1/50,000 As^{75}; As^{74}, 1/20,000 As^{75}; As^{73}, As^{72}, and As^{71}, 1/100,000 As^{75}.

Iodine

Aston[10] showed that iodine consisted of a single isotope of mass 127. This, too, is confirmed by the present investigation. Iodine vapor was admitted into the apparatus and a search was made around the 127 and 254 peaks. It was possible to set the following upper limits for the abundances of other isotopes relative to I^{127}: I^{131}, 1/250,000; I^{130}, 1/120,000; I^{129}, 1/40,000; I^{128}, 1/15,000; I^{126}, 1/25,000; I^{125}, I^{124}, and I^{123}, 1/50,000.

Caesium

Caesium was found to be single by Aston.[11] Bainbridge[12] searched for other isotopes, but was unable to find any. Because of the much higher sensitivity of the present apparatus it seemed worth while to search further. Caesium was introduced in its vapor form into the apparatus. No new isotopes were found. The following upper limits can be set for the abundances of hypothetical isotopes relative to Cs^{133}: Cs^{137} and Cs^{136}, 1/100,000; Cs^{135}, 1/50,000; Cs^{134}, 1/6000; Cs^{132}, 1/4000; Cs^{131}, 1/20,000; Cs^{130} and Cs^{129}, 1/100,000.

The construction of the large electromagnet used in this work was made possible by a grant from the Milton fund. The author wishes to express his appreciation to Professor Kenneth T. Bainbridge for his encouragement and assistance, to Mr. H. L. Leighton for his design and construction of the glass parts of the apparatus, and to Mr. David Mann for his suggestions and aid in the construction of the electromagnet.

[10] Aston, *Mass Spectra and Isotopes*, p. 154.
[11] Aston, *Mass Spectra and Isotopes*, p. 111.
[12] Bainbridge, Phys. Rev. **36**, 1668 (1930).

NOVEMBER 1, 1937 PHYSICAL REVIEW VOLUME 52

Wide-Angle Interference of Multipole Radiation

O. Halpern and F. W. Doermann
Department of Physics, New York University, University Heights, New York, N. Y.

(Received June 11, 1937)

The paper presents a discussion of wide-angle interference phenomena and their dependence upon the nature of the light. Formulas are derived representing the interference pattern for various geometrical arrangements and for arbitrarily composed sources. It is shown that the coherence properties of beams emerging from a point source should depend in a marked manner upon the type of poles which compose it. Special discussion is given for dipole, quadrupole, and octopole sources. The treatment is limited to the use of ideal mirrors; extensions are indicated.

I. Introduction

The coherence properties of light rays emitted from a point source under a finite angle were first investigated to some extent in the well-known experiment of Schrödinger[1] on wide-angle interference. This experiment was originally undertaken to decide between the concepts of unidirectional and spherical emission of light; it was thus primarily concerned with a side of the question which today possesses only historical importance, since the new quantum theory has reinterpreted the apparent conflict between the two old points of view.

Schrödinger observed that the two rays which passed through two holes in a screen produced an interference pattern on the other side. In the *final* interpretation given to his results he remarked that this observation follows alone from the assumption that the interference phenomenon can be described with the help of the Huygens-Kirchhoff principle by the assignment of proper values to the light vectors in the two holes: such a description, according to Schrödinger, leads with necessity to an interference pattern behind the screen.

It is at this point that our opinion differs. We believe that even at the present state of theoretical physics extended importance can be

[1] E. Schrödinger, Ann. d. Physik **61**, 69 (1920).

given to interference experiments with beams diverging under a sizable angle from a point source.

Whether interference patterns will be observable or not depends, as far as we can see, on the *phase relations* between the components of the electric and magnetic vectors of the two beams in their corresponding holes. Whether or not we should expect phase relations to be present will depend on our theoretical view concerning the emission of light. According to quantum optics the phase relations for solid point sources can be calculated in the same way that the classical wave theory would prescribe.

We shall show in the following that the phase relations in equivalent experiments, which allow of a simple ideal treatment, will depend both on the angle between the rays emerging from the point source and upon its nature, i.e., whether it is composed of dipoles, quadrupoles, or higher multipoles. Interference experiments should thus allow us to obtain information, by observations *on the light rays only*, as to the nature of the emitting light source.

Before proceeding to the mathematical treatment it might seem advisable to present a short physical picture which will allow us to verify qualitatively the statement made above. Let us first imagine a single dipole with arbitrary direction of oscillation and let us fix our attention on two rays sent off in two definite directions. As these two rays come from a single dipole they will be coherent by hypothesis, but as they make in general different angles with the direction of oscillation they will show different polarization and different intensities. Since our point source of light is supposed to emit in every direction an equal intensity of unpolarized light, we have to picture it as the superposition of an ensemble of dipoles with random directions of oscillation. A second dipole arbitrarily chosen will emit, in the two fixed directions, light which will also be coherent, but which will again show different states of polarization and intensities, etc. The two rays emitted by the ensemble will have equal intensities, but as the contribution of each individual dipole will in general be different for the two, they will not in general be completely coherent. The amount of coherence will depend on the angle which the rays make with each other and also, as we shall see, on the nature of the source. This is physically plausible because dipoles and quadrupoles, for example, show a different functional dependence of energy and polarization on the angle made by the "axis of the atom" and the direction of emission.

The mathematical treatment can be carried out in either of two essentially equivalent ways. We might, in the first place, determine the radiation emitted by a multipole of definite orientation in two fixed directions and then average certain intensities over all orientations of the multipole. This process is straightforward, but becomes somewhat cumbersome for higher poles. On the other hand we can, according to well-known methods of quantum optics, characterize the field of the multipole (2^l pole) by $2l+1$ complex amplitudes. We can then obtain a comprehensive model for a point source emitting unpolarized light of equal intensity in all directions by assuming that the average values of products of these amplitudes vanish whenever the two amplitudes refer to different states of oscillation, and are equal to one and the same constant when they refer to the same state of oscillation. This second method of presentation follows closely a paper of W. Heitler[2] on the angular momentum of light, and will be employed in the following.

II. Definition of the Interference Parameter

Let B and B' represent the two interfering beams, and let J and J' represent their individual intensities at the center of the interference pattern, i.e., at a point of zero difference of optical paths; finally let J^S represent the intensity of the superposed beams at this same point. It is then clear that if we define

$$Q=J^S/(J+J')-1 \qquad (1)$$

the illumination in the pattern will be proportional to $1+Q\cos\delta$ so that the value of $|Q|$ may be taken as a convenient measure of the visibility of the fringes. For $|Q|=0$ no fringes will be visible, and for $|Q|=1$ they will have their maximum sharpness; the central fringe will be light or dark according as $Q\gtreqless 0$.

[2] W. Heitler, Proc. Camb. Phil. Soc. 32, 112 (1936).

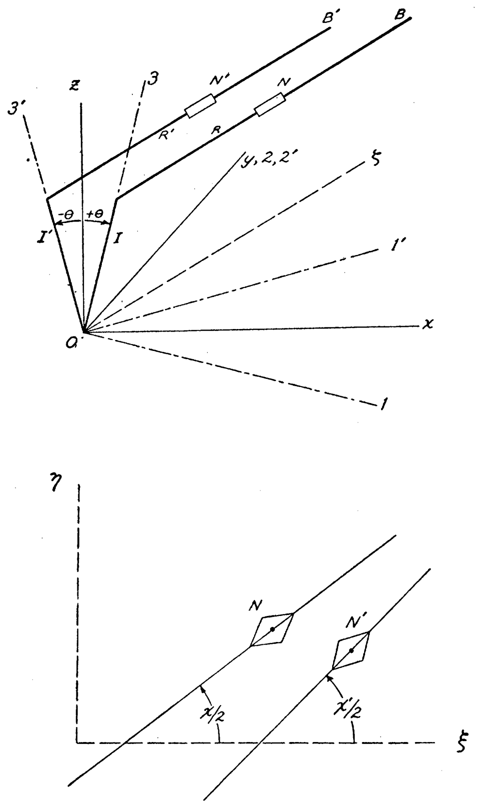

FIG. 1. General interference arrangement.

The intensities will be proportional to the average, over the time and the constituents of the source, of the square of the magnetic vector. Thus Q may be calculated by the general formula:

$$Q+1=\langle(\mathbf{H}^B+\mathbf{H}^{B'})^2\rangle_{\text{Av}}/\langle(\mathbf{H}^{B2})_{\text{Av}}+(\mathbf{H}^{B'2})_{\text{Av}}\rangle. \quad (2)$$

III. GENERAL INTERFERENCE ARRANGEMENT

Our point source is located at O (Fig. 1)[3] and emits unpolarized spherically symmetrical radiation from which two beams I and I' are delimited. These beams are then made parallel to become eventually the B, B' of Section II. To simplify the treatment we suppose that I, I' first become R, R' by a process (such as reflection at an ideal metallic mirror[4]) which involves no polarization or absorption,[5] and that these become in turn B, B' on passing, without deflection, through ideal Nicol prisms N, N'. We shall find it convenient to employ three sets of axes: $Oxyz$, $O\xi\eta\zeta$, $O123$ ($O1'2'3'$), of which the respective orientations are clear from the figure. It is important to notice that $O3$ is parallel to I ($O3'$ to I') and $O\zeta$ to B (and B'); also that Oy is parallel to $O2$ (and $O2'$).

If $\mathbf{H}$ and $\mathbf{H}'$ correspond to the beams I, I' then it is obvious from the assumptions just made that $\mathbf{H}^R$, $\mathbf{H}^{R'}$ will be linear vector functions of $\mathbf{H}$, $\mathbf{H}'$, respectively, with $\mathbf{H}^{R2}=\mathbf{H}^2$, $\mathbf{H}^{R'2}=\mathbf{H}'_2$. The mathematical statement of this fact may, because of the transversality, be put in the form:

$$\left.\begin{aligned} H_\xi^R&=aH_1+bH_2 \\ H_\eta^R&=\epsilon bH_1-\epsilon aH_2 \end{aligned}\right\}, \qquad \begin{aligned} a^2+b^2&=1, \\ \epsilon&=\pm1; \end{aligned} \quad (3)$$

$$\left.\begin{aligned} H_\xi^{R'}&=a'H_{1'}'+b'H_{2'}' \\ H_\eta^{R'}&=\epsilon'b'H_{1'}'-\epsilon'a'H_{2'}' \end{aligned}\right\}, \qquad \begin{aligned} a'^2+b'^2&=1, \\ \epsilon'&=\pm1. \end{aligned} \quad (3')$$

We suppose the Nicols N, N' set so that the planes of polarization of B, B' (i.e., the planes containing the vectors $\mathbf{H}^B$, $\mathbf{H}^{B'}$) make angles $\chi/2$, $\chi'/2$ with the $\xi\zeta$ plane. The relation between $\mathbf{H}^B$ and $\mathbf{H}^R$ is then:

$$\left.\begin{aligned} 2H_\xi^B&=(1+\cos\chi)H_\xi^R+\sin\chi H_\eta^R \\ 2H_\eta^B&=\sin\chi H_\xi^R+(1-\cos\chi)H_\eta^R \end{aligned}\right\} \quad (4)$$

and an analogous relation involving χ' connects $\mathbf{H}^{B'}$ and $\mathbf{H}^{R'}$.

We must finally remark upon the relation between $\mathbf{H}$ and $\mathbf{H}'$ as functions of θ. In the following section we shall show how $\mathbf{H}(\theta)$ is defined for the range $-\pi<\theta\leqq\pi$. From our construction it is then clear that $\mathbf{H}'=\mathbf{H}(-\theta)$; furthermore, because of the way the triples $O123$, $O1'2'3'$ have been defined: $H_{1'}'=H_1(-\theta)$, $H_{2'}'=H_2(-\theta)$. These lead at once to the formulas:

$$\left.\begin{aligned} H_1&=H_{1+}+H_{1-} \\ H_2&=H_{2+}+H_{2-} \end{aligned}\right\}, \quad \left.\begin{aligned} H_{1'}'&=H_{1+}-H_{1-} \\ H_{2'}'&=H_{2+}-H_{2-} \end{aligned}\right\}, \quad (5)$$

in which the inferior index "+" or "−" indicates the part even or odd in θ, respectively.

It is now a lengthy and tedious, but essentially simple and straightforward matter to introduce (3), (4), and (5) into (2) and so express Q in terms of averaged products of the type $(H_{1+}H_{1-})_{\text{Av}}$,

[3] The authors are very grateful to Dr. Frank E. Myers for preparing the figures accompanying this article.

[4] See Section V.

[5] If, by an absorptive process that involves neither polarization nor change of phase, the intensities of the beams are brought into the ratio $1:\alpha^2$, the value of Q will simply suffer a reduction in the ratio $1+\alpha^2:2\alpha$.

etc. Marked simplifications will arise from our idealized assumptions, *viz.:* $(\mathbf{H}^{R^2})_{\mathrm{Av}}=(\mathbf{H}^{R'^2})_{\mathrm{Av}}=(\mathbf{H}^2)_{\mathrm{Av}}$, $(\mathbf{H}^{B^2})_{\mathrm{Av}}=(\mathbf{H}^{B'^2})_{\mathrm{Av}}=\frac{1}{2}(\mathbf{H}^2)_{\mathrm{Av}}$; and also from the fact, which will be obvious as soon as the explicit formulas are written down, that for multipole radiation:

$$(H_{1+}H_{1-})_{\mathrm{Av}}=(H_{1+}H_{2+})_{\mathrm{Av}}=(H_{1+}H_{2-})_{\mathrm{Av}}=(H_{1-}H_{2+})_{\mathrm{Av}}=(H_{1-}H_{2-})_{\mathrm{Av}}=(H_{2-}H_{2+})_{\mathrm{Av}}=0. \quad (6)$$

From these, and from the relations

$$\langle(H_{1+}+H_{1-})^2\rangle_{\mathrm{Av}}=\langle(H_{2+}+H_{2-})^2\rangle_{\mathrm{Av}}=\tfrac{1}{2}(\mathbf{H}^2)_{\mathrm{Av}},$$

there also follows immediately:

$$[(H_{1+}{}^2)_{\mathrm{Av}}+(H_{1-}{}^2)_{\mathrm{Av}}]/(\mathbf{H}^2)_{\mathrm{Av}}=[(H_{2+}{}^2)_{\mathrm{Av}}+(H_{2-}{}^2)_{\mathrm{Av}}]/(\mathbf{H}^2)_{\mathrm{Av}}=\tfrac{1}{2}. \quad (7)$$

Finally, it will be convenient to set

$$\begin{aligned}D_1&=[(H_{1+}{}^2)_{\mathrm{Av}}-(H_{1-}{}^2)_{\mathrm{Av}}]/(\mathbf{H}^2)_{\mathrm{Av}},\\ D_2&=[(H_{2+}{}^2)_{\mathrm{Av}}-(H_{2-}{}^2)_{\mathrm{Av}}]/(\mathbf{H}^2)_{\mathrm{Av}}.\end{aligned} \quad (8)$$

In the manner indicated we then obtain the fundamental formula:

$$\begin{aligned}2Q=&(D_1+\epsilon\epsilon' D_2)[(aa'+\epsilon\epsilon' bb')(1+\cos(\chi-\chi'))\\&+(\epsilon a'b-\epsilon' ab')\sin(\chi-\chi')]+(D_1-\epsilon\epsilon' D_2)\\&\times[(aa'-\epsilon\epsilon' bb')(\cos\chi+\cos\chi')\\&+(\epsilon a'b+\epsilon' ab')(\sin\chi+\sin\chi')],\end{aligned} \quad (9)$$

and in case no Nicols are employed:

$$Q=(D_1+\epsilon\epsilon' D_2)(aa'+\epsilon\epsilon' bb'). \quad (10)$$

IV. The Field of a Multipole[6]

It will be convenient to introduce the abbreviations:

$$\begin{aligned}H_x{}^{l,\,m}&=[(l-m+1)(l+m)]^{\frac{1}{2}}P_{l,\,m-1}+[(l+m+1)(l-m)]^{\frac{1}{2}}P_{l,\,m+1},\\ H_y{}^{l,\,m}&=-[(l-m+1)(l+m)]^{\frac{1}{2}}P_{l,\,m-1}+[(l+m+1)(l-m)]^{\frac{1}{2}}P_{l,\,m+1},\\ H_z{}^{l,\,m}&=-2mP_{l,\,m},\end{aligned} \quad (11)$$

where

$$P_{l,\,m}=\left[\frac{2l+1}{2}\right]^{\frac{1}{2}}\left[\frac{(l-m)!}{(l+m)!}\right]^{\frac{1}{2}}\frac{(2l)!}{2^l l!}\sin^m\theta\times\frac{d^{l+m}(\mu^2-1)^l}{d\mu^{l+m}}$$

is the associated Legendre function ($\mu\equiv\cos\theta$) normalized and defined also for negative m as indicated.[7] $P_{l,\,m}$ is customarily employed for the range $0\leqq\theta\leqq\pi$; we, however, shall remain in the xz plane and may therefore without ambiguity extend the range to $-\pi<\theta\leqq\pi$.

