terms of the area between the magnetization curves. Values are given in Table II.

TABLE II. *Values of anisotropy constants.*

Method	K_1 (erg cm^{-3})	K_2 (erg cm^{-3})
Measurement of areas between magnetization curves	272,000	150,000
Fitting theoretical magnetization curves to the experimental curves	280,000	100,000
Torque measurements	287,000	100,000

Constants from Torque Measurements

A third method of determining anisotropy constants is by means of torque measurements. A single crystal disk 1.77 cm in diameter and 0.222 cm thick was cut so the plane of the disk was parallel to (110). Torque measurements were made on this disk by means of a torsion magnetometer. Using the relation $L=-dE/d\alpha$ and substituting the direction cosines for a vector lying in the (110) plane gives the following expression for the torque.

$$L=-K_1(2\sin 2\alpha+3\sin 4\alpha)/8 \\ -K_2(\sin 2\alpha+4\sin 4\alpha-3\cos 6\alpha)/64. \quad (7)$$

Fig. 5 shows the theoretical and experimental torque curves. K_1 and K_2 were determined by trial to make the theoretical curve fit the experimental curve as closely as possible.

The values of K_1 and K_2 determined by the three methods are given in Table II.

I wish to thank F. E. Haworth for the careful x-ray determination of the crystal orientations, and R. M. Bozorth for valuable discussions.

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The Viscosity of Air

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The viscosity of air has been remeasured with the rotating cylinder apparatus used by Day and Bleakney. All of the constants have been redetermined, and especial attention has been given to the corrections to the simple theory. The value obtained at 22°C is $\eta=1.8243\pm0.0045$ c.g.s. units. This raises the oil drop value of e to 4.796 e.s.u. It is shown in the appendix that a correction must be applied for the opening between the suspended cylinder and the guard cylinders, and also a correction must be applied for the moment of inertia of the air carried around by the cylinder when determining its moment of inertia. Neglect of these corrections has introduced some additional uncertainty into other work.

The viscosity of air has become of special interest within the last few years since the suggestion of Shiba[1] that an error in the adopted value of this quantity was responsible for the discrepancy between the values of e determined by the oil drop and the x-ray methods. The work of Harrington[2] was supposed to have established its value with the necessary precision, but Shiba concluded from an examination of other determinations that this precision might have been considerably overestimated. Because of this interest and the presence in this laboratory of a rotating cylinder apparatus suited to this purpose it seemed worth while to make a complete redetermination of the viscosity. Although since the work was started, about a year ago, the very careful work of Kellström[3] and a preliminary note from Bearden have appeared and have confirmed Shiba's supposition, it may be still of interest to have an independent determination.

It is not necessary to enter into a detailed description of the apparatus since it has been previously described.[4] No essential changes have been

[1] K. Shiba, Sci. Papers Inst. Phys. Chem. Res. Tokyo, **19**, 97 (1932).
[2] E. L. Harrington, Phys. Rev. **8**, 738 (1916).
[3] G. Kellström, Phil. Mag. **23**, 313 (1937); J. A. Bearden, Phys. Rev. **51**, 378 (1937).
[4] R. K. Day, Phys. Rev. **40**, 281 (1932); W. M. Bleakney, Physics **3**, 123 (1932).

made since the work of Bleakney but all of the dimensions have been redetermined.

The expression for the viscosity when determined by this method is

$$\eta = \pi(b^2 - a^2)I\theta/la^2b^2P^2\Omega. \quad (1)$$

For convenience in discussion this may be divided into three parts. Let

$$K = QI = \{\pi(b^2 - a^2)/la^2b^2\}I,$$
$$D = \theta/P^2\Omega,$$

where a is the radius of the inner suspended cylinder, b is the radius of the outer rotating cylinder, l is the effective length of the inner cylinder, I is the moment of inertia and P the period of oscillation of the inner cylinder, Ω is the angular velocity with which the outer cylinder is turned, and θ is the angular deflection of the inner cylinder. The quantity Q is determined by direct measurement, I by observation of periods of vibration, and the determinations of D constitute the measurement of the viscosity. K may be considered a constant of the instrument independent of the suspension used.

Determination of Q

The diameter of the inner cylinder was measured directly with two different micrometers. Each micrometer was checked on a standard block. Although no investigation of the standards was made it is certain that their error is entirely negligible compared with other errors which are present. Measurements were taken near the top, near the middle, and near the bottom. These are given in Table I. The uncertainties given are the root-mean-square deviations from the mean. The uncertainty indicated in the final value, in this as in all other tables, is either the root-mean-square deviation of the mean values given or $1/n^{\frac{1}{2}}$ times the square root of the mean square of the given uncertainties, whichever is the larger. There was evidence of slight ellipticity at the top and at the bottom.

Table I. *Diameter of inner cylinder.*

Readings	Position	Mean
20	Top	10.7173 ± 0.0014
20	Middle	10.7176 ± 0.0014
20	Bottom	10.7173 ± 0.0006
	Mean	10.7174 ± 0.0007

The inside diameter of the rotating cylinder was measured with two inside micrometers. The results are given in Table II.

Table II. *Diameter of outer cylinder.*

Readings	Position	Mean
14	Top	11.9070 ± 0.0014
14	Middle	11.9078 ± 0.0006
14	Bottom	11.9082 ± 0.0009
	Mean	11.9077 ± 0.0006

The length of the cylinder was measured with a steel scale and with a glass scale on a cathetometer. Both of these scales had previously been compared with a standard meter.[5] With the cathetometer, nine readings gave a mean of 25.482 ± 0.018 cm when corrected for the error of the scale. Six readings with the steel scale gave 25.495 ± 0.005. The mean is 25.489 ± 0.013 cm.

The use of the length of the cylinder in Eq. (1) implies that all of the torque is applied to the outer surface of the cylinder. However, the air penetrates the opening between it and the guard cylinder and exerts a torque on the end and in fact on the inside surface of the suspended cylinder. This effect is small since the distance between the cylinders is small compared with the length, but it is not entirely negligible. The distance between the guard cylinders as measured with the steel scale was 25.587 ± 0.015 cm so that the sum of the two openings was 0.098 cm. If the suspended cylinder and the guard cylinder had been identical in wall thickness, just half of this opening should be added to the length of the inner cylinder. However, the guard cylinder had walls some 3 mm thick while the thickness of the suspended cylinder wall was less than 0.5 mm. On this account more of the torque was applied to the guard cylinder than to the other and the correction added was 0.47 of the opening or 0.046 cm. A further discussion of this correction is given in the appendix.

With this correction the effective length of the cylinder was

$$l = 25.535 \pm 0.015 \text{ cm}.$$

[5] J. S. Campbell and W. V. Houston, Phys. Rