the problem, its complete wave mechanical understanding can thus be partly replaced by requirements of invariance to order v^2/c^2. In the present as well as in the previous paper the possibilities of making analogous extensions are investigated for arbitrary interactions.

In the previous work the possibilities in classical relativistic dynamics have been the starting point. The classical approximately relativistic equations have then turned out to be invariant in the sense that for each particle the motion, as obtained by means of the equations in one system, is the transform of the motion as obtained by the same equations in another system. Using the picture of wave packets subjected to small accelerations, some necessary conditions for wave equations have been also derived and the corresponding forms of spin orbit interactions have been obtained.

In the present paper approximately relativistic equations are investigated using two different methods of approach. One of them consists in using the first approximation of Born's method for the description of the collision process. It is then possible to devise matrix elements that give an invariant description of the collision process not only in the v^2/c^2 approximation but in all orders of v/c. These matrix elements are not applicable, however, to ordinary physical systems in higher approximations of Born's method. The second method of approach consists in working only with the v^2/c^2 approximation. It is then found possible to have relatively simple equations that give an invariant description of the collision process *exactly*, i.e., independently of Born's first approximation. The equations correct to order v^2/c^2 are set up for Wigner, Majorana, Heisenberg and Wheeler forces using wave functions with two components per particle. It is also found possible to have equations for exchange forces with four components per particle. An equation of this type has been used by Share and the writer in a calculation of the relativistic effects in the deuteron and shows that spin-spin interactions of relativistic origin can be appreciable and should be expected to have a range of force different from the nonspin dependent part. The interpretation of wave equations is discussed. It is shown how, even in approximately relativistic discussions, these equations should be considered only as approximations to equations with four components per particle. The limitations of the classical spin model arising from this cause are pointed out and the magnetic interaction energy of the deuteron is discussed from this point of view. It is concluded that most of the existing estimates are not sound, since the assignment of a magnetic moment to an elementary particle has to be defined in a physical way, and since the spin current, as obtained from Dirac's equation, is the particle current rather than the electric current.

INTRODUCTION

THE nature of interactions between nuclear particles is understood at present rather poorly. It is frequently supposed that it has its origin in the electron-neutrino field which can be postulated in order to explain the phenomena of β-ray disintegration. This hypothesis is very attractive since it promises to relate in an intimate manner the observations on mass defects with those on β-decay. Another argument in its favor is the adaptability of a field theory to the requirements of relativistic invariance.

In this respect the structure of the field interaction theories is much superior to the attempts to introduce interaction forces with finite ranges between the heavy nuclear particles, for it is known that interactions with finite range are not relativistically invariant. This decided preference for field theories practically disappears, however, when it is considered that the forces obtained so far from the field point of view become infinite as the distance between the particles is decreased and that the energy becomes infinite as well. It is, therefore, customary to cut off all integrations at some suitably chosen wave-length.

As is well known, the cut-off wave-length essentially determines the range of the interaction between heavy particles. The cut-off procedure is arbitrary and destroys the relativistic invariance besides. One is, therefore, no better off with the field theory in this respect than one is in postulating an interaction with finite range to begin with. The well-known difficulty of obtaining the empirical values of the interactions satisfying simultaneously the requirements of range, of approximate symmetry, of neutron-proton interactions and of β-decay make this approach still more questionable. Since it is clear that the proper point of view for field theories has not yet been found and since the agreement of theory and experiment for β-decay is at present unsatisfactory, it appears to be worth while to investigate in more detail the consequences of forces introduced more directly from empirical material. The usual equations obviously give a description that is not invariant under Lorentz transformations and it is presumably impossible to make them completely invariant by any simple modification without introducing some sort of field. This, however, was also the case in the development of the older atomic theory in which the electrostatic interaction between charged particles was at first used exclusively. The electromagnetic field has not been introduced since in a satisfactory manner into the theory. Nevertheless it is possible to have self-consistent discussions of all effects up to the order v^2/c^2 in the velocity v of the atomic electrons. It thus appears reasonable to attempt the same type of relativistic generalization in the theory of nuclear particles even though one knows to begin with that a completely relativistic theory must involve a field. In atomic theory a treatment by means of a Hamiltonian referring only to electrons is possible only if one is not interested in effects that involve the radiation reaction such as is responsible for the natural breadth of lines. The relatively large value of the v^2/c^2 effects on the energy in comparison with the width of levels due to radiation damping is thus essential for the applicability of the ordinary field-less theory. The analogous condition in nuclear theories is that the width of nuclear levels due to β-disintegration be small in comparison with their displacements due to the v^2/c^2 effects. This condition is apparently very well satisfied in most cases since the v^2/c^2 effects amount to several 100 kv while the β-ray effects range from seconds to years. It is not possible, however, to tell *a priori* that heavy particles can be treated consistently without explicit reference to a field even in the order v^2/c^2. For a too high density of heavy particles it is probable that no such treatment will apply since then the interference between the fields of different particles may be of primary importance. It is not known, however, that this is the case and there is as yet no very strong reason for abandoning the simpler though admittedly provisional point of view of individual interactions between pairs of particles.

In a previous paper possible forms of A.R.E. (approximate relativistic equations) have been discussed.[1] The first treatment was incomplete in several respects. For exchange forces only the spin orbit part of the interaction was considered and the possibility of having similar equations with exchange forces was left open. It will be shown below that it is possible to form theories with exchange forces as well as with the more general velocity dependent interactions introduced by Wheeler. It also turned out possible to make generalizations for the treatment of collisions that can be expected to be restricted to the first order of the Born approximation but on the other hand accurate for any velocity. The consideration of the collision between two particles forms a convenient starting point for both types of generalization.

[1] G. Breit, Phys. Rev. **51**, 248 (1937); Phys. Rev. **51**, 778 (1937).

Collision Between Two Particles

Let the two colliding particles be 1 and 2. Their masses will be supposed equal. Their collision will be treated using the first approximation of Born's method. The initial momenta will be designated by $\mathbf{p}_1^0$, $\mathbf{p}_2^0$, the final momenta by $\mathbf{p}_1$, $\mathbf{p}_2$. Consider first the more familiar case of electromagnetic interaction between charged particles. The interaction energy is then[2] in the order v^2/c^2

$$H=(e^2/r)[1-(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)/2-(\boldsymbol{\alpha}_1\mathbf{r})(\boldsymbol{\alpha}_2\mathbf{r})/2r^2], \quad (1)$$

where e is the charge, $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ is the displacement vector from particle 2 to particle 1, $r=|\mathbf{r}|$ and $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ are respectively Dirac's matrices that represent by their expectation values the quantity $-\mathbf{v}/c$ for the two particles. On performing the calculation one finds that the number of collisions per second, per cubic centimeter, per unit density of both kinds of incident particles that take place so as to have the y and z components of 1 lie in $dp_1^y dp_1^z$ is

$$N=\frac{4\pi^2}{h^4c^2}\frac{E_1E_2|\mathfrak{M}|^2}{p_1^xE_2-p_2^xE_1}dp_1^y dp_1^z, \quad (2)$$

where E_1, E_2 are the energies and $\mathfrak{M}$ is essentially the matrix element of the interaction energy divided by the δ function that expresses conservation of momentum. In a fundamental volume, we have

$$(\mathbf{p}_1^0\mathbf{p}_2^0 s_1^0 s_2^0|H|\mathbf{p}_1\mathbf{p}_2 s_1 s_2)=\mathfrak{M}\delta/V, \quad (3)$$

where $\delta=1$ if $\mathbf{p}_1^0+\mathbf{p}_2^0=\mathbf{p}_1+\mathbf{p}_2$ and $\delta=0$ otherwise. Here the initial and final states are specified by assigning the values of the momenta as well as of the spin variables s for the two particles. So far the relations are general and do not depend on the special form of H assumed in Eq. (1). Use of this special interaction energy gives

$$\mathfrak{M}=e^2\int[(a_1^{0*}a_1)(a_2^{0*}a_2)-(a_1^{0*}\boldsymbol{\alpha}a_1)(a_2^{0*}\boldsymbol{\alpha}a_2)/2$$
$$-(a_1^{0*}\boldsymbol{\alpha}\mathbf{r}a_1)(a_2^{0*}\boldsymbol{\alpha}\mathbf{r}a_2)/2r^2]r^{-1}\exp(i\mathbf{k}\mathbf{r})d\mathbf{r} \quad (4)$$

$$\mathbf{k}=(\mathbf{p}_1-\mathbf{p}_1^0)/\hbar, \quad (4.1)$$

where $$(a^*\alpha b)=\sum a_\mu^*\alpha_{\mu\nu}b_\nu, \quad (4.2)$$

the initial states being for each particle of the form

$$\psi_\mu=V^{-\frac{1}{2}}\exp(i\mathbf{p}\mathbf{r}/\hbar), \quad (4.3)$$

where μ is Dirac's spin index. The normalization corresponding to unit density is

$$(a^*a)=(a^{0*}a^0)=1. \quad (4.4)$$

Replacing the Coulomb interaction $1/r$ by e^{-ar}/r and making $a\to 0$, one has

$$\mathfrak{M}=4\pi e^2k^{-2}[(a_1^{0*}a_1)(a_2^{0*}a_2)-(a_1^{0*}\boldsymbol{\alpha}a_1)(a_2^{0*}\boldsymbol{\alpha}a_2)$$
$$+k^{-2}(a_1^{0*}\boldsymbol{\alpha}\mathbf{k}a_1)(a_2^{0*}\boldsymbol{\alpha}\mathbf{k}a_2)], \quad (4.5)$$

which within terms of order v^2/c^2 is the same as[3]

$$\mathfrak{M}=\pi e^2h^2[(a_1^{0*}a_1)(a_2^{0*}a_2)-(a_1^{0*}\boldsymbol{\alpha}a_1)(a_2^{0*}\boldsymbol{\alpha}a_2)]/$$
$$[(\mathbf{p}_1-\mathbf{p}_1^0)^2-(E_1-E_1^0)^2c^{-2}]. \quad (4.6)$$

If the interaction energy H' is to give a relativistic description of the collision process then the number of collisions per cm^3 per sec. that take place in a reference system K must be equal to the corresponding number in the system K'. By the "corresponding number" one must mean here the number that would be observed in K' if the waves that were used in K are viewed in K' because a collision is a four-dimensional event and the space time density of events is invariant. It should not be required, therefore, that the number of the collisions per cm^3 per sec. per unit density of the incident beams which we have denoted by N be unchanged by the transformation because the density of a plane wave changes on passing from K to K'. Let this "corresponding number" of collisions in K' be N'. Then every satisfactory H must be such that

$$N=N'. \quad (5)$$

The number N' can be obtained by applying

[2] G. Breit, Phys. Rev. **34**, 553 (1929).

[3] C. Møller, Zeits. f. Physik **70**, 786 (1931); H. Bethe, Zeits. f. Physik **76**, 293 (1932). Invariant matrix elements of the above type occur first in the papers of Møller and Bethe. They are in agreement with the results obtained from the Heisenberg-Pauli electrodynamics for high velocities in reference 2 to which Eq. (1) in the text is the v^2/c^2 approximation. Møller's extension is more general than the work of reference 2 inasmuch as it deals with particles that need not be identical and shows that the matrix elements have a simple mathematical form in momentum space which is especially convenient in the treatment of collision problems.

Eq. (2) in the system K' and using, therefore, in $\mathfrak{M}$ unit densities for the initial as well as final states to give a number N''. In terms of this number

$$N'=\rho_1^{0\prime}\rho_2^{0\prime}N''/\rho_1^0\rho_2^0, \tag{5.1}$$

where the ρ stand for the densities. Since N''/N is entirely defined by the transformation formulas every satisfactory interaction energy must lead to the same transformation properties of $|\mathfrak{M}|^2$ as those that follow from Eq. (4.6), *viz.* that $|\mathfrak{M}|^2$ be invariant if *all* the amplitudes a are transformed by the formulas of the Lorentz transformation. This can also be seen directly through the fact that $N/|\mathfrak{M}|^2$ transforms as $\rho_1\rho_2$ so that

$$N''=\rho_1'\rho_2'N|\mathfrak{M}''|^2/\rho_1\rho_2|\mathfrak{M}|^2, \tag{5.2}$$

where in $\mathfrak{M}''$ waves of unit density in K' are used. From Eqs. (5.1) and (5.2)

$$N'/N=\rho_1'\rho_2'\rho_1^{0\prime}\rho_2^{0\prime}|\mathfrak{M}''|^2/[\rho_1\rho_2\rho_1^0\rho_2^0|\mathfrak{M}|^2], \tag{5.3}$$

which is $=1$ for (4.6). It is now clear that an invariant description of collision processes will be obtained for any interaction energy in which the matrix element $\mathfrak{M}$ transforms as (4.6); i.e., for all interactions in which $|\mathfrak{M}|^2$ is unchanged when the quantities a_μ, a_μ^* are transformed by Dirac's formulas.

The invariance of $|\mathfrak{M}|^2$ does not by itself necessitate the invariance of $\mathfrak{M}$. It is nevertheless simplest to require the latter as well and it will be seen at the end of the paper that the equations obtained are invariant to order v^2/c^2 quite independently of Born's first approximation. In asking that $\mathfrak{M}$ be invariant there is an essential difference between the use of Møller forms of type (4.6) and v^2/c^2 forms of type (4.4). The high velocity forms (4.6) are essentially restricted in their application to those matrix elements for which conservation of energy holds because it is not possible to have conservation of momentum in all frames without having conservation of energy as well. They do not have, therefore, an immediate usefulness for cases requiring more accuracy than the first approximation of Born's method because in such cases it is necessary to know the whole interaction matrix rather than just the part of it corresponding to a constant energy. If one is satisfied with v^2/c^2 approximations, however, then it is possible to define $\mathfrak{M}$ so that it is invariant also for $E_1+E_2\neq E_1^0+E_2^0$. In this order of approximation the lack of conservation of energy destroys conservation of momentum only partially, the approximate relation being

$$\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1^{0\prime}-\mathbf{p}_2^{0\prime}= -\mathbf{v}(E_1+E_2-E_1^0-E_2^0)/c^2, \tag{5.4}$$

where $\mathbf{v}$ is the velocity with which K' moves with respect to K. It will be seen that $\mathfrak{M}=\mathfrak{M}'$ provided $\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}_1^0+\mathbf{p}_2^0$ and also that $\mathfrak{M}=\mathfrak{M}'$ to order $v^2\mathfrak{M}/c^2$ even if conservation of momentum holds only approximately as in Eq. (5.4). It should be noted that the invariance of the part of $\mathfrak{M}$ referring to elements with approximate conservation of momentum and no conservation of energy does not establish as yet the invariance of the physical predictions of the equations because the intermediate states in two reference systems are not connected by the Lorentz transformation that transforms the systems into each other.

The verification of this type of invariance is carried out just as easily using a generalization[1] of Eq. (1)

$$-H=J-\tfrac{1}{2}(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)J+\tfrac{1}{2}(\boldsymbol{\alpha}_1\mathbf{r})(\boldsymbol{\alpha}_2\mathbf{r})dJ/rdr. \tag{6}$$

Here J is a function only of the distance between the particles.

$$\mathfrak{M}=[(a_1^{0*}a_1)(a_2^{0*}a_2)-(a_1^{0*}\boldsymbol{\alpha}a_1)(a_2^{0*}\boldsymbol{\alpha}a_2) -\tfrac{1}{2}(a_1^{0*}\boldsymbol{\alpha}\mathbf{k}a_1)(a_2^{0*}\boldsymbol{\alpha}\mathbf{k}a_2)d/kdk]I \tag{6.1}$$

with
$$I=4\pi\int_0^\infty (rJ/k)\sin krdr \tag{6.2}$$

and
$$\mathbf{k}=(\mathbf{p}_1-\mathbf{p}_1^0+\mathbf{p}_2^0-\mathbf{p}_2)/2\hbar. \tag{6.3}$$

The symmetrical choice[4] (6.3) is the satisfactory

[4] In the calculations that follow the invariance of $\mathfrak{M}$ is proved by using conservation of momentum but without using conservation of energy. The symmetric choice of Eq. (6.3) gives expressions for $\mathfrak{M}$ that are invariant in this sense. The unsymmetric expression of Eq. (4.1) requires, on the other hand, conservation of energy for the invariance of $\mathfrak{M}$. The two expressions for $\mathbf{k}$ (symmetric and unsymmetric) are, of course, equal in the reference system K. It may appear strange that they lead to different transformation properties of $\mathfrak{M}$. The explanation is that the expressions for $\mathbf{k}$ given by the symmetric and unsymmetric formulas (6.3), (4.1) are equal only on account of con-

one and will be used here. Under a Lorentz transformation the quantity $\mathbf{k}$ changes by an amount $\delta\mathbf{k}=\mathbf{k}'-\mathbf{k}$ such that

$$(\mathbf{k}\delta\mathbf{k})=[(\mathbf{vk})^2-\tfrac{1}{2}(\mathbf{vk})(\mathbf{kv}_1+\mathbf{kv}_2+\mathbf{kv}_1{}^0+\mathbf{kv}_2{}^0)]/(2c^2) \quad (6.4)$$

and gives rise only to second order effects. Here $\mathbf{v}_1$, $\mathbf{v}_2$ stand for the velocities of the particles. Since I changes only as a result of a change in $\mathbf{k}$, it is not necessary to take into account the change in I when one considers the last term in the brackets in the expression for $\mathfrak{M}$. The first two terms inside the brackets in Eq. (6.1) form a Lorentz invariant combination and, therefore, the change in I is the only material one in this connection. Since the change in I is of second order it need not be multiplied by the second term in brackets because the product is of fourth order. The third term in the brackets involves the velocities and changes, therefore, by amounts comparable with its own magnitude. This change can be computed using the identity

$$(E+E^0)(a^{0*}\boldsymbol{\alpha}a)+c(\mathbf{p}+\mathbf{p}^0)(a^{0*}a)+ic(a^{0*}[(\mathbf{p}-\mathbf{p}^0)\times\boldsymbol{\sigma}]a)=0,$$

which follows from Dirac's equations that are satisfied by a, a^{0*}. Here $\boldsymbol{\sigma}$ is Dirac's vector spin matrix. The quantities $\mathbf{p}$ are, of course, c numbers. With sufficient accuracy

$$(a^{0*}\boldsymbol{\alpha}a)=-\frac{\mathbf{p}+\mathbf{p}^0}{2Mc}(a^{0*}a)-\frac{i}{2Mc}(a^{0*}[(\mathbf{p}-\mathbf{p}^0)\times\boldsymbol{\sigma}]a) \quad (6.5)$$

$$\delta(a^{0*}\boldsymbol{\alpha}a)=\mathbf{v}(a^{0*}a)/c.$$

Hence

$$-\tfrac{1}{2}\delta(a_1{}^{0*}\boldsymbol{\alpha}\mathbf{k}a_1)(a_2{}^{0*}\boldsymbol{\alpha}\mathbf{k}a_2)=-(1/2c^2)[(\mathbf{vk})^2-\tfrac{1}{2}(\mathbf{vk})(\mathbf{kv}_1+\mathbf{kv}_2+\mathbf{kv}_1{}^0+\mathbf{kv}_2{}^0)](a_1{}^{0*}a_1)(a_2{}^{0*}a_2). \quad (6.6)$$

Here it was essential to use the relation

$$\mathbf{k}[(\mathbf{p}-\mathbf{p}^0)\times\boldsymbol{\sigma}]=0,$$

which is true only if conservation of momentum is satisfied. If conservation of momentum is satisfied only within the limits of Eq. (5.4) no harm is done because then only a fourth-order term results. On the other hand

$$\delta I=(\mathbf{k}\delta\mathbf{k})dI/kdk. \quad (6.7)$$

Using this together with (6.4) and comparing with (6.6) it is seen that

$$\delta\mathfrak{M}=0, \quad (6.8)$$

which proves the invariance of $\mathfrak{M}$.

Two Component Equations

An exact relativistic theory satisfying all the necessary requirements and in particular that of invariance under reflections can be formulated only using wave functions with at least four components for each particle. In applications it is often convenient to have only two components and the essential requirements for invariance to order v^2/c^2 are seen best in the two component forms. A satisfactory two component theory is obtained by equating these to the two large components of the four component function in the well-known manner. In many applications this two component function, Ψ, can be treated as an ordinary wave function and its square integral (Ψ, Ψ) can then be set equal to unity. Such a procedure is, however, an essentially non-relativistic one and may lead to incorrect results as will be seen further on in this section. Keeping ψ for the four component function the two small components of ψ in terms of the large ones are given by

$$\Phi=-(\boldsymbol{\sigma}\mathbf{p})\Psi/2Mc \quad (7)$$

in sufficient approximation for most v^2/c^2 applications. Here $\mathbf{p}$ is the usual differential operator and $\boldsymbol{\sigma}$ is Pauli's spin matrix. One has then

$$(\psi, \psi)=(\Psi, \Psi)+(\Phi, \Phi), \quad (7.1)$$

which differs from (Ψ, Ψ) by terms of order v^2/c^2. These terms due to (Φ, Φ) must be taken into account in the normalization integrals that enter expressions for matrix elements. The alternative procedure of equating (Ψ, Ψ) to unity would not be correct. Transformations of Ψ to new two component functions $\Psi^{(1)}$ chosen so as to have

servation of momentum, while according to Eq. (5.4) conservation of momentum cannot be valid in K and K' unless conservation of energy is satisfied as well. Therefore, by adding a suitable term in $\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1{}^0-\mathbf{p}_2{}^0$ it is possible to introduce at will in the transformed equation a term in $-\mathbf{v}(E_1+E_2-E_1{}^0-E_2{}^0)/2c^2$ and to cancel by means of it terms requiring the assumption of conservation of energy for the invariance of $\mathfrak{M}$.

$(\Psi^{(1)}, \Psi^{(1)})=1$ are possible and convenient if one wishes to put the wave equation in a form suitable for perturbation calculations. This has been done for example in reference 5. Care must be used, however, *not* to give to $(\Psi^{(1)*}\Psi^{(1)})$ the interpretation of particle density as one is tempted to do by analogy with the Galilean case. Thus a suitable $\Psi^{(1)}$ is

$$\Psi^{(1)}=[1+p^2/(8M^2c^2)]\Psi. \tag{7.2}$$

Attempting for the moment to interpret $(\Psi^{(1)*}\Psi^{(1)})$ as the particle density one can obtain the expression for the particle current density by performing a Lorentz transformation. For a system K' moving with velocity v along the x axis one obtains

$$(\Psi^{(1)*}\Psi^{(1)})'=(1+v^2/2c^2)(\Psi^{(1)*}\Psi^{(1)})-vJ_x/c^2$$

with

$$J_x=(\hbar/2iM)[\Psi_\alpha{}^{(1)*}\partial\Psi_\alpha{}^{(1)}/\partial x-\Psi_\alpha{}^{(1)}\partial\Psi_\alpha{}^{(1)*}/\partial x]$$
$$+(\hbar/4M)(\sigma_{\alpha\beta}{}^z\partial/\partial y-\sigma_{\alpha\beta}{}^y\partial/\partial z)\Psi_\alpha{}^{(1)*}\Psi_\beta{}^{(1)}, \tag{7.3}$$

where summations are performed over indices occurring twice and σ are Pauli's matrices. In the v^2/c^2 approximation the relation between the particle density ρ and the particle current density $\mathbf{J}$ must be $\rho'=(1+v^2/2c^2)\rho-vJ_x/c^2$. Hence the assumption $\rho=(\Psi^{(1)*}\Psi^{(1)})$ forces one to the conclusion that the above J_x is the expression for the particle current density correct to terms of order $v\rho/c$. On the other hand the particle current density expression following from Dirac's theory is

$$J_x=-c(\psi^*\alpha^x\psi)=(\hbar/2iM)(\Psi_\alpha{}^*\partial\Psi_\alpha/\partial x$$
$$-\Psi_\alpha\partial\Psi_\alpha{}^*/\partial x)+(\hbar/2M)\cdot$$
$$\times(\sigma_{\alpha\beta}{}^z\partial/\partial y-\sigma_{\alpha\beta}{}^y\partial/\partial z)\Psi_\alpha{}^*\Psi_\beta. \tag{7.4}$$

The functions Ψ and $\Psi^{(1)}$ agree in the lowest order terms. The two expressions (7.3), (7.4) for J_x, therefore, disagree in the spin terms. It is the latter expression, following from Dirac's theory, that agrees with experiments on the magnetic deflection of hydrogen atoms[6] as well as the measured magnetic moment of the "electron orbit."[7] The expression $\rho=(\Psi^{(1)*}\Psi^{(1)})$ is thus seen to be in contradiction with experience since it requires (7.3) even though

$$\int(\Psi^{(1)*}\Psi^{(1)})dxdydz=1.$$

The situation indicated above is a natural one since according to Eq. (7.1) the charge density contains terms in the spin when it is expressed in terms of Ψ.

The disagreement between the relativistic expressions for current densities in the order v^2/c^2 and the ordinary nonrelativistic spin model has already been known from the point of view of hyperfine structure[8] where it is essential to use Dirac's equation for s terms of single electron spectra in order to obtain results agreeing in sign with those for other terms. This disagreement is of a different character from the one just mentioned. When the wave equation is reduced from the four component to the two component form one obtains a term which has no ordinary classical analogy.[9] This term is responsible for the splitting of the s terms. The terms which look like the ordinary classical spin-spin interactions have the expectation value zero for s terms because they are modified by factors which make the result converge so that they disappear by symmetry. On the other hand Eq. (7.4) can be used to calculate the interaction energy with the nucleus directly as the energy of the nuclear magnetic moment in the magnetic field due to the electronic current. It has been verified by Casimir[10] that the result is identical with the four component calculation of Fermi's[11] and the re-

[5] Reference 1, Eq. (9) for one particle. Eq. (17.1) for two particles. Applied in Eq. (17.6) to Eq. (1) of present paper.

[6] R. Frisch and O. Stern, Zeits. f. Physik **85**, 4 (1933). I. Estermann and O. Stern, Zeits. f. Physik **85**, 17 (1933). I. I. Rabi, J. M. B. Kellogg and J. R. Zacharias, Phys. Rev. **46**, 157 (1934); **46**, 163 (1934); **49**, 421 (1936).

[7] The molecular beam experiments give *approximately* the same values for the magnetic moment of the proton and deuteron as the atomic beams. This indicates that the magnetic field *at* the nucleus is given correctly by Dirac's theory. The atomic beam experiments in the stronger fields show further that the magnetic moments of the electron configuration are given correctly by Dirac's theory.

[8] G. Breit and F. W. Doermann, Phys. Rev. **36**, 1732 (1930). See Eq. (12) and following paragraph.

[9] G. Breit, Phys. Rev. **37**, 51 (1931).

[10] Casimir, Physica **3**, 936 (1936). See also footnote 14 reference 1 for effect of spin-spin coupling of different origin. Detailed calculation by S. Share and G. Breit, Phys. Rev. **52**, 546 (1937).

[11] E. Fermi, Zeits. f. Physik **60**, 320 (1930).

sults were then extended by him to the magnetic interaction of a proton and a neutron. It should be noted, however, that the use of an ordinary spin-spin interaction operator based on classical analogy cannot be expected to be equivalent to the interaction with the spin currents represented by the last terms in Eqs. (7.3), (7.4). It is generally and perhaps incorrectly claimed that the classical spin model has no sense and that one "must" use Dirac's equation. There is no doubt that the classical model of two interacting magnets is inapplicable to the interaction of an electron with a nucleus. This does not mean, however, that it is completely inapplicable to the interaction of a proton with a neutron. For it should be recalled that the magnetic moment in Dirac's theory is not properly thought of as an intrinsic property of the particle but rather as a consequence of the particle current density due to the last term in Eq. (7.4). It has, therefore, no direct meaning to multiply the result of calculations obtained by means of (7.4) by the magnetic moments of the proton and neutron as measured in nuclear magnetons. This procedure is allowable only for the calculation of the magnetic field due to one "nuclear magneton" of the proton but not for the calculation of the remaining field. According to the calculation of Breit and Doermann[8] the classical spin model leads for s terms to $(-\frac{1}{2})$ of the hyperfine structure splitting which is obtained on Dirac's theory. In this calculation both the nucleus and the electron were supposed to be magnets of finite size with an intensity of magnetization distributed uniformly throughout the volume which was assumed to be small for the nucleus as well as the electron. The reversal of sign as compared with Dirac's theory was recently noticed independently by Lamb.[12] The reversal of sign is easily visualized by considering the direction of the magnetic field due to the electron at the proton. On the classical spin model the field is opposite to the magnetic moment of the electron, while on Dirac's theory the field is due to the current of the electron and is therefore parallel to the magnetic moment of the current distribution. The same picture can be applied to the interaction between proton and neutron. The fraction of the total magnetic moment of the proton that exists per unit volume at a point having a displacement $\mathbf{R}_p$ with respect to the center of the proton will be denoted by $D(\mathbf{R}_p)$; $D(\mathbf{R}_n)$ stands for the corresponding quantity for the neutron. The functions D are supposed to be spherically symmetric. The absolute values of the magnetic moments will be written $g_p\mu_0 I_p$, $g_n\mu_0 I_n$ where the g factors are g_p, g_n, the nuclear Bohr magneton is $\mu_0=e\hbar/2Mc$ and the I are the absolute values of the nuclear spins in units $\hbar$. Each I could have been set $=\frac{1}{2}$. The corresponding interaction energy operator is

$$H'=g_p g_n \mu_0^2 \int\int\left[\frac{\mathbf{I}_p\mathbf{I}_n}{|\mathbf{r}+\mathbf{R}_n-\mathbf{R}_p|^3}-\frac{3(\mathbf{I}_p(\mathbf{r}+\mathbf{R}_n-\mathbf{R}_p))(\mathbf{I}_n(\mathbf{r}+\mathbf{R}_n-\mathbf{R}_p))}{|\mathbf{r}+\mathbf{R}_n-\mathbf{R}_p|^5}\right]\times D(\mathbf{R}_n)D(\mathbf{R}_p)d\mathbf{R}_n d\mathbf{R}_p.$$

Here $\mathbf{r}$ is the vector representing the displacement of the neutron with respect to the proton and $\mathbf{I}_n$, $\mathbf{I}_p$ are vector matrices of absolute value I_n, I_p. The expectation value of H' for a strong field state in which there are definite eigenverte $I_n{}^z$, $I_p{}^z$ is for s states

$$\Delta W=-\tfrac{1}{3}I_p{}^z I_n{}^z g_p g_n \mu_0^2 \int D(\mathbf{R}_n)D(\mathbf{R}_p)\times\frac{\mathbf{r}(\mathbf{r}+\mathbf{R}_n+\mathbf{R}_p)}{r|\mathbf{r}+\mathbf{R}_n-\mathbf{R}_p|^3}\frac{d\psi^2(r)}{dr}d\mathbf{r}d\mathbf{R}_n d\mathbf{R}_p,$$

where $\psi(r)$ is the wave function representing relative motion of the proton and neutron. The integration over $\mathbf{R}_n$, $\mathbf{R}_p$ amounts to the calculation of the electrostatic force acting on the charge distribution D_n due to the distribution D_p in the direction from p to n. This force is conveniently denoted by

$$\{1/r^2\}_{pn}$$

and then

$$\Delta W=-\tfrac{1}{3}I_p{}^z I_n{}^z g_p g_n \mu_0^2 \int \{1/r^2\}_{pn}\frac{d\psi^2(r)}{dr}d\mathbf{r}.$$

If D_p has a value only through a small sphere and D_n through a thin spherical shell of radius r_0

[12] W. E. Lamb, to be published in the Physical Review. The writer is grateful to Dr. Lamb for correspondence on this subject.

large compared to the radius of the sphere representing the proton then $\{1/r^2\}_{pn}=1/r^2$ for $r>r_0$ and $\{1/r^2\}_{pn}=0$ for $r<r_0$. Then

$$\Delta W=(4\pi/3)I_p{}^zI_n{}^zg_pg_n\mu_0{}^2\psi^2(r_0).$$

If r_0 is small this is just $(-\frac{1}{2})$ of the corresponding result of Casimir's. The above calculation cannot be applied to all of the interaction since part of the proton's magnetic moment is probably due to the spin current.

The use of the spin current in order to represent the magnetic moment of the neutron is logically also not completely correct since the neutron has no charge. Since the spin current of Eq. (7.4) is thought of more simply by Dirac's equation for the motion of the proton it is perhaps clearer to talk about this part of the interaction energy by means of the current density expression in Dirac's theory. The interaction energy can then be reasonably postulated to be given by the same operator as in hyperfine structure theory with the neutron moment replacing the moment of the nucleus and the proton current replacing that of the electron. Fermi's calculation for an electron in the field of a nucleus[11] can then be repeated without any essential difference. The fact that the ratio of the neutron and proton masses is nearly unity while that of the nucleus to the electron is much larger does not matter because only the distance between the particles enters the interaction energy. The elimination of small components by means of the large ones can be made for the proton as Fermi did it for the electron. The interaction energy due to the proton's charge, assumed to be given by the expectation value of the above operator, can thus be obtained without a new calculation using the results of hyperfine structure. One obtains in this manner the same result as Casimir even though the spin current of the neutron is not used. This is analogous to the comparison between electron proton interactions with four and with two components for the proton which has been made by Lowen.[13] It should be remembered, however, that the correspondence holds only if the neutron's magnetism is extended through regions small compared with the size of the deuteron. If one uses $\mu_p=g_p/2=2.8$; $\mu_n=g_n/2=-2$, Casimir's equation gives $-5.6C$ where C depends on the binding energy of the deuteron, the range of force and fundamental constants. According to the above discussion one can tentatively estimate the interaction energy by using $\mu_p=1$ to interact with $\mu_n=-2$ by Casimir's formula and to give $-2C$. The remaining $\mu_p'=1.8$ interacting as a classical magnet then gives $(-\frac{1}{2})(-2\times1.8)C=1.8C$. This leaves a net effect of $\Delta W=-0.2C$ or about 1/30 of Casimir's. For $\mu_p=3$, $\mu_n=-2$ the first order perturbation would vanish. These estimates cannot be expected to be reliable since the classical spin model is probably incorrect, since the magnetism is in all probability distributed in some sense through space, and since according to Lamb,[12] field theories lead one to expect lack of constancy of magnetic moments.

A composite system can be described by a linear combination of products of four component functions. For each particle it is sufficient, however, to specify only the two larger components since the small components are determined then by Eq. (7). It is, therefore, sufficient to have equations in the $2n$ component function which will also be denoted by Ψ for n particles. In making calculations with such equations it must be remembered that the normalizing condition is

$$(\psi,\psi)=(\Psi^{(1)*}\Psi^{(1)})=1$$

for each state. This means that the probability of a state n can be taken as $|c_n|^2$ in the expansion $\psi=\sum c_n\psi_n/(\psi_n,\psi_n)^{\frac{1}{2}}$. When one works with Ψ it is more convenient to deal with coefficients C_n defined by the expansion $\Psi=\sum C_n\Psi_n/(\Psi_n,\Psi_n)^{\frac{1}{2}}$. Since the 2^n large components of ψ are equal to the components of Ψ and similarly for components of ψ_n, Ψ_n it follows that

$$c_n(\Psi_n,\Psi_n)^{\frac{1}{2}}=C_n(\psi_n,\psi_n)^{\frac{1}{2}}.$$

From a wave equation in Ψ there follows a set of equations on the C_n and hence also on the c_n. The latter are of the form $\hbar dc_n/idt+\sum\mathfrak{M}_{nm}c_m\delta=0$. Since the c_n are coefficients of 4^n component functions the $\mathfrak{M}$ must have the transformation property

$$\rho_1'\rho_2'\rho_1{}^{0'}\rho_2{}^{0'}|\mathfrak{M}''|^2=\rho_1\rho_2\rho_1{}^0\rho_2{}^0|\mathfrak{M}|^2 \quad (7.5)$$

[13] I. S. Lowen, Phys. Rev. **51**, 190 (1937).

as follows from Eq. (5.3). The connection between c_n and C_n gives

$$\mathfrak{M}_{nm}\delta=\left[\frac{(\psi_n,\psi_n)}{(\Psi_n,\Psi_n)}\right]^{\frac{1}{2}}\frac{(\Psi_n,H\Psi_m)}{[(\Psi_n,\Psi_n)(\Psi_m,\Psi_m)]^{\frac{1}{2}}}\times[(\Psi_m,\Psi_m)/(\psi_m,\psi_m)]^{\frac{1}{2}}. \quad (7.6)$$

These matrix elements are not Hermitean if H is Hermitean. It is, therefore, necessary to use non-Hermitean H in agreement with the fact that four component forms lead on reduction to non-Hermitean two component equations. [Cf. Eqs. (9), (17.1) reference 1.] The first and last factor in $\mathfrak{M}_{nm}$ give together for states of constant momenta

$$1+(-p_1^2-p_2^2+p_1^{02}+p_2^{02})/(8M^2c^2).$$

On transformation this factor changes by

$$\mathbf{v}(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1^0-\mathbf{p}_2^0)/(4M^2c^2)=0,$$

which vanishes by the conservation of momentum. It is, for this reason, sufficient to consider the transformation of the remaining part of $\mathfrak{M}$. In Eq. (7.5) $\mathfrak{M}'$ is the matrix element for normalized wave functions. It is obtained by taking the value of $\mathfrak{M}_{nm}$ as given by Eq. (7.6) in the primed reference system. One has as a necessary condition for invariance

$$[\rho_1'\rho_2'\rho_1^{0\prime}\rho_2^{0\prime}R_1R_2R_1^0R_2^0]^{\frac{1}{2}}(\Psi_n,H\Psi_m)'/\delta'=[\rho_1\rho_2\rho_1^0\rho_2^0R_1'R_2'R_1^{0\prime}R_2^{0\prime}]^{\frac{1}{2}}(\Psi_n,H\Psi_m)/\delta, \quad (7.7)$$

where $R_i=(\Psi_i,\Psi_i)$, $R_i^0=(\Psi_i^0,\Psi_i^0)$

and the δ expresses conservation of momentum. The state m is referred to by 0 and $\Psi_m=\Psi_1^0\Psi_2^0$. Similarly for Ψ_n. The relations (7.6), (7.7) are independent of the normalization of Ψ_1, Ψ_2, Ψ_1', Ψ_2' and they can be used by letting Ψ_1' be that wave function which corresponds to viewing in K' the state represented by Ψ_1 in K. The ρ_1, ρ_1', have the same relationship. Since $\rho_1/\rho_1'=(\psi_1,\psi_1)/(\psi_1',\psi_1')$ the factor $[\ ]^{\frac{1}{2}}$ consists of ratios like

$$[(\psi_1,\psi_1)/(\Psi_1,\Psi_1)]^{\frac{1}{2}}=1+p_1^2/(8M^2c^2).$$

The condition for invariance is thus that

$$\mathfrak{M}=[1+(p_1^2+p_2^2+p_1^{02}+p_2^{02})/(8M^2c^2)]\times(\Psi_1^0\Psi_2^0,H\Psi_1\Psi_2)/\delta=\text{Invariant} \quad (7.8)$$

be invariant. By means of this result it is possible to test a given equation for invariance. We apply the test first to the case of ordinary interactions already dealt with in a previous paper.[1]