The magnetic vector $\mathbf{H}$ at a point on the xz plane in the wave zone of an oscillating 2^l pole may now be described by:

$$H_s=C\Re\{e^{2\pi i\nu t}\sum_{m=-l}^{+l}\epsilon_s a_l{}^m H_s{}^{l,\,m}\},\quad (s=x,y,z;\ \epsilon_x=\epsilon_z=i,\ \epsilon_y=1), \quad (12)$$

where C depends upon the radius vector r, but may by us be treated as a constant.[8] Since, further, $P_{l,\,m}$ is an even or odd function of θ according as m is even or odd, we may obtain from (12) the formulas:

$$\begin{aligned}H_{1\pm}&=-\csc\theta H_{z\mp}=C\Re[e^{2\pi i\nu t}\textstyle\sum^{\mp}(-i\csc\theta)a_l{}^m H_z{}^{l,\,m}],\\ H_{2\pm}&=H_{y\pm}=C\Re[e^{2\pi i\nu t}\textstyle\sum^{\mp}a_l{}^m H_y{}^{l,\,m}],\end{aligned} \quad (13)$$

where $\sum^+$, $\sum^-$ signify summations over only even values and only odd values of m, respectively.

In the relations given above the $a_l{}^m$ are the complex amplitudes referred to in Section I; according to the assumptions there stated we have for averages taken over the constituents of the source:

$$(a_l{}^{m*}a_l{}^{m'})_{\mathrm{Av}}=\delta_{mm'}k_l, \quad (14)$$

Table I.

	D_1	D_2
$2\theta=0$	$\frac{1}{2}$	$\frac{1}{2}$
$2\theta=\pi$	$-\frac{1}{2}(-1)^l$	$\frac{1}{2}(-1)^l$
$l=1$	$\frac{1}{2}$	$\frac{1}{2}(2\mu^2-1)$
$l=2$	$\frac{1}{2}(2\mu^2-1)$	$\frac{1}{2}(8\mu^4-8\mu^2+1)$
$l=3$	$\frac{1}{2}(5\mu^4-5\mu^2+1)$	$\frac{1}{2}(30\mu^6-45\mu^4+17\mu^2-1)$

[6] We here employ the field of the *electric* 2^l pole. The field, in the wave zone, of the *magnetic* 2^l pole can readily be obtained from it by writing $\mathbf{E}$ for $\mathbf{H}$ (reference 2). Since, moreover, $H_1=-E_2$, $H_2=E_1$, it is at once evident that for multipoles of the same order: $(D_1)_{\mathrm{mag}}=(D_2)_{\mathrm{elect}}$; $(D_2)_{\mathrm{mag}}=(D_1)_{\mathrm{elect}}$.

[7] See Bethe's treatment of spherical harmonics in the *Handbuch der Physik*, Vol. 24.

[8] For points on the xz plane our relations (12) are identical with Heitler's formulas (22 a, b), reference 2.

where k_l is independent of m. Relations (6) may now be verified at sight, and relations (8) may be expanded to give:

$$\begin{aligned} 2l(l+1)(2l+1)D_1 &\\ &= \csc^2\theta[\sum{}^- H_z^{l,m^2} - \sum{}^+ H_z^{l,m^2}], \\ 2l(l+1)(2l+1)D_2 &= \sum{}^- H_y^{l,m^2} - \sum{}^+ H_y^{l,m^2}. \end{aligned} \tag{15}$$

We shall show in the Appendix that the evaluation of the four summations in (15) may, with the help of a recursion formula, be reduced to the carrying out of the single simpler summation $\sum{}^+ P^2_{l,m}$. Here we shall merely give the results for arbitrary multipole for $2\theta = 0$ and $2\theta = \pi$ and for arbitrary angle 2θ for the dipole, quadrupole, and octopole.

