# Internal Friction in Solids

## V. General Theory of Macroscopic Eddy Currents

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In vibrating ferromagnetic metals macroscopic eddy currents tend to shield the interior of the metal from changes in magnetic induction. The dissipation of energy by these eddy currents contributes to the internal friction of the metal. This internal friction has previously been investigated theoretically only for longitudinal vibrations, and only the asymptotic formulae for high and low frequencies have been found. In this paper this internal friction is calculated for all frequencies for both longitudinal and transverse vibrations. The theory of internal friction due to macroscopic eddy currents is shown to be formally identical with the theory of internal friction due to macroscopic thermal currents. The methods developed by the author for the study of thermoelastic internal friction are thus directly applicable to the study of the macroscopic electric eddy currents.

### §1. INTRODUCTION

PART of the internal friction of ferromagnetic metals arises from the magnetic-elastic coupling. Under certain conditions this part is of a larger order of magnitude than the internal friction of nonferromagnetic origin. In spite of numerous experimental investigations,[1] only one source of ferromagnetic internal friction has been theoretically investigated[2] for small strains, namely that due to the macroscopic eddy currents. These eddy currents tend to shield the interior of the sample from changes in magnetic induction. Further, this effect has only been investigated for longitudinal vibrations, and only the asymptotic values for high and low frequencies have been obtained. In this paper the internal friction due to macroscopic eddy currents is calculated at all frequencies for the longitudinal and transverse vibrations of circular rods, and for the transverse vibrations of reeds.

It is found that the internal friction due to the eddy currents may be written in the form

$$Q^{-1}=[(E_B-E_H)/E_B]f(\nu). \qquad (1)$$

Here $Q^{-1}$ is $(1/2\pi)$ times the fraction of the vibrational energy dissipated per cycle. $E_B$ and $E_H$ are Young's moduli for constant magnetic induction and constant magnetic field, respectively. The first factor may be calculated by means of Eq. (14). The second factor is given in §3 for the three cases, and is plotted in Fig. 1. The asymptotic expressions for $f$ at high and low frequencies are shown in Fig. 1 by broken lines, and are likewise given explicitly in §3. The frequency of maximum internal friction is determined by the magnetic diffusion constant $D_H$. This is given by

$$D_H=10^8/(0.4\pi\mu\sigma), \qquad (2)$$

where $\mu$ is the differential magnetic permeability, and the electric conductivity $\sigma$ is in $\text{ohm}^{-1}\,\text{cm}^{-1}$. The area beneath all three curves is the same, namely $\frac{1}{2}\pi\log_{10}e=0.682$. This is a direct consequence of the general formula

$$\int_0^\infty f(\nu)\nu^{-1}d\nu=\pi/2, \qquad (3)$$

valid for a specimen of arbitrary cross section in which the only stress associated with the vibration is a tensile stress along the axis of the specimen.

The method used in the present investigation is that developed by the author for studying the internal friction due to macroscopic thermal currents.[3] The differential equations for the two problems become identical when $T$ and $S$ are replaced by $H$ and $B/4\pi$, respectively. The two problems would be identical if the adiabatic boundary condition for thermal flow were replaced by an isothermal boundary condition.

[1] E. Giebe and E. Blechschmidt, Ann. d. Physik **11**, 905 (1931); O. v. Auwers, Ann. d. Physik **17**, 83 (1933); J. Zacharias, Phys. Rev. **44**, 116 (1933); W. T. Cooke, Phys. Rev. **50**, 1158 (1936); Mary D. Waller, Proc. Phys. Soc. **50**, 144 (1938).

[2] M. Kersten, Zeits. f. tech. Physik **15**, 463 (1934); W. F. Brown, Phys. Rev. **50**, 1165 (1936).

[3] C. Zener, Phys. Rev. **52**, 90 (1938).

Thus the thermoelastic internal friction due to macroscopic thermal currents would, if the surface were maintained at constant temperature, be given by Eq. (1) and Fig. 1, the subscripts $H$ and $B$ being replaced by $T$ and $S$, respectively.

## §2. General Theory

The theory for longitudinal and transverse vibrations is particularly simple when the wave-length of vibration is large compared with the transverse dimensions of the specimen. In the first place, all stresses are then negligible compared with the tensile stress parallel to the specimen's axis. If this axis is denoted as the $z$ axis, the energy loss per cycle per unit volume is then

$$\Delta w = \nu^{-1} \text{ time average of } Z_z de_{zz}/dt. \quad (4)$$

In the second place, the transverse components of the oscillating part of the magnetic field and magnetic induction may then be neglected compared with the parallel components. Thus $Z_z$ may be written as a linear function of only $e_{zz}$ and $\Delta H_z$, the fluctuation of $H_z$ from its mean.

$$Z_z = (\partial Z_z/\partial e_{zz})_{H(z)} e_{zz} + (\partial Z_z/\partial H_z)_{e(zz)} \Delta H_z. \quad (5)$$

We are here neglecting a term proportional to the temperature fluctuation. The effect of this thermoelastic term has already been investigated by the author.

By substituting (5) into (4) we obtain

$$\Delta w = \nu^{-1}(\partial Z_z/\partial H_z)_{e(zz)} \text{ time average of } \Delta H_z de_{zz}/dt. \quad (6)$$

We must now find the differential equation which $\Delta H_z$ obeys. This is obtained of course from Maxwell's equations. Here the displacement current may be neglected compared with the real current, since their ratio is given by $\epsilon\sigma/(1.8\times 10^{12}\sigma)$, where $\sigma$ is the electric conductivity in ohm$^{-1}$ cm$^{-1}$. Elimination of the electric field from Maxwell's equations then gives

$$(d/dt)\mathbf{B} = -(10^8/0.4\pi\sigma) \text{ curl curl } \mathbf{H}.$$

If we take the $z$ component of this vector equation, and use our previous approximation of neglecting $H_y$ and $H_x$, we obtain

$$(d/dt)B_z = (10^8/0.4\pi\sigma)(\partial^2/\partial x^2 + \partial^2/\partial y^2)H_z.$$

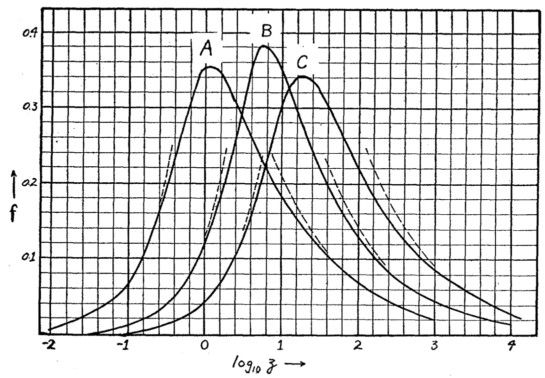


FIG. 1. $f$ functions associated with macroscopic eddy currents. Curve $A$, transverse vibration of reed. $z = D_H/2\pi d^2$, $d$ = transverse width of reed. Curve $B$, longitudinal vibration of circular rod. $z = D_H/2\pi a^2$, $a$ = radius of rod. Curve $C$, transverse vibration of circular rod. $z = D_H/2\pi a^2$, $a$ = radius of rod.

On introducing the relation

$$\delta B_z = (\partial B_z/\partial H_z)_{e(zz)}\delta H_z + (\partial B_z/\partial e_{zz})_{H(z)}\delta e_{zz},$$

we obtain

$$(d/dt)\Delta H_z = D_H(\partial^2/\partial x^2 + \partial^2/\partial y^2)\Delta H_z + (\partial H_z/\partial e_{zz})_{B(z)} de_{zz}/dt. \quad (7)$$

Here $D_H$ is defined by Eq. (2) with

$$\mu = (\partial B_z/\partial H_z)_{e(zz)}.$$

The first term on the right side represents the change in $H_z$ arising from diffusion of the magnetic induction. The second term represents the change arising solely from the change in strain. This would be the only term in the absence of diffusion, i.e., if the electric conductivity were infinite.

The function $de_{zz}/dt$ is affected only very slightly by the magnetic-elastic coupling. Hence very little error is introduced in $\Delta w$ by regarding $de_{zz}/dt$ in Eqs. (6) and (7) as specified by the elastic equations in the absence of this coupling. The function $\Delta H_z$ is then completely determined by Eq. (7) and by the appropriate boundary condition.

We have previously assumed the wave-length of the vibration to be long compared with the transverse dimensions of the sample, and also the transverse components of the oscillating part of $H$ and $B$ to be negligible compared with the parallel component. To this same approximation we may neglect $\Delta H_z$ outside the specimen. The appropriate boundary condition is thus

$$\Delta H_z = 0 \text{ at surface.} \tag{8}$$

Equations (6), (7), (8) contain the physics of our problem. The mathematics involved in the solution of Eq. (7) subject to the boundary condition (8), in taking the time average in Eq. (6), and finally in evaluating the integral

$$Q^{-1} = \frac{\int \Delta w dv}{2\pi \text{ energy of vibration}}$$

has been formally solved in the appendix of reference 3. The result is

$$Q^{-1} = [(E_B - E_H)/E_B] \sum f_k \nu_k \nu / (\nu_k^2 + \nu^2), \tag{9}$$

where $f_k$ satisfies the condition

$$\sum f_k = 1. \tag{10}$$

The constants $f_k$ and $\nu_k$ are interpreted in terms of the differential equation

$$\{D_H(\partial^2/\partial x^2 + \partial^2/\partial y^2) + 2\pi\nu_k\} U_k(x, y) = 0 \tag{11}$$

together with the boundary condition

$$U_k = 0 \text{ at surface.} \tag{12}$$

The $\nu_k$ are the eigenwert of this equation. The $f_k$ are the squares of the coefficients of the expansion of the normalized function

$$e_{zz} \Big/ \left[\int e_{zz}^2 dx\, dy\right]^{\frac{1}{2}} \tag{13}$$

in terms of the normalized eigenfunctions $U_k$. Here $e_{zz}$ is taken at an arbitrary $z$ and $t$.

The ratio $(E_B - E_H)/E_B$ may be calculated in terms of the magnetostriction constant and the permeability by means of the equation[2]

$$(E_B - E_H)/E_B = 4\pi E_H (\partial e_{zz}/\partial H_{zz})_{Z(z)}{}^2 / (\partial B_z/\partial H_z)_{Z(z)}. \tag{14}$$

This equation may be derived in exactly the same way as the equation

$$(E_S - E_T)/E_S = E_T(\partial e_{zz}/\partial T)_{Z(z)}{}^2 / (\partial S/\partial T)_{Z(z)}.$$

## §3. Application to Special Cases

In the previous section we found that the $Q^{-1}$ of a rod of arbitrary cross section vibrating either longitudinally or transversely is given by $(E_B - E_H)/E_B$ times the function

$$f(\nu) = f_k \nu_k \nu / (\nu_k^2 + \nu^2). \tag{15}$$

In this section we describe in detail the evaluation of this function for one particular case, the transverse vibration of a circular rod, and then give the results for the longitudinal vibration of a circular rod, and for the transverse vibration of a reed.

### Circular rod: transverse vibration

Let $a$ be the radius of the circular rod. Then those eigenfunctions of Eq. (11) which satisfy the boundary condition (12), and whose associated coefficients in the expansion of

$$e_{zz} = \text{constant} \times r \cos \varphi$$

do not vanish, are given by

$$\cos \varphi J_1(q_k r/a).$$

Here $q_k$ is the $k$th root of

$$J_1(q) = 0,$$

and determines the eigenwert $\nu_k$ by the equation

$$q_k = (2\pi\nu_k/D_H)^{\frac{1}{2}} a. \tag{16}$$

We may now write $f_k$ explicitly as

$$f_k = \frac{\left(\int_0^a r^2 J_1(q_k r/a) dr\right)^2}{\int_0^a r^3 dr \int_0^a r J_1^2(q_k r/a) dr}.$$

With the aid of the integral formulae for Bessel functions[4] we obtain, as in reference 3, §2,

$$f_k = \frac{8 J_2^2(q_k)}{q_k^2 \{J_1^2(q_k) - J_0(q_k) J_2(q_k)\}}.$$

Since $J_1(q_k) = 0$, the recurrence formulae for Bessel functions[5] gives

$$J_2(q_k) = -J_0(q_k).$$

Hence
$$f_k = 8/q_k^2. \tag{17}$$

[4] See E. Jahnke and F. Emde, *Funktionentafeln* (Teubner, 1933), pp. 213–214.

[5] See G. N. Watson, *Theory of Bessel Functions* (Cambridge, 1922), p. 17.

As a check upon Eq. (10), we note that[6]

$$\sum q_k^{-2}=\tfrac{1}{8}.$$

On substituting Eqs. (16) and (17) into Eq. (15), we obtain

$$f(\nu)=8z\sum(q_k^2+z^2)^{-1}, \tag{18}$$

where

$$z=(2\pi a^2/D_H)\nu.$$

The asymptotic expressions for small and large $z$ may readily be obtained. From[6]

$$\sum q_k^{-4}=1/192$$

we obtain

$$f(\nu)=z/24 \quad \text{when} \quad z\ll 1.$$

By observing that when $k$ is large, successive $k$'s differ by $\pi$, we obtain, for large $z$'s,

$$f(\nu)=(8z/\pi)\int_0^\infty (q^4+z^2)^{-1}dz.$$

Hence $\quad f(\nu)=2^{\frac{3}{2}}z^{-\frac{1}{2}} \quad \text{when} \quad z\gg 1.$

When $z$ is of the order of magnitude of unity, $f(\nu)$ may be evaluated by means of the rapidly converging series (18). In plotting this function in Fig. 1, the values of $q_k$ were taken from reference 5, p. 748.

**Circular rod: longitudinal vibration**

The constant $f_k$ is given by

$$f_k=4/q_k^2,$$

where $q_k$ is the $k$th root of

$$J_0(q)=0.$$

The constant $\nu_k$ is related to $q_k$ by Eq. (16). The function $f$ is given in terms of $z=(2\pi a^2/D_H)\nu$ by

$$4z\sum_{k=1}^{\infty}(q_k^4+z^2)^{-1} \text{ exactly},$$

by $\quad z/8 \quad$ when $\quad z\ll 2\pi,$

by $\quad (2/z)^{\frac{1}{2}} \quad$ when $\quad z\gg 2\pi.$

These two asymptotic expressions have previously been obtained.[2]

**Reed: transverse vibrations**

Let $d$ be the width of the reed in the plane of vibration. Then

$$f_k=6/\pi^2k^2,$$

$$\nu_k=k^2D_H/2\pi d^2.$$

The function $f$ is given in terms of $z=(2\pi d^2/D_H)\nu$ by

$$6\pi^{-2}z\sum_{k=1}^{\infty}(k^4+z^2)^{-1} \text{ exactly},$$

by $\quad (\pi^2/15)z \quad$ when $\quad z\ll 4\pi,$

by $\quad (3/\pi\sqrt{2})z^{-\frac{1}{2}} \quad$ when $\quad z\gg 4\pi.$

[6] Reference 5, p. 502.