Ordinary Interactions

Let the interaction energy be given by the operator

$$H'=J-\frac{\mathbf{p}_1\mathbf{p}_2J+J\mathbf{p}_1\mathbf{p}_2}{4M^2c^2}+\frac{p_1^ap_2^bx^ax^bf+fx^ax^bp_1^ap_2^b}{4M^2c^2}-\frac{\hbar f}{4M^2c^2}\{[\mathbf{r}\times(2\mathbf{p}_2-\mathbf{p}_1)]\boldsymbol{\sigma}_1-[\mathbf{r}\times(2\mathbf{p}_1-\mathbf{p}_2)]\boldsymbol{\sigma}_2\} \quad (8)$$

$$f=dJ/rdr, \quad \mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$$

in the coordinate space of the two particles. The corresponding operator in momentum space can be obtained by the substitution

$$\Psi_i=A_i\exp(i\mathbf{p}_i\mathbf{r}/\hbar) \quad (i=1,2), \quad (8.1)$$

where in the last equation $\mathbf{p}_1$, $\mathbf{p}_2$ are c number vectors equal to the momenta of the two particles. The quantities A_i are two row one column matrices corresponding to the fact that for each particle there are two components of the wave function. The quantities A_i transform according to the formulas

$$A'=\left\{1+\frac{v^2}{8c^2}-\frac{\mathbf{vp}}{4Mc^2}-\frac{i\mathbf{v}}{4Mc^2}[\mathbf{p}\times\boldsymbol{\sigma}]\right\}A,$$
$$A'\dagger=A\dagger\left\{1+\frac{v^2}{8c^2}-\frac{\mathbf{vp}}{4Mc^2}+\frac{i\mathbf{v}}{4Mc^2}[\mathbf{p}\times\boldsymbol{\sigma}]\right\} \quad (8.2)$$

for each particle. Hence also

$$(A^0,A)'=\left(A^0,\left\{1+\frac{v^2}{4c^2}-\frac{\mathbf{v}(\mathbf{p}+\mathbf{p}^0)}{4Mc^2}+\frac{i\mathbf{v}}{4Mc^2}[(\mathbf{p}^0-\mathbf{p})\times\boldsymbol{\sigma}]\right\}A\right). \quad (8.3)$$

Discarding constant factors of no interest we have to test for invariance

$$\mathfrak{M}=\int(A_1{}^0A_2{}^0,\Big\{J-\frac{\mathbf{p}_1{}^0\mathbf{p}_2{}^0+\mathbf{p}_1\mathbf{p}_2}{4M^2c^2}J$$
$$+\frac{(\mathbf{p}_1{}^0\mathbf{r})(\mathbf{p}_2{}^0\mathbf{r})+(\mathbf{p}_1\mathbf{r})(\mathbf{p}_2\mathbf{r})}{4M^2c^2}f$$
$$-\frac{hf}{4M^2c^2}[\mathbf{r}\times(2\mathbf{p}_2-\mathbf{p}_1)]\boldsymbol{\sigma}_1$$
$$+\frac{hf}{4M^2c^2}[\mathbf{r}\times(2\mathbf{p}_1-\mathbf{p}_2)]\boldsymbol{\sigma}_2$$
$$+\frac{J}{8M^2c^2}(p_1{}^2+p_2{}^2+p_1{}^{02}+p_2{}^{02})\Big\}A_1A_2\Big)$$
$$\times\exp(i\mathbf{kr})d\mathbf{r} \quad (8.4)$$

with $\mathbf{k}=(\mathbf{p}_1-\mathbf{p}_1{}^0+\mathbf{p}_2{}^0-\mathbf{p}_2)/2h.$

In this and the following equations the $\mathbf{p}_i$ are c number vectors so that the order of factors in the curly brackets is immaterial. The last term in the above formula is due to the four component normalization which introduced the corresponding factor in square brackets in Eq. (7.8). The above form of $\mathfrak{M}$ can be transformed by partial integrations which suppose sufficient regularity of J. The result is

$$\mathfrak{M}=\Big(A_1{}^0A_2{}^0,\Big\{1-\frac{\mathbf{p}_1{}^0\mathbf{p}_2{}^0+\mathbf{p}_1\mathbf{p}_2}{2M^2c^2}$$
$$-\frac{(\mathbf{p}_1{}^0\mathbf{k})(\mathbf{p}_2{}^0\mathbf{k})+(\mathbf{p}_1\mathbf{k})(\mathbf{p}_2\mathbf{k})}{4M^2c^2}\frac{d}{kdk}$$
$$+\frac{p_1{}^2+p_2{}^2+p_1{}^{02}+p_2{}^{02}}{8M^2c^2}+\frac{hi\boldsymbol{\sigma}_1}{4M^2c^2}[\mathbf{k}\times(2\mathbf{p}_2-\mathbf{p}_1)]$$
$$-\frac{hi\boldsymbol{\sigma}_2}{4M^2c^2}[\mathbf{k}\times(2\mathbf{p}_1-\mathbf{p}_2)]\Big\}A_1A_2\Big)I, \quad (8.5)$$

where I is given by Eq. (6.2). Calculating $\delta\mathfrak{M}$ all terms are found to cancel each other so that

$$\delta\mathfrak{M}=0 \quad (8.6)$$

and the test is satisfied. For simplicity only one type of relativistic equation for a Wigner interaction was considered above. The proof is quite similar for the other types.[1] No new calculation is required if it is noticed that the other types are obtainable from the one tried by the addition of relativistic invariants. It may be noted that in the discussion of (8) just as in that of (4) the invariance followed without making use of conservation of energy.

Exchange and Velocity Dependent Interactions

The above way of testing equations for invariance is more suitable for the discussion of the invariance of exchange interactions than the method of the variational integrals previously used. It appears at first sight that the very idea of an exchange interaction is at variance with the postulate of relativity. A proton and a neutron exchanging places at a certain time t in the reference system K can be viewed in the system K'. Since the section $t=$const. contains in it world points with continuously variable values of t' there are some values of t' for which the system appears to consist of two protons and some other values for which it appears to consist of two neutrons. Such a state of affairs would, however, be in contradiction with the principle of conservation of charge to which there is no known exception, and a theory with this consequence is not acceptable. The situation is nevertheless not much worse than with ordinary forces acting at a distance because for these also no invariance is obtained for finite ranges. If one only requires approximate invariance to order v^2/c^2 the lack of simultaneity in K' of observations made simultaneously in K is of the order $vr/c^2=\Delta t$ where r is the distance between the two heavy particles. This should be compared with the reciprocal of the frequency characteristic of exchange. This frequency has the order of magnitude $\nu=(\psi(12),\ J\psi(21))/h$ where $\psi(12)$ is the wave function representing the two particles. If the wave packets representing the two particles do not overlap, the frequency of exchange is zero and no difficulty with conservation of charge can arise since the proton and neutron do not change places. Wave packets of the two particles that overlap partially give rise to an exchange occurring with a small frequency. As long as $\nu\Delta t\ll 1$ the lack of simultaneity introduces no absurdity in the description of the behavior of the wave packets because the time at which the exchange occurs cannot be specified to better than approximately $1/\nu$. The upper limit of ν

is of the order J/h. In this case

$$\nu\Delta t = Jvr/hc^2 \sim (60mc^2/h)(ve^2/mc^4)$$
$$= 60(e^2/hc)(v/c) \sim 1/137$$

for $v/c \sim 1/10$. The viewpoint of exchange forces leads thus to a slight but not very serious inconsistency even under these conditions. The time during which lack of conservation of charge could be observed classically is still only a small fraction of the inaccuracy that is present in our knowledge of the time at which the proton-neutron exchange occurs. The difficulty with conservation of charge that has been just discussed is, therefore, not in the way of considering formulations which are partially invariant. On the other hand extensions to higher velocities appear questionable.

By a procedure exactly similar to that used in discussing Eq. (8) one verifies that

$$H' = -JP^M + Q^M \tag{9}$$

satisfies the invariance test (7.8) where P^M is the Majorana exchange operator and Q^M is given by

$$4M^2c^2Q^M = -b(p_1{}^i p_2{}^j x^i x^j f P^M + f P^M x^i x^j p_1{}^i p_2{}^j)$$
$$-\tfrac{1}{2}(1-b)(p_1{}^i p_1{}^i x^i x^j f P^M + f P^M x^i x^j p_1{}^i p_1{}^i$$
$$+p_2{}^i p_2{}^j x^i x^j f P^M + f P^M x^i x^j p_2{}^i p_2{}^j) + a(\mathbf{p}_1 \mathbf{p}_2 J P^M$$
$$+J P^M \mathbf{p}_1 \mathbf{p}_2) + (1-a)(\mathbf{p}_1 J P^M \mathbf{p}_2 + \mathbf{p}_2 J P^M \mathbf{p}_1)$$
$$+i[\mathbf{p}_1 \times J P^M \mathbf{p}_1][a'\boldsymbol{\sigma}_1 - (1-a'')\boldsymbol{\sigma}_2]$$
$$+i[\mathbf{p}_2 \times J P^M \mathbf{p}_2][a''\boldsymbol{\sigma}_2 - (1-a')\boldsymbol{\sigma}_1]. \tag{9.1}$$

The invariance of $\mathfrak{M}$ can be seen to be independent of the restriction of conservation of energy by using

$$\mathbf{k}^M = (\mathbf{p}_2{}^0 - \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1{}^0)/2\hbar. \tag{9.2}$$

The matrix elements in momentum space take then the form

$$\mathbf{p}_1\mathbf{p}_2 J P^M + J P^M \mathbf{p}_1\mathbf{p}_2 \rightarrow (\mathbf{p}_1{}^0\mathbf{p}_2{}^0 + \mathbf{p}_1\mathbf{p}_2)$$
$$\times (A_1{}^0 A_2{}^0, A_1 A_2) \int J \exp(i\mathbf{k}^M \mathbf{r}) d\mathbf{r} \tag{9.3}$$

and similarly for the other quantities occurring in Eq. (9.1). The quantities a, a', a'', b are arbitrary constants. Their values are immaterial for the transformation properties of Q^M. For the Heisenberg interaction one similarly finds that

$$H' = -JP^H + Q^H \tag{10}$$

satisfies the invariance test. Here P^H is the Heisenberg exchange operator and Q^H is given by

$$4M^2c^2Q^H = -b(p_1{}^i p_2{}^j x^i x^j f P^H + P^H f x^i x^j p_1{}^i p_2{}^j)$$
$$-\tfrac{1}{2}(1-b)(p_1{}^i p_1{}^i x^i x^j f P^H + P^H f x^i x^j p_1{}^i p_1{}^i$$
$$+p_2{}^i p_2{}^j x^i x^j f P^H + P^H f x^i x^j p_2{}^i p_2{}^j) + a(\mathbf{p}_1 J P^H \mathbf{p}_2$$
$$+\mathbf{p}_2 J P^H \mathbf{p}_1) + (1-a)(\mathbf{p}_1\mathbf{p}_2 J P^H + P^H J \mathbf{p}_1\mathbf{p}_2)$$
$$-\hbar[a'\mathbf{p}_1 + (1-a')\mathbf{p}_2][\boldsymbol{\nabla}_1 J \times \boldsymbol{\sigma}_1] P^H$$
$$-\hbar[a''\mathbf{p}_2 + (1-a'')\mathbf{p}_1][\boldsymbol{\nabla}_2 J \times \boldsymbol{\sigma}_2] P^H. \tag{10.1}$$

The last two terms in the Eq. (9.1), (10.1) represent the spin orbit interactions for exchange forces. They agree with the results previously obtained.[14] The other terms required by invariance represent what may be called orbit-orbit interactions even though this classical language is perhaps misleading when applied to exchange forces. The presence of such terms is more difficult to infer by the method previously used. As for ordinary interactions one can obtain one possible form from another by the addition of invariants.

For two particles the Majorana interaction $-JP^M$ is equivalent to the ordinary interaction $-J$ for states with even orbital angular momentum and it is equivalent to an ordinary interaction $+J$ for states with odd orbital angular momentum. The orbit-orbit interactions do not change the orbital angular momentum of the two particles. It is thus natural that these interactions have essentially the same form for exchange and for non-exchange interactions. The spin-orbit interactions on the other hand are known to produce a mixing effect between different values of L which in atomic spectra gives rise in extreme cases to jj coupling. For them no similarity of as close a character need be expected, therefore, as for the Majorana interaction. For the Heisenberg interaction it will be noticed that the spin orbit interactions are the same as for ordinary forces except for the presence of P^H. For identical particles $P^H = -1$ there is then no essential difference between the transformation properties of the Heisenberg and

[14] Reference 1, Eqs. (15.7), (15.8), (15.9).

the ordinary interaction which is in agreement with the formal similarity of the results obtained for the two cases.