V. Discussion of Special Cases

The value of Q will depend not only upon l and θ but also upon the way in which the original beams I and I' are made parallel, i.e., upon the values of the coefficients a, b, a', b'. Let us now suppose that R is obtained from I by a single reflection at an ideal metallic mirror M. The boundary conditions at such a mirror show that $\mathbf{H}^R = \mathbf{H} - 2(\mathbf{n}\cdot\mathbf{H})\mathbf{n}$ where $\mathbf{n}$ is a unit vector normal to it. The simplest vector algebra then gives:

$$\begin{aligned} a &= \frac{-c_{\eta y} - c_{\xi z}\sin\theta + c_{\xi x}\cos\theta}{1 - c_{\zeta x}\sin\theta - c_{\zeta z}\cos\theta}, \\ b &= \frac{c_{\xi y} - c_{\eta z}\sin\theta + c_{\eta x}\cos\theta}{1 - c_{\zeta x}\sin\theta - c_{\zeta z}\cos\theta}, \\ \epsilon &= +1, \end{aligned} \tag{16}$$

where the c's are elements of the matrix $(c_{\sigma s})$ of direction cosines relating the $O\xi\eta\zeta$ and $Oxyz$ axes. If I' is reflected into R' at M', the corresponding formulas for a', b' may be obtained from (16) by setting $-\theta$ for θ.

In case A (Fig. 2), I and I' are reflected to become parallel to Oz. It is here possible to choose the $O\xi\eta\zeta$ axes identical with the $Oxyz$ axes so that $(c_{\sigma s}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Formula (16) then gives: $a = a' = -1$, $b = b' = 0$, $\epsilon = \epsilon' = +1$, and substitution into (9) and (10) gives as a final

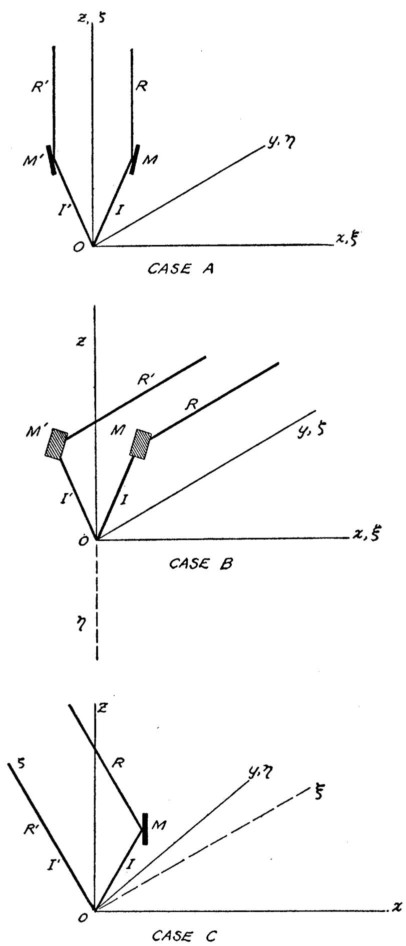

Fig. 2. Particular interference arrangements.

result:

$$\begin{aligned} 2Q &= (D_1 + D_2)(1 + \cos(\chi - \chi')) \\ &\quad + (D_1 - D_2)(\cos\chi + \cos\chi'); \end{aligned} \tag{17}$$

$Q = D_1 + D_2$ (without Nicols).

In case B, I and I' are reflected to become parallel to Oy, i.e. normal to their plane. Here we may take $(c_{\sigma s}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ so that (16) gives: $a = a' = \cos\theta$, $b = -b' = \sin\theta$, $\epsilon = \epsilon' = +1$ and our general formula becomes:

$$\begin{aligned} 2Q &= (D_1 + D_2)[\cos 2\theta(1 + \cos(\chi - \chi')) \\ &\quad + \sin 2\theta \sin(\chi - \chi')] \\ &\quad + (D_1 - D_2)(\cos\chi + \cos\chi'); \end{aligned} \tag{18}$$

$Q = (D_1 + D_2)\cos 2\theta$ (without Nicols).