Wheeler[15] introduced for consideration interactions of a more general type than the usual exchange forces which he calls velocity dependent forces. His generalization will be considered here only insofar as it concerns forces without spin dependence. An extension with spin dependence can be made along the same lines. The interaction potential between two particles 1, 2 is defined as a linear operator occurring in the wave equation

$$H_0'\psi(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_3, \cdots)=\int R(\mathbf{r}', \mathbf{r}'')\delta(\mathbf{r}_1'+\mathbf{r}_2'-\mathbf{r}_1''-\mathbf{r}_2'')\psi(\mathbf{r}_1'', \mathbf{r}_2'', \mathbf{r}_3, \cdots)d\mathbf{r}_1''d\mathbf{r}_2'', \quad (11)$$

where

$$\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2 \quad (11.1)$$

and δ is a three-dimensional Dirac delta function. This form is the most general one consistent with the requirement of the conservation of momentum. The isotropic character of space and the absence of preferential axes in the bodies of the elementary particles makes it also necessary to suppose that R depends only on $\mathbf{r}'$, $\mathbf{r}''$ and $(\mathbf{r}'\mathbf{r}'')$ so that R is invariant under rotations. Substitution of Eq. (11) into Eq. (7.8) gives

$$\mathfrak{M}=(A_1^0A_2^0, A_1A_2)\int d\mathbf{r}'d\mathbf{r}''R(\mathbf{r}', \mathbf{r}'')\times[1+(p_1^2+p_2^2+p_1^{02}+p_2^{02})/8M^2c^2]e^{i\alpha} \quad (11.2)$$

with α defined by Eq. (11.5). The application of the transformation formulas (8.2), (8.3) gives

$$\delta\mathfrak{M}=(A_1^0A_2^0, \int d\mathbf{r}'d\mathbf{r}''R(\mathbf{r}', \mathbf{r}'')\times\left\{\frac{v^2}{c^2}-\frac{\mathbf{v}(\mathbf{p}_1+\mathbf{p}_1^0+\mathbf{p}_2+\mathbf{p}_2^0)}{2Mc^2}+\frac{i\mathbf{v}[(\mathbf{p}_1^0-\mathbf{p}_1)\times\boldsymbol{\sigma}_1]+i\mathbf{v}[(\mathbf{p}_2^0-\mathbf{p}_2)\times\boldsymbol{\sigma}_2]}{4Mc^2}+\frac{i}{4\hbar c^2}\Big[(\mathbf{v}\mathbf{r}'')(\mathbf{v}(\mathbf{p}_1-\mathbf{p}_2))-(\mathbf{v}\mathbf{r}')(\mathbf{v}(\mathbf{p}_1^0-\mathbf{p}_2^0)+(\mathbf{v}\mathbf{r}')(p_1^{02}-p_2^{02})/M-(\mathbf{v}\mathbf{r}'')(p_1^2-p_2^2)/M\Big]\right\}e^{i\alpha}\times A_1A_2). \quad (11.3)$$

[15] J. A. Wheeler, Phys. Rev. **50**, 643 (1936).

The problem now is to make $\delta\mathfrak{M}=0$ by choosing the function R properly. This can be accomplished by starting with an arbitrary continuous and differentiable function R_0 and adding to it terms that will compensate after a Lorentz transformation the terms occurring in Eq. (11.3). Let

$$\mathbf{k}=(\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_1^0+\mathbf{p}_2^0)/2\hbar;\qquad \boldsymbol{\kappa}=(\mathbf{p}_2^0-\mathbf{p}_1^0-\mathbf{p}_1+\mathbf{p}_2)/2\hbar. \quad (11.4)$$

Then

$$(\mathbf{p}_1-\mathbf{p}_2)\mathbf{r}''-(\mathbf{p}_1^0-\mathbf{p}_2^0)\mathbf{r}'=\hbar\mathbf{k}(\mathbf{r}''+\mathbf{r}')+\hbar\boldsymbol{\kappa}(\mathbf{r}'-\mathbf{r}'')=2\hbar\alpha. \quad (11.5)$$

Define

$$S=\int e^{i\alpha}R_0(\mathbf{r}', \mathbf{r}'')d\mathbf{r}'d\mathbf{r}''. \quad (11.6)$$

The function S must be expressible in terms of the absolute values of $\mathbf{k}$, $\boldsymbol{\kappa}$ and the scalar product $(\mathbf{k}\boldsymbol{\kappa})$ because R is invariant under rotations. Thus

$$S=S(k, \kappa, \mathbf{k}\boldsymbol{\kappa}). \quad (11.7)$$

Consider a part of $\delta\mathfrak{M}$

$$\delta\mathfrak{M}_1=(A_1^0A_2^0, A_1A_2)\delta I_1, \quad (11.8)$$

$$\delta I_1=\int d\mathbf{r}'d\mathbf{r}''e^{i\alpha}R_0(\mathbf{r}', \mathbf{r}'')[(\mathbf{v}\mathbf{r}'')(\mathbf{v}(\mathbf{p}_1-\mathbf{p}_2))-(\mathbf{v}\mathbf{r}')(\mathbf{v}(\mathbf{p}_1^0-\mathbf{p}_2^0))+(\mathbf{v}\mathbf{r}')(p_1^{02}-p_2^{02})/M-(\mathbf{v}\mathbf{r}'')(p_1^2-p_2^2)/M](i/4\hbar c^2), \quad (11.9)$$

which corresponds to the square brackets in Eq. (11.3). The factors $\mathbf{r}''$, $\mathbf{r}'$ occurring in δI_1 can be expressed[16] in terms of derivatives of S with respect to the components of $\mathbf{k}$, $\boldsymbol{\kappa}$. Calculating δI_1 in this way and making use of conservation of momentum it is found that

$$2c^2\delta I_1=\left[(\mathbf{v}\mathbf{k})-\frac{(\mathbf{k}\mathbf{P})}{2M}\right]\left[\frac{(\mathbf{v}\mathbf{k})\partial S}{k\partial k}+\frac{(\mathbf{v}\boldsymbol{\kappa})\partial S}{\partial(\mathbf{k}\boldsymbol{\kappa})}\right]+\left[(\mathbf{v}\boldsymbol{\kappa})-\frac{(\boldsymbol{\kappa}\mathbf{P})}{2M}\right]\left[\frac{(\mathbf{v}\boldsymbol{\kappa})\partial S}{\kappa\partial\kappa}+(\mathbf{v}\mathbf{k})\frac{\partial S}{\partial(\mathbf{k}\boldsymbol{\kappa})}\right] \quad (12)$$

$$\mathbf{P}=\mathbf{p}_1+\mathbf{p}_1^0+\mathbf{p}_2+\mathbf{p}_2^0.$$

[16] This is a natural thing to do since δI_1 is the change in S due to the transformation. The intermediate step of Eq. (11.9) can be avoided if desired using this fact.

The change $\delta\mathfrak{M}_1$ can be compensated by adding to R_0 a suitably chosen $-Q_1$ which gives rise to a contribution

$$-\delta\mathfrak{M}_{1Q}=-(A_1{}^0A_2{}^0, A_1A_2)\delta I_{1Q}. \quad (12.1)$$

It is sufficient to make

$$\delta I_1=\delta I_{1Q} \quad (12.2)$$

in order that $\delta\mathfrak{M}_1$ be compensated. This can be accomplished by

$$4M^2c^2I_{1Q}=[(\mathbf{p}_1{}^0\mathbf{k})(\mathbf{p}_2{}^0\mathbf{k})+(\mathbf{p}_1\mathbf{k})(\mathbf{p}_2\mathbf{k})]\frac{\partial S}{k\partial k}$$
$$+[(\mathbf{p}_1{}^0\boldsymbol{\kappa})(\mathbf{p}_2{}^0\boldsymbol{\kappa})+(\mathbf{p}_1\boldsymbol{\kappa})(\mathbf{p}_2\boldsymbol{\kappa})]\frac{\partial S}{\kappa\partial\kappa}$$
$$+[(\mathbf{p}_1{}^0\mathbf{k})(\mathbf{p}_2{}^0\boldsymbol{\kappa})+(\mathbf{p}_1{}^0\boldsymbol{\kappa})(\mathbf{p}_2{}^0\mathbf{k})+(\mathbf{p}_1\mathbf{k})(\mathbf{p}_2\boldsymbol{\kappa})$$
$$+(\mathbf{p}_1\boldsymbol{\kappa})(\mathbf{p}_2\mathbf{k})]\frac{\partial S}{\partial(\mathbf{k}\boldsymbol{\kappa})}. \quad (12.3)$$

The substitution of S by means of Eq. (11.6) yields on transformation

$$I_{1Q}=-\frac{1}{16M^2c^2}\int d\mathbf{r}'d\mathbf{r}''(p_2{}^{0a}p_1{}^{0b}+p_1{}^{0a}p_2{}^{0b}$$
$$+p_2{}^ap_1{}^b+p_1{}^ap_2{}^b)e^{i\alpha}\left[\left(\frac{\partial}{\partial x'^b}+\frac{\partial}{\partial x''^b}\right)(x'^a+x''^a)\right.$$
$$\left.+\left(\frac{\partial}{\partial x'^b}-\frac{\partial}{\partial x''^b}\right)(x'^a-x''^a)\right]R. \quad (12.4)$$

Define $$K_0=R_0\delta(\mathbf{r}_1'+\mathbf{r}_2'-\mathbf{r}_1''-\mathbf{r}_2''). \quad (12.5)$$

When one makes a change of K_0 into K_0-K_1 which corresponds to changing R_0 into R_0-Q_1, one has

$$K_1=Q_1\delta(\mathbf{r}_1'+\mathbf{r}_2'-\mathbf{r}_1''-\mathbf{r}_2''). \quad (12.6)$$

Eq. (12.4) suggests the following form for K_1

$$K_1=\frac{\hbar^2}{16M^2c^2}\left\langle\left(\frac{\partial^2}{\partial x_2'{}^a\partial x_1'{}^b}+\frac{\partial^2}{\partial x_1'{}^a\partial x_2'{}^b}+\frac{\partial^2}{\partial x_2''{}^a\partial x_1''{}^b}\right.\right.$$
$$\left.+\frac{\partial^2}{\partial x_1''{}^a\partial x_2''{}^b}\right)\left[\left(\frac{\partial}{\partial x'^b}+\frac{\partial}{\partial x''^b}\right)(x'^a+x''^a)\right.$$
$$\left.\left.+\left(\frac{\partial}{\partial x'^b}-\frac{\partial}{\partial x''^b}\right)(x'^a-x''^a)\right]K_0\right\rangle. \quad (12.7)$$

Here $\langle\ \rangle$ indicate that the differential operators do not extend beyond these brackets. The partial differentiations $\partial/\partial x'$, $\partial/\partial x''$ are understood respectively in the sense of keeping $x_1'+x_2'$ and $x_1''+x_2''$ constant. The partial differentiations $\partial/\partial x_1{}^a$ are understood in the sense of keeping all $x_2{}^a$ constant and all $x_1{}^b$ const. with $b\neq a$. Eq. (12.7) defines an operator in coordinate space which produces I_{1Q} as is seen by partial integration from Eq. (11). It is clear from the form of Eq. (12.7) that the kernel K_1 involves derivatives of the δ function and that Q_1 is singular. It appears probable, although it has not been proved, that it is impossible to make $\delta\mathfrak{M}=0$ with a regular R. In the calculations that led to Eq. (12.7) essential use has been made of the isotropy of space by means of Eq. (11.7). Conservation of energy implies for any collision process that $(\mathbf{k}\boldsymbol{\kappa})=0$. No use of this relation was made. On substitution into Eq. (11) the differentiations $\partial/\partial x_i{}^a$ can be transferred to the wave function by partial integration and are equivalent to the introduction of momentum operators. These commute with the total momentum operator and conservation of momentum is not destroyed by its introduction.

Again let

$$\delta\mathfrak{M}_2=(A_1{}^0A_2{}^0, A_1A_2)\delta I_2, \quad (13)$$

with

$$\delta I_2=\int d\mathbf{r}'d\mathbf{r}''e^{i\alpha}R(\mathbf{r}'\mathbf{r}'')[v^2/c^2-\mathbf{vP}/2Mc^2]. \quad (13.1)$$

This represents the first two terms in Eq. (11.3). Compensation can be effected by introducting an addition $-K_2$ to K which gives rise to a contribution $-\delta\mathfrak{M}_{2Q}=-(A_1{}^0A_2{}^0, A_1A_2)\delta I_{2Q}$ similarly to Eq. (12.1). It will be sufficient to make

$$\delta I_{2Q}=\delta I_2. \quad (13.2)$$

This can be done by making

$$K_2=-\frac{\hbar^2}{2M^2c^2}\left[\frac{\partial^2K_0}{\partial x_1''{}^a\partial x_2''{}^a}+\frac{\partial^2K_0}{\partial x_1'{}^a\partial x_2'{}^a}\right] \quad (13.3)$$

as well as in other ways corresponding to different combinations of derivatives with respect to $\partial x_1'\partial x_2''$, $\partial x_1'\partial x_1''$, etc. These forms are immediately obtained by inspection, no reference to S being necessary. The contributions $-K_1-K_2$ to K_0 correspond to the orbit-orbit interactions. In addition spin orbit interactions are required

by the remaining terms in δI which must be inserted after the comma of the scalar product rather than after the product as in Eq. (13)

$$\delta I_3=\int d\mathbf{r}'d\mathbf{r}''e^{i\alpha R(\mathbf{r}',\mathbf{r}'')}(i\mathbf{v}/4Mc^2)$$
$$\times\{[(\mathbf{p}_1^0-\mathbf{p}_1)\times\boldsymbol{\sigma}_1]+[(\mathbf{p}_2^0-\mathbf{p}_2)\times\boldsymbol{\sigma}_2]\}. \quad (13.4)$$