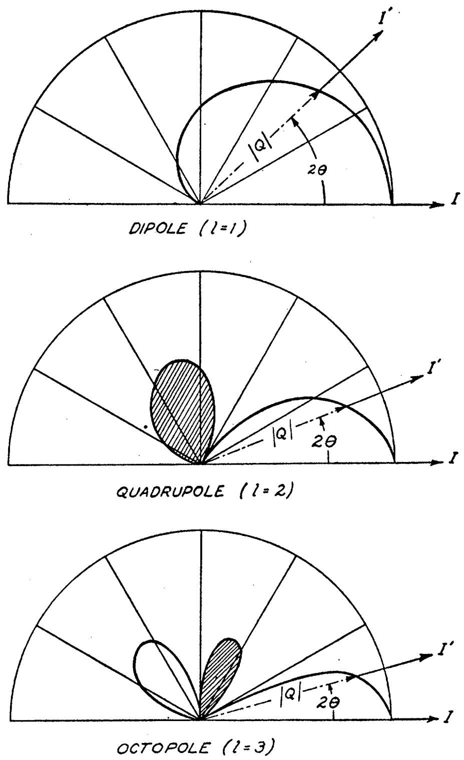

FIG. 3. Polar graphs of $|Q|$ as functions of 2θ (case A).

In case C, I' is not reflected at all so that $\mathbf{H}^{R'}=\mathbf{H}'$. If we take $O\eta$ parallel to Oy the axes $O1'2'3'$ and $O\xi\eta\zeta$ will become coincident, so that $a'=1$, $b'=0$, $\epsilon'=-1$. With this determination of $O\eta$ we have

$$(c_{\sigma s})=\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix};$$

whence if I is now reflected to become parallel to I', we find $a=-1$, $b=0$, $\epsilon=+1$. Finally:

$$2Q=(D_2-D_1)(1+\cos(\chi-\chi'))-(D_1+D_2)(\cos\chi+\cos\chi');\quad (19)$$

$Q=D_2-D_1$ (without Nicols).[9]

Having given these examples to demonstrate the nature of the dependence upon experimental arrangement, we proceed to discuss case A in greater detail. For vanishing angle between I and I', the insertion of the values given in Table I into our formula gives $Q=\frac{1}{2}(1+\cos(\chi-\chi'))$ and this is entirely in keeping with our expectations since it means that $Q=1$ for parallel and $Q=0$ for crossed Nicols. Without Nicols $Q=1$ for $2\theta=0$. If, on the other hand, no Nicols are employed for $2\theta=\pi$ then $Q=0$; and this seems at first somewhat surprising since we should expect such rays to be capable of producing fringes. They will do so as soon as properly oriented Nicols are inserted, for then $Q=-\frac{1}{2}(-1)^l(\cos\chi+\cos\chi')$. Let us now set the Nicols parallel at an arbitrary angle $\chi_0/2$ $(=\chi/2=\chi'/2)$ and denote the corresponding value of Q by Q_0. It is immediately evident that if, keeping the Nicols parallel, we turn them through $\pi/2$ to make $\chi=\chi'=\chi_0+\pi$, then the new value of Q will be $-Q_0$. This means that the two patterns of fringes will be of the same intensity, but will be *complementary*, the light fringes of one falling on the dark fringes of the other so that if they were superposed, no fringes would be visible. The absence of fringes when no Nicols are employed may be explained by such superpositions.

The results just stated are independent of the order of the multipole radiation; however a strong dependence on l will become immediately evident as soon as we study the values of Q for intermediary angles. Without Nicols, our formulas give for dipoles, quadrupoles, and octopoles:

$$\begin{aligned} l&=1,\quad Q=\mu^2,\\ l&=2,\quad Q=4\mu^4-3\mu^2,\qquad (20)\\ l&=3,\quad Q=15\mu^6-20\mu^4+6\mu^2.\end{aligned}$$

Fig. 3 shows most clearly how Q behaves as 2θ is varied from 0° to 180°. For dipoles Q falls off monotonically and the fringes do not disappear until $2\theta=180°$. For quadrupoles Q falls off much more rapidly, and the fringes already disappear for $2\theta=60°$. They then reappear with a dark central fringe ($Q<0$, hatched in the figure) and attain a maximum sharpness ($|Q|=0.56$) for $2\theta=105°$. For octopoles the fringes disappear at $2\theta=35°$, and again at $2\theta=95°$; a maximum ($|Q|=0.46$) with dark central fringe occurs at $2\theta=67°$ and a slightly greater one ($|Q|=0.52$)

[9] In this case, according to reference 6, the interference patterns for the electrical multipole and the magnetic multipole of the same order will be complementary. In case A (without Nicols) they will be identical.

with bright central fringe at $2\theta=128°$. These examples show clearly how strongly the constitution of the source can affect the qualitative characteristics of wide-angle interference phenomena.

In the treatment given it has been assumed that the reflections occur at the surface of *ideal* mirrors. For nonideal mirrors additional polarization effects would have to be considered which in some arrangements might appreciably alter the results as stated. The effects would then depend upon the angle of incidence as well as on the optical constants of the mirrors and might in turn throw light upon the numerical values and dispersion of these constants. It is intended to present a treatment of nonideal reflections in a separate investigation. The theory would thus also be brought into closer proximity with actual experimental conditions.

Appendix

It will in the following be convenient to employ the notation:

$$S_l^{\pm}=\frac{2}{2l+1}\sum_{-l}^{l}{}^{\pm}P^2_{l,m};\; T_l^{\pm}=\frac{2}{2l+1}\sum_{-l}^{l}{}^{\pm}m^2\csc^2\theta P^2_{l,m}. \quad (21)$$

Each harmonic component of **H** is transverse, i.e.

$$\sin\theta H_x^{l,m}+\cos\theta H_z^{l,m}=0,$$

a fact which may easily be checked by the usual formulas for expressing $\sin\theta P_{l,m}$ and $\cos\theta P_{l,m}$. If, now, (11) be introduced into the equivalent relation

$$H_x^{l,m2}+H_y^{l,m2}+H_z^{l,m2}=H_y^{l,m2}+\csc^2\theta H_z^{l,m2}$$

there results

$$2(l-m+1)(l+m)P^2_{l,m-1}+2(l+m+1)(l-m)P^2_{l,m+1}+4m^2P^2_{l,m}=H_y^{l,m2}+4m^2\csc^2\theta P^2_{l,m}.$$

from which in turn follows, on alteration of the summation index in certain terms:

$$\frac{1}{4}\frac{2}{2l+1}\sum{}^{\pm}H_y^{l,m2}=l(l+1)S_l^{\mp}-\sin^2\theta T_l^{\mp}-\cos^2\theta T_l^{\pm}. \quad (22)$$

We can, on the other hand, again by the usual formulas, show that

$$2m(2l+1)^{\frac{1}{2}}\csc\theta P_{l+1,m}=(2l+3)^{\frac{1}{2}}\{[(l+m+1)(l+m)]^{\frac{1}{2}}P_{l,m-1}+[(l-m+1)(l-m)]^{\frac{1}{2}}P_{l,m+1}\}$$

and by employing the square of this obtain also:

$$\frac{1}{4}\frac{2}{2l+1}\sum{}^{\pm}H_y^{l,m2}=(l+1)^2S_l^{\mp}-T^{\pm}_{l+1}. \quad (23)$$

Comparison of (22) with (23) gives at once the recursion formulas:

$$T^{\pm}_{l+1}=(1-\mu^2)T_l^{\mp}+\mu^2T_l^{\pm}+(l+1)S_l^{\mp}. \quad (24)$$

If we now introduce the quantities

$$S_l=S_l^+-S_l^-;\quad T_l=T_l^+-T_l^- \quad (25)$$

we can write our fundamental formulas (15) in the form:

$$l(l+1)D_1=-T_l;\quad l(l+1)D_2=(l+1)^2S_l+T_{l+1}. \quad (26)$$

Combination of (24) with (25) leads at once to:

$$T_{l+1}=(2\mu^2-1)T_l-(l+1)S_l; \quad (27)$$

which, together with the fact that $T_1=-1$, gives a ready method for calculating the T_l once the S_l are known. Finally

$$S_l^++S_l^-=1;\quad T_l^++T_l^-=\tfrac{1}{2}l(l+1); \quad (28)$$

the former being a well-known theorem, and the latter readily demonstrable from (24) by its aid. Combination of (25) with (28) gives

$$S_l=2S_l^+-1=1-2S_l^-, \quad (29)$$

which together with (27) and (26) gives all the relations needed for the calculation of D_1 and D_2 by performing the summations S_l^+ (or S_l^-) alone.

As an illustration we show the order of calculation for $2\theta=0$ and $2\theta=\pi$ in Table II.

Table II.

	$P^2_{l,m}$	S_l^+	S_l^-	S_l	T_l
$\mu=1$	0 for $m\neq 0$		0	1	$-\frac{1}{2}l(l+1)$
$\mu=0$	0 for $l-m$ odd	0 for l odd	0 for l even	$(-1)^l$	$(-1)^l\frac{1}{2}l(l+1)$