Two possibilities suggest themselves for terms in Q such as to give, respectively,

$$I_{3Q}^{\mathrm{I}}=\int d\mathbf{r}'d\mathbf{r}''e^{i\alpha R(\mathbf{r}',\mathbf{r}'')}(-i/8M^2c^2)$$
$$\times\{[(\mathbf{p}_2^0+\mathbf{p}_2)\times(\mathbf{p}_1^0-\mathbf{p}_1)]\boldsymbol{\sigma}_1$$
$$+[(\mathbf{p}_1^0+\mathbf{p}_1)\times(\mathbf{p}_2^0-\mathbf{p}_2)]\boldsymbol{\sigma}_2\}, \quad (13.5)$$

$$I_{3Q}^{\mathrm{II}}=\int d\mathbf{r}'d\mathbf{r}''e^{i\alpha R(\mathbf{r}',\mathbf{r}'')}(-i/8M^2c^2)$$
$$\times\{[(\mathbf{p}_1^0+\mathbf{p}_1)\times(\mathbf{p}_1^0-\mathbf{p}_1)]\boldsymbol{\sigma}_1$$
$$+[(\mathbf{p}_2^0+\mathbf{p}_2)\times(\mathbf{p}_2^0-\mathbf{p}_2)]\boldsymbol{\sigma}_2\}. \quad (13.6)$$

Corresponding forms of K_3 are easily found. A simpler form for K_3^I is obtained, however, by using conservation of momentum to substitute $\mathbf{p}_2-\mathbf{p}_2^0$ for $\mathbf{p}_1^0-\mathbf{p}_1$ and $\mathbf{p}_1-\mathbf{p}_1^0$ for $\mathbf{p}_2^0-\mathbf{p}_2$. One has then

$$I_{3Q}^{\mathrm{I}}=(-i/4M^2c^2)\int e^{i\alpha R(\mathbf{r}',\mathbf{r}'')}$$
$$\times\{[\mathbf{p}_1^0\times\mathbf{p}_1]\boldsymbol{\sigma}_2+[\mathbf{p}_2^0\times\mathbf{p}_2]\boldsymbol{\sigma}_1\}d\mathbf{r}'d\mathbf{r}'', \quad (13.7)$$

$$I_{3Q}^{\mathrm{II}}=(i/4M^2c^2)\int e^{i\alpha R(\mathbf{r}',\mathbf{r}'')}$$
$$\times\{[\mathbf{p}_1^0\times\mathbf{p}_1]\boldsymbol{\sigma}_1+[\mathbf{p}_2^0\times\mathbf{p}_2]\boldsymbol{\sigma}_2\}d\mathbf{r}'d\mathbf{r}''. \quad (13.8)$$

In this form Eq. (13.7) corresponds to $a'=a''=0$ in Eq. (9.1) for the Majorana interaction while I_{3Q}^{II} is recognized from Eq. (13.8) to correspond to $a'=a''=1$.

A generalization with two constants a', a'' such as is present in Eq. (9.1) is also possible here. Similarly for ordinary interactions one can write the spin orbit terms of Q as

$$Q_0=\frac{ib}{4M^2c^2}\{[\mathbf{p}_1\times J\mathbf{p}_1]\boldsymbol{\sigma}_1+[\mathbf{p}_2\times J\mathbf{p}_2]\boldsymbol{\sigma}_2\}$$
$$-\frac{i(1-b)}{4M^2c^2}\{[\mathbf{p}_1\times J\mathbf{p}_1]\boldsymbol{\sigma}_2+[\mathbf{p}_2\times J\mathbf{p}_2]\boldsymbol{\sigma}_1\},$$

which is equivalent to Eq. (15.4) of the previous calculation.[1] This is seen to be a linear combination of terms corresponding to Eqs. (13.7), (13.8). The Majorana and ordinary spin orbit interactions are thus seen to be properly special cases of the more general results for Wheeler's forces. The forms of K_3 that correspond to Eqs. (13.7), (13.8) are

$$K_3^{\mathrm{I}}=(-i\hbar^2/4M^2c^2)\langle[\boldsymbol{\nabla}_1'\times\boldsymbol{\nabla}_1'']\boldsymbol{\sigma}_2K_0$$
$$+[\boldsymbol{\nabla}_2'\times\boldsymbol{\nabla}_2'']\boldsymbol{\sigma}_1K_0\rangle, \quad (13.7')$$

$$K_3^{\mathrm{II}}=(i\hbar^2/4M^2c^2)\langle[\boldsymbol{\nabla}_1'\times\boldsymbol{\nabla}_1'']\boldsymbol{\sigma}_1K_0$$
$$+[\boldsymbol{\nabla}_2'\times\boldsymbol{\nabla}_2'']\boldsymbol{\sigma}_2K_0\rangle. \quad (13.8')$$

The whole K is $K=K_0-K_1-K_2-K_3$. (13.9)

It should be noted that the kernel K_0 was supposed to be the product of a regular kernel R_0 multiplied by a δ function. If R_0 is not regular, the above discussion may be inapplicable because then the velocities of the particles may enter explicitly into the interaction energy. The presence of $\delta(\mathbf{r}'-\mathbf{r}'')$ and $\delta(\mathbf{r}'+\mathbf{r}'')$ in R_0 does not spoil the results, as is seen from the fact that satisfactory forms were obtained for ordinary and Majorana forces. The interactions are thus not applicable to the most general velocity dependent potentials but are restricted to either ordinary and Majorana forces or else to regular R. For as long as R is regular, $\mathfrak{M}_0$ is a function only of $\mathbf{k}$ and $\boldsymbol{\kappa}$ and these quantities change on transformation only in the order v^2/c^2. If, however, K_0 should contain a factor,

$$\delta(\mathbf{r}_1'+\mathbf{r}_2'-\mathbf{r}_1''-\mathbf{r}_2'')\delta(\mathbf{r}'-\mathbf{r}'')\partial^2/\partial^2x_1''\partial x_2'',$$

the corresponding $\mathfrak{M}_0$ will contain $p_1^xp_2^x$ which changes on transformation to the order v/c. In this case the above discussion does not apply but then also Eq. (12.5) connecting K_0 and R_0 is not satisfied unless R_0 is made highly singular.

Exchange Equation with Four Components per Particle

It is probable, although uncertain, that Diracian wave equations with four components per particle have more direct significance than equations of the Pauli type. This is suggested by their success for electrons as well as by the fact that two component equations are essentially

nonrelativistic. Inasmuch as experiment[6] shows that the magnetic moment of the proton does not have the value predicted by Dirac's theory one may doubt the four component theories as well. However, β-decay indicates that field phenomena of the general nature of the electron-neutrino field are intimately connected with the behavior of protons and neutrons. It appears likely that this field has to do with the magnetic moment of the proton and that a partial description can be obtained using a four component theory together with a β-ray field. From this point of view it appears useful to know something about possible approximately invariant forms of exchange equations with four components per particles. No attempt is made here at enumerating and classifying all possible forms and only one example will be considered. The equation in question has been already used in a calculation of relativistic effects in the deuteron by S. Share and the writer.[10] The interaction energy is

$$H'=-\{J-\tfrac{1}{2}(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)J+\tfrac{1}{2}(\boldsymbol{\alpha}_1\mathbf{r})(\boldsymbol{\alpha}_2\mathbf{r})f\}P^M$$
$$-\frac{i}{4Mc}\{([(\mathbf{p}_2-\mathbf{p}_1)\times\boldsymbol{\sigma}_1]\mathbf{r})(\boldsymbol{\alpha}_2\mathbf{r})f$$
$$+f(\boldsymbol{\alpha}_1\mathbf{r})(\mathbf{r}[(\mathbf{p}_1-\mathbf{p}_2)\times\boldsymbol{\sigma}_2])+[(\mathbf{p}_2-\mathbf{p}_1)\times\boldsymbol{\sigma}_1]\boldsymbol{\alpha}_2J$$
$$+J\boldsymbol{\alpha}_1[(\mathbf{p}_1-\mathbf{p}_2)\times\boldsymbol{\sigma}_2]\}P^M. \quad (14)$$

Here the meaning of P^M is the customary one of an operator which exchanges the space coordinates of the two particles and it is thought of as the operator $\delta(\mathbf{r}_1'-\mathbf{r}_2'')\delta(\mathbf{r}_2'-\mathbf{r}_1'')$ in the 16 component equation and the $\boldsymbol{\sigma}_i$ are four component Dirac's spin matrices. As in the discussion of Eq. (6) plane waves are introduced together with four component column matrices a_1, a_2. The matrices $\boldsymbol{\alpha}_1$, $\boldsymbol{\sigma}_1$ are supposed to operate on a_1. Thus $\boldsymbol{\alpha}_1$, when operating on $P^M\chi(\mathbf{r}_1)\varphi(\mathbf{r}_2)$ operates on components of χ even though $P^M\chi(\mathbf{r}_1)\varphi(\mathbf{r}_2)=\chi(\mathbf{r}_2)\varphi(\mathbf{r}_1)$. The part

$$[-J+\tfrac{1}{2}(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)J-\tfrac{1}{2}(\boldsymbol{\alpha}_1\mathbf{r})(\boldsymbol{\alpha}_2\mathbf{r})f]P^M \quad (14.1)$$

of H' is formally analogous to Eq. (16) but it does not correspond to an invariant interaction in the sense used here. The part of $\mathfrak{M}$ due to it is

$$\mathfrak{M}_0=[-(a_1^{0*}a_1)(a_2^{0*}a_2)+(a_1^{0*}\boldsymbol{\alpha}a_1)(a_2^{0*}\boldsymbol{\alpha}a_2)$$
$$+\tfrac{1}{2}(a_1^{0*}\boldsymbol{\alpha}_1\mathbf{k}a_1)(a_2^{0*}\boldsymbol{\alpha}\mathbf{k}a_2)d/kdk]I \quad (14.2)$$
$$\mathbf{k}=(\mathbf{p}_2^0-\mathbf{p}_1-\mathbf{p}_1^0+\mathbf{p}_2)/2\hbar.$$

On transformation

$$\delta\mathbf{k}=\mathbf{v}(\mathbf{vk})/2c^2-\mathbf{v}(E_2^0-E_1-E_1^0+E_2)/2c^2\hbar. \quad (14.3)$$

The first two terms in the brackets of Eq. (14.2) form an invariant. The change in $\mathfrak{M}$ comes about therefore, due only to the change in I multiplied by these two terms as well as due to the change in the third term with I and $\mathbf{k}$ treated as constants. The change in I can be neglected when considering the third term because it gives rise to a term of order v^4/c^4. Similarly the second term in the brackets multiplied by δI can be dropped. Thus

$$\delta\mathfrak{M}_0=(a_1^{0*}a_1)(a_2^{0*}a_2)(\mathbf{kv})(E_2^0-E_1-E_1^0+E_2)$$
$$\times(dI/dk)/2kc^2\hbar+(\mathbf{kv})[(a_1^{0*}\boldsymbol{\alpha}_1\mathbf{k}a_1)(a_2^{0*}a_2)$$
$$+(a_1^{0*}a_1)(a_2^{0*}\boldsymbol{\alpha}_2\mathbf{k}a_2)](dI/dk)/2kc. \quad (14.4)$$

Solutions a, a^0 of Dirac's equation for a single particle not subjected to external forces for energies E, E^0 satisfy[17]

$$(E+E^0)(a^{0*}\boldsymbol{\alpha}a)+c(\mathbf{p}^0+\mathbf{p})(a^{0*}a)$$
$$+ic(a^{0*}[(\mathbf{p}-\mathbf{p}^0)\times\boldsymbol{\sigma}]a)=0. \quad (14.5)$$

This can be used to express the terms containing $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ by means of terms in the $\boldsymbol{\sigma}$. It is legitimate, in making the replacement, to approximate $E+E^0$ by $2\ Mc^2$. This substitution simplifies $\delta\mathfrak{M}_0$ and we have

$$\delta\mathfrak{M}_0=-(i(\mathbf{vk})/4Mc^2)(a_1^{0*}a_2^{0*}\{\mathbf{k}[(\mathbf{p}_2-\mathbf{p}_2^0)\times\boldsymbol{\sigma}_2]$$
$$+\mathbf{k}[(\mathbf{p}_1-\mathbf{p}_1^0)\times\boldsymbol{\sigma}_1]a_1a_2)dI/kdk, \quad (14.6)$$

using the fact that $E_2^0-E_1+E_2-E_1^0-(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_1^0+\mathbf{p}_2^0)(\mathbf{p}_2^0-\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1^0)/4M$ in virtue of conservation of momentum. The whole

$$\mathfrak{M}=\mathfrak{M}_0+\mathfrak{M}_1 \quad (14.7)$$

and the remaining part $\mathfrak{M}_1$ due to terms in $i/4\ Mc$ of H' in Eq. (14) is simplified by partial integration to

$$\mathfrak{M}_1=(i/4Mc)(a_1^{0*}a_2^{0*}\{([(\mathbf{p}_2^0-\mathbf{p}_1^0)\times\boldsymbol{\sigma}_1]\mathbf{k})(\boldsymbol{\alpha}_2\mathbf{k})$$
$$+([(\mathbf{p}_2-\mathbf{p}_1)\times\boldsymbol{\sigma}_2]\mathbf{k})(\boldsymbol{\alpha}_1\mathbf{k})\}a_1a_2)dI/kdk. \quad (14.8)$$

Hence

$$\delta\mathfrak{M}_1=(i(\mathbf{vk})/4Mc^2)(a_1^{0*}a_2^{0*}\{([\mathbf{p}_2^0-\mathbf{p}_1^0)\times\boldsymbol{\sigma}_1]\mathbf{k})$$
$$+([(\mathbf{p}_2-\mathbf{p}_1)\times\boldsymbol{\sigma}_2]\mathbf{k})\}a_1a_2)dI/kdk.$$

[17] E. Schroedinger, Berl. Ber. 24, 422 (1930).

On combining the last equation with Eq. (14.6) and using Eq. (14.7) one finds that $\delta\mathfrak{M}=0$, conservation of momentum having been used once more. By means of Eqs. (14.2), (14.8) it may be verified that H' is Hermitean.

Invariant Matrix Elements

Collisions between particles occurring with a high relative velocity can be treated, in the electromagnetic case, by means of Born's method, as has been shown by Møller.[3] This can be done only if the nonrelativistic matrix element is suitably generalized as has been done in coordinate space using formal electrodynamics[2] and in momentum space using less formal arguments by Møller.[3] Although the knowledge of the electromagnetic relations is essential for the understanding of the problem it is possible to test it for invariance quite independently of the knowledge of that field. This was discussed in connection with Eq. (4.6). It is not possible to ascertain the proper generalization of the nonrelativistic Coulombian matrix element by considerations confined to transformation properties combined with a knowledge of the nonrelativistic limit. Thus one could use instead of $(a_1{}^{0*}a_1)(a_2{}^{0*}a_2)$ $-(a_1{}^{0*}\boldsymbol{\alpha}a_1)(a_2{}^{0*}\boldsymbol{\alpha}a_2)$ in the numerator of Møller's form Eq. (4.6) the quantity $(a_1{}^{0*}\beta a_1)(a_2{}^{0*}\beta a_2)$. The transformation properties and the nonrelativistic limit of the new expression for $\mathfrak{M}$ will then be the same as those of Eq. (4.6). It is thus seen that the interactions at high velocities are not determinable from nonrelativistic limits and require a more intimate knowledge of the field responsible for them. Nevertheless it is possible to eliminate those which are not suitable. It is also conceivable that it is not possible to describe the collision between two heavy particles by means of a matrix element. This, for instance, would correspond to a condition in which at high relative velocities two protons are more likely to disintegrate each other into neutrons, electrons and neutrinos than to scatter each other. Experiment shows that at moderate velocities the opposite is the case.[18] Whether at infinite velocities the ratio of elastic to inelastic scattering approaches zero or not is not known. The ultimate usefulness of generalizations such as Eq. (4.6) for nuclear interactions is seen to be doubtful both from the point of view of lack of uniqueness as well as on account of a possible disappearance of elastic scattering to a more rapid degree than such equations imply. The generalizations are helpful on the other hand for the understanding of the mathematical connections of the theory as well as crude estimates of relativistic effects at high velocities.

The extensions considered are restricted to matrix elements for which conservation of energy and conservation of momentum hold between initial and final states. In this respect they differ from the forms invariant to order v^2/c^2. Clearly it is not possible in a completely covariant theory to have conservation of momentum without having also conservation of energy. For forms invariant to order v^2/c^2 the expression $\mathfrak{M}$ was, nevertheless, found to be invariant independently of whether the initial and final states had the same energy. It should be remembered here that $\mathfrak{M}$ is not the matrix element and that it must be multiplied by a δ function of the change in total momentum $\boldsymbol{\pi}$ before it is used in a perturbation calculation. Even though $\boldsymbol{\pi}^0-\boldsymbol{\pi}=0$ and $\boldsymbol{\pi}^{0\prime}-\boldsymbol{\pi}'=0$ do not correspond to the same points in the space of total momentum and energy the quantity $\mathfrak{M}$ can be discussed in either system. It will be remembered that in the proofs of invariance for forms of order v^2/c^2 the conservation of momentum was essential in the discussion of terms of the highest order that was taken into account. In the proofs, as presented, the equation $\boldsymbol{\pi}^0=\boldsymbol{\pi}$ was postulated. The proofs would not be of practical value, however, if it were not for the fact that in the transformed system also $\boldsymbol{\pi}^0=\boldsymbol{\pi}'$. According to Eq. (5.4) this is not an exact relation but its inaccuracy does not matter in the terms of highest order. It is clearly impossible to make use of such a condition for exact relations. It is also clear that Eq. (4.6) corresponds to a non-Hermitean matrix element, the Hermitean character of the corresponding Eq. (4.5) arising through relations expressing the conservation theorem for the charge current density.

A given ordinary interaction energy J in coordinate space gives rise to a matrix element in

[18] M. A. Tuve, N. P. Heydenburg and L. R. Hafstad, Phys. Rev. **50**, 806 (1936).

momentum space of the form $\mathfrak{M}\delta$ with

$$\mathfrak{M}=\int J(r)\exp(i\mathbf{kr})d\mathbf{r}$$

$$=4\pi\int(rJ/k)\sin krdr=\varphi(k^2), \quad (15)$$

where $$\mathbf{k}=(\mathbf{p}_1-\mathbf{p}_1^0)/\hbar, \quad (15.1)$$

$$\epsilon=(E_1-E_1^0)/\hbar c. \quad (15.2)$$

Then $$\mathfrak{M}=\varphi(k^2-\epsilon^2)(a_1^{0*}\beta a_1)(a_2^{0*}\beta a_2) \quad (15.3)$$

is a relativistic generalization of Eq. (15) which automatically agrees with Eq. (15) for low velocities. Similarly

$$\mathfrak{M}=\varphi(k^2-\epsilon^2)[(a_1^{0*}a_1)(a_2^{0*}a_2)-(a_1^{0*}\boldsymbol{\alpha}a_1)(a_2^{0*}\boldsymbol{\alpha}a_2)] \quad (15.4)$$

is such a generalization.

For exchange forces one can write down similar generalizations. Thus for Majorana exchange

$$\mathfrak{M}=\varphi(k_M^2) \quad (16)$$

with $$\mathbf{k}_M=(\mathbf{p}_2^0-\mathbf{p}_1)/\hbar \quad (16.1)$$

is the nonrelativistic form of $\mathfrak{M}$. Here φ is the same function of k_M as in Eq. (15). If one defines

$$\epsilon_M=(E_2^0-E_1)/\hbar c \quad (16.2)$$

the relativistic generalizations of Eq. (16) are

$$\mathfrak{M}=\varphi(k_M^2-\epsilon_M^2)(a_1^{0*}\beta a_1)(a_2^{0*}\beta a_2), \quad (16.3)$$

$$\mathfrak{M}=\varphi(k_M^2-\epsilon_M^2)[(a_1^{0*}a_1)(a_2^{0*}a_2)-(a_1^{0*}\boldsymbol{\alpha}a_1)(a_2^{0*}\boldsymbol{\alpha}a_2)]. \quad (16.4)$$

The above forms of $\mathfrak{M}$ give invariant results for collisions when applied in the first approximation of Born's method. The fact that they are not Hermitean does not matter when they are used that way. They can be generalized so as to give Hermitean matrix elements by using

$$\mathbf{k}=(\mathbf{p}_1-\mathbf{p}_1^0-\mathbf{p}_2+\mathbf{p}_2^0)/2\hbar,$$
$$\epsilon=(E_1-E_1^0-E_2+E_2^0)/2\hbar c$$

in Eqs. (15.1), (15.2) and

$$\mathbf{k}_M=(\mathbf{p}_2^0-\mathbf{p}_1-\mathbf{p}_1^0+\mathbf{p}_2)/2\hbar,$$
$$\epsilon_M=(E_2^0-E_1-E_1^0+E_2)/2\hbar c$$

in Eqs. (16.1), (16.2). Nothing is gained by doing so as long as the $\mathfrak{M}$ are used in calculations with Born's first approximation. For then the symmetrized expressions are identical with those in Eqs. (15.1), (15.2), (16.1), (16.2). For more general calculations one obtains, however, in this way generalizations of $\mathfrak{M}$ which are Hermitean and invariant independently of whether either conservation of energy or that of momentum holds between initial and final states. Such generalizations do not appear to have any physical interest for arbitrary interactions because they do not necessarily imply in general an invariant description of the collision process in higher than the first Born approximation. Born's second approximation brings in sums over intermediate states having an energy different from that of either the initial or the final level.[19] The intermediate states entering the Born formula in K', when observed in K, have total momenta that depend on their energy. They cannot be obtained from the intermediate states used in K by the Lorentz transformation $K\rightarrow K'$ since the latter have the same momentum as observed in K. The possible existence of special interaction energies in which the result is nevertheless invariant is apparently not excluded. A trivial example is offered by $\mathfrak{M}$ matrices diagonal in the energy. In such cases calculation shows that Born's second and first approximations transform alike. However, this has no practical value because Born's second approximation diverges. An exact relativistic description of the collision process along the above lines appears to be difficult and may be impossible.

Invariance of Collision Treatment with v^2/c^2 Approximation

It will now be shown that the v^2/c^2 approximation gives invariant predictions for collisions of

[19] Dirac's method of variation of constants is frequently referred to as Born's method. In the first approximation these methods give the same result. In higher approximations Dirac's method gives the solution corresponding to the interaction energy being $=0$ for $t<0$ and having the value demanded by physical circumstances for $t>0$. The time dependent solution obtained by Dirac's method thus corresponds to a physical condition which varies with the reference system. Time dependent solutions obtained by Dirac's method in two reference systems should not be the transforms of each other. *Stationary* solutions obtained by Born's method, on the other hand, can be the transforms of each other in two systems since the question of simultaneity in two systems is immaterial for them.

two particles not only in the first approximation of Born but exactly. This will be done by using a transformation of Cartesian coordinates, suggested by the Lorentz contraction, which takes into account automatically the relations between measured momenta and energies in different reference systems. The considerations previously published suggest the existence of such a transformation because the equations derived there approach the proper limit if many particles form a heavy conglomerate in the field of which the remaining particle may move. The limit approached is the equation for a single particle subjected to a field. In this limit the theory is very much better than Born's first approximation because the v^2/c^2 correction terms are then exact. On the other hand the collision between two particles of equal mass when treated by successive approximations using Born's method is not obviously invariant in the higher approximations. According to Eq. (5.4) the intermediate states in two different systems of reference are not the transforms of each other the total momenta of the two sets of states being different to order v^2/c^2 as measured in one frame. A discussion by the formal method of Born and the use of Dirac's arrangement of the calculation with time dependent coefficients appear to be involved on account of this circumstance as well as the complexity of formulas for the higher approximations. Instead the wave equation corresponding to Eq. (8) will be discussed directly in two reference systems and the results for the collisions problem will be compared. The wave equation to be satisfied is

$$E\Psi=\Big\{2Mc^2+\frac{p_1^2+p_2^2}{2M}-\frac{p_1^4+p_2^4}{8M^3c^2}-J+\frac{\mathbf{p}_1\mathbf{p}_2J+J\mathbf{p}_1\mathbf{p}_2}{4M^2c^2}-\frac{p_1^ap_2^bx^ax^bf+fx^ax^bp_1^ap_2^b}{4M^2c^2}+\frac{\hbar f}{4M^2c^2}[\mathbf{r}\times(2\mathbf{p}_2-\mathbf{p}_1)]\boldsymbol{\sigma}_1-\frac{\hbar f}{4M^2c^2}[\mathbf{r}\times(2\mathbf{p}_1-\mathbf{p}_2)]\boldsymbol{\sigma}_2\Big\}\Psi. \quad (17)$$

In order to avoid complications with different times of the two particles this equation will be considered here only with E having a definite numerical value. By means of it one can discuss the collision of two particles even though the wave equation does not contain the time. One may in fact look for a solution which consists of a product of a function describing the motion of the center of gravity and of a function describing the relative motion. The latter can be arranged to be asymptotic to a plane wave for large values of the relative distance and its outgoing parts can be used to determine the collision cross section in the usual manner. The question as to whether this is a justifiable procedure does not enter with the point of view taken here; for the point under discussion is whether Eq. (17), when used for the treatment of collisions by the method of stationary states, will give results transforming themselves properly. In the frame of reference K for which Eq. (17) has been written down one may make the transformation

$$\mathbf{R}=\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2);\quad \mathbf{r}=\mathbf{r}_1-\mathbf{r}_2;\quad \mathbf{P}=\mathbf{p}_1+\mathbf{p}_2;\quad \mathbf{p}=\tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_2). \quad (17.1)$$

Since the operator $\mathbf{P}$ commutes with the right side of Eq. (17) it is possible to have solutions for which $\mathbf{P}$ has a definite numerical value having the significance of the total momentum of the system. Solutions of the form

$$\Psi=\Phi(\mathbf{R})\chi(\mathbf{r});\quad \Phi=\exp(i\mathbf{PR}/\hbar) \quad (17.2)$$

are possible and lead on substitution into Eq. (17) to

$$W_r\chi=\Big\{L-\frac{P^2p^2}{8M^3c^2}-\frac{(\mathbf{Pp})^2}{4M^3c^2}+\frac{P^2J}{8M^2c^2}-\frac{f(\mathbf{Pr})^2}{8M^2c^2}+\frac{\hbar f}{8M^2c^2}[\mathbf{r}\times\mathbf{P}](\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\Big\}\chi, \quad (17.3)$$

$$L=\frac{p^2}{M}-\frac{p^4}{4M^3c^2}-\frac{p^2J+Jp^2}{4M^2c^2}+\frac{fx^ax^bp^ap^b+p^ap^bx^ax^bf}{4M^2c^2}-\frac{3\hbar f}{4M^2c^2}[\mathbf{r}\times\mathbf{p}](\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)-J, \quad (17.4)$$

$$W_r=E-2Mc^2-W_R;\qquad W_R=\frac{P^2}{4M}-\frac{P^4}{8(2M)^3c^2}. \quad (17.5)$$

In these equations $\mathbf{P}=(P_x, P_y, P_z)$ is a set of three c numbers while the vector $\mathbf{p}$ is an operator. The quantity W_R+2Mc^2 is the kinetic energy of a single particle of mass $2M$ and momentum $\mathbf{P}$. By means of Eq. (17.3) one can study the collision process in the frame K. The operator on the right side of this equation contains the "relative momentum" $\mathbf{p}$ and the relative displacement $\mathbf{r}$ in L. In addition the momentum of the center of mass is contained in terms which are not included in L. For large values of r the waves for Ψ are asymptotic to a product of two plane waves representing respectively incident states of particles 1 and 2. The momenta in these states will be called $\mathbf{p}_1{}^0$, $\mathbf{p}_2{}^0$ and the energies $E_1{}^0$, $E_2{}^0$. By conservation of energy and momentum

$$E=E_1{}^0+E_2{}^0, \quad \mathbf{P}=\mathbf{p}_1{}^0+\mathbf{p}_2{}^0. \tag{17.6}$$

In a manner similar to the above the wave equation for the same incident waves can be written down in a reference system K' chosen so that $\mathbf{P}'=0$. This is the reference system of the center of mass. The equation for the new function χ' is

$$W_r'\chi'(\mathbf{r}')=L(\mathbf{p}', \mathbf{r}', \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)\chi'(\mathbf{r}') \tag{18}$$

with $$W_r'=E'-2Mc^2, \quad W_R'=0 \tag{18.1}$$

and $$E'=E_1{}^{0\prime}+E_2{}^{0\prime}; \quad \mathbf{p}_1{}^{0\prime}=-\mathbf{p}_2{}^{0\prime}. \tag{18.2}$$

Initial states are referred to as before by the superscript ^{0}. Quite independently of any question as to the time dependence of the wave function the transformation formulas relate $(E_1{}^0, \mathbf{p}_1{}^0)$ with $(E_1{}^{0\prime}, \mathbf{p}_1{}^{0\prime})$, $(E_2{}^0, \mathbf{p}_2{}^0)$ with $(E_2{}^{0\prime}, \mathbf{p}_2{}^{0\prime})$ and hence also $(E, \mathbf{P})$ with $(E', \mathbf{P}')$ because, in regions with sufficiently large r, portions of the wave front can be passed through slits and examined without affecting the main scattering mechanism in the region of small r. Since $\mathbf{P}'=0$ both E and P are determined by E' and the transformation velocity. Through the definitions of W_r, W_R, W_r' as given in Eqs. (17.5), (18.1) the difference W_r-W_r' can be obtained to the order v^2W_r/c^2. Let $\mathbf{v}$ be the velocity of K' as observed in K. Then

$$\mathbf{P}=(\mathbf{v}E'/c^2)(1-v^2/c^2)^{-\frac{1}{2}}=\mathbf{v}E/c^2. \tag{18.3}$$

The use of these relations gives

$$W_r-W_r'=-P^2W_r/8M^2c^2$$
$$=-P^2(p^2/M-J)/8M^2c^2, \tag{18.4}$$

where the last expression is to be used only if W_r-W_r' operates on a wave function. Since W_r-W_r' is of the order v^2W_r/c^2 the difference between p^2/M and p'^2/M can be neglected in Eq. (18.4). On comparing Eq. (18.4) with Eqs. (17.3), (18) one notes that the first and third terms involving $\mathbf{P}$ are directly accounted for by W_r-W_r' and are simply called for by the fact that in K one uses the energy W_r rather than W_r'.

It is now necessary to take into account the fact that the directions and wave-lengths of the plane waves are subject to change on account of the Doppler effect that is caused by the transformation. For large values of $\mathbf{r}$ the wave can be resolved into plane waves by Fourier's theorem. The major term in the analysis refers to the incident waves $\mathbf{p}_1{}^0$, $\mathbf{p}_2{}^0$. However, it is also necessary to pay attention to the other terms for they are essential in the description of scattering. It is thus desirable to introduce a correlation between all possible relative momenta $\mathbf{p}$ in the two reference systems in such a way as to agree with the formulas of the Lorentz transformation for large r. For any state in which the momenta and energy have the values $(\mathbf{p}_1, E_1)$, $(\mathbf{p}_2, E_2)$ in K and $(\mathbf{p}_1', E_1')$, $(\mathbf{p}_2', E_2')$ in K' (relative velocity along x axis),

$$p_{1x}=(p_{1x}'+vE_1'/c^2)(1-v^2/c^2)^{-\frac{1}{2}};$$
$$p_{2x}=(p_{2x}'+vE_2'/c^2)(1-v^2/c^2)^{-\frac{1}{2}};$$
$$p_{1y}=p_{1y}', \quad \text{etc.}$$

Hence

$$p_{1x}-p_{2x}=(1+v^2/2c^2)(p_{1x}'-p_{2x}')$$
$$+v(p_1'^2-p_2'^2)/2Mc^2;$$
$$p_{1y}-p_{2y}=p_{1y}'-p_{2y}'.$$

Only those waves can be leaving the system for which the total momentum is $=0$ in K'. Therefore, the description of the collision process is concerned only with the correct calculation of states for which $\mathbf{p}_1'=-\mathbf{p}_2'$. For such states the last term in the expression for $p_{1x}-p_{2x}$ disappears and

$$\mathbf{p}=\mathbf{p}'+\mathbf{v}(\mathbf{v}\mathbf{p}')/2c^2. \tag{18.5}$$

So far this is an equation between c number vectors. It shows how the relative momenta $\mathbf{p}$

in the solutions of Eq. (17.3) should be associated with the relative momenta $\mathbf{p}'$ in the solutions of Eq. (18). It is complicated to be calculating the solutions and their Fourier analyses for both of these equations. Instead a transformation of coordinates will be made in Eq. (17.3) in such a way as to make Eq. (18.5) follow. Let

$$x=(1-v^2/2c^2)x', \quad y=y', \quad z=z'. \tag{18.6}$$

Then

$$\begin{aligned}&\mathbf{p}=\mathbf{p}'+\mathbf{v}(\mathbf{vp}')/2c^2;\\ &\mathbf{p}=(\hbar/i)(\partial/\partial x, \partial/\partial y, \partial/\partial z);\\ &\mathbf{p}'=(\hbar/i)(\partial/\partial x', \partial/\partial y', \partial/\partial z').\end{aligned} \tag{18.7}$$

Here it will be noted that the quantities $\mathbf{p}$, $\mathbf{p}'$ are operators. For large r, when they operate on the plane waves that exist there, they may be replaced by the coefficients of $i\mathbf{r}/\hbar$, $i\mathbf{r}'/\hbar$ and give thus automatically Eq. (18.6).

The transformation used in Eqs. (18.6), (18.7) is of the same type as has been used in the classical discussions of the electromagnetic field of a moving electron by Abraham and Lorentz. From Eq. (18.6)

$$J(r)=J(r')-\frac{(\mathbf{vr}')^2}{2c^2}\frac{dJ}{r'dr'}. \tag{18.8}$$

The difference $\mathbf{p}-\mathbf{p}'$ affects p^2/M so as to remove the term in Eq. (17.3) and similarly the term in $f(\mathbf{Pr})^2$ is canceled by $-J(r)+J(r')$. Collecting (18.4), (18.5), (18.8), substituting into Eq. (17.3), and neglecting throughout quantities of order v^4J/c^4 one obtains

$$W_r'\chi=\left\{L(\mathbf{p}', \mathbf{r}', \sigma_1, \sigma_2)+\frac{\hbar f}{8M^2c^2}[\mathbf{r}'\times\mathbf{P}](\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\right\}\chi. \tag{18.9}$$

The comparison of this with Eq. (18) shows that the two differ only in a term containing $\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2$ as well as in the fact that Eq. (18.9) is on χ while Eq. (18) is on χ'. For large r the wave function for each particle is transformed by[1]

$$\Psi_i=\left\{1+\frac{v^2}{8c^2}+\frac{\mathbf{vp}_i'}{4Mc^2}+\frac{i\mathbf{v}}{4Mc^2}[\mathbf{p}_i'\times\boldsymbol{\sigma}_i]\right\}\Psi_i'; \quad (i=1, 2). \tag{19}$$

Since the whole wave function is a sum of products of wave functions for the two particles and since for large r each factor must transform by Eq. (19) in order to represent correctly the behavior of current and particle densities for that particle, it is necessary for large r to transform the whole wave function by

$$\Psi=\left\{1+\frac{v^2}{4c^2}+\frac{\mathbf{v}(\mathbf{p}_1'+\mathbf{p}_2')}{4Mc^2}+\frac{i\mathbf{v}}{4Mc^2}[\mathbf{p}_1'\times\boldsymbol{\sigma}_1]+\frac{i\mathbf{v}}{4Mc^2}[\mathbf{p}_2'\times\boldsymbol{\sigma}_2]\right\}\Psi'. \tag{19.1}$$

It does no harm, however, to extend this transformation to the whole of space because one is only interested in the calculation of collisions so that it does not matter what the transformation is as long as it relates correctly the incident and outgoing waves. Of course, one cannot be sure as yet that the function Ψ' defined by Eq. (19.1) is equal everywhere to the function Ψ' as defined by the original wave equation (17) applied in K'. However it will turn out that they satisfy the same wave equation. Since they are equal at infinity they must be equal everywhere. For large r, Ψ' is just the wave function in K'. Therefore in this region $\mathbf{p}_1'+\mathbf{p}_2'=0$ and

$$\Psi=\left\{1+\frac{v^2}{4c^2}+\frac{i\mathbf{v}}{4Mc^2}[\mathbf{p}'\times(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)]\right\}\Psi'. \tag{19.2}$$

Here it should be noted that the only operations are confined to $\mathbf{p}'$, $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ while $\mathbf{v}$ is a c number. *This* is the convenient form for the extension of Eq. (19.1). Substituting Eq. (19.2) into Eq. (18.9), observing that the operator in curly brackets does not affect Φ in the highest order of magnitude, multiplying finally the resultant equation by the reciprocal of the curly brackets, one obtains additional terms on the right side as

$$-J\frac{i\mathbf{v}}{4Mc^2}[\mathbf{p}'\times\boldsymbol{\sigma}_1]+\frac{i\mathbf{v}}{4Mc^2}[\mathbf{p}'\times\boldsymbol{\sigma}_1]J=\frac{\hbar\mathbf{v}f}{4Mc^2}[\mathbf{r}'\times\boldsymbol{\sigma}_1]=-\frac{\hbar f}{8Mc^2}[\mathbf{r}'\times\mathbf{P}]\boldsymbol{\sigma}_1,$$

where $\mathbf{v}=\mathbf{P}/2M$ is a sufficiently close approximation. This term cancels the corresponding

term in $[\mathbf{r}'\times\mathbf{P}]\sigma_1$ occurring in Eq. (18.9). In a similar way the term in $[\mathbf{r}'\times\mathbf{P}]\sigma_2$ is canceled. It is thus seen that the transformation of Eq. (18.6) together with the wave equation transformation of Eq. (19.2) transforms the wave equation in the frame K into the wave equation in the frame K'. Since at a large distance the values of $\mathbf{p}'$ used in the transformed Eq. (17.3) and the center of mass Eq. (18) are the same the numbers of collisions taking place into corresponding solid angles are equal. No special consideration of the geometrical factors is necessary here anyway since they are the same for the Born first approximation as for the general case and since they have been found to be satisfactory in a previous section.

Note added in proof: Considerations similar to those made for Eq. (17) have been carried out also for the Majorana and Heisenberg exchange equations.

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On Multiple Scattering of Neutrons

I. Theory of the Albedo of a Plane Boundary

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The paper contains the rigorous solution of the following problem in multiple scattering: a beam of particles impinges with arbitrary velocity distribution upon the plane boundary surface of a body which extends towards infinity on the other side of the boundary. In this body the particles have a finite probability of being either captured or scattered without loss of energy. The probability of scattering shall be spherically symmetrical in the laboratory frame of reference. Number and velocity distribution of the returning particles are given explicitly; density as well as velocity distribution of the particles inside the body are determined by the formulae but not worked out in detail since they lack direct physical interest. The result is found to depend on the ratio of the capture to the scattering cross section and on the velocity distribution of the incident particles. Applying the theory to the diffuse reflection of slow neutrons at paraffin surfaces it is found that agreement with observations and previous determinations of the capture cross section can exclusively be obtained, if the active level of the "deuteron with spin zero" is virtual. The connection of these results with some other experiments on the velocity and magnetic moment of the neutrons is discussed.

INTRODUCTION

NEUTRONS before reaching the point of observation usually have to travel through various layers of different materials in which they undergo collisions depending on the nature of the material penetrated. These collisions can be elastic, inelastic, or capture collisions. Quantum mechanics has supplied us with a large amount of information concerning the *single* processes while the problem of the effect of consecutive collisions on the beam has not yet in our opinion been solved satisfactorily. A large number of authors (Fermi,[1] etc., Yost and Dickinson,[2] Wick,[3] Ornstein,[4] etc.) have treated special cases like the stationary state of neutrons that are losing their energy through collisions in hydrogenated substances, elastic diffusion of neutrons accompanied by capture, albedo, etc. Without attempting to enter into any detailed discussions of these papers we think that the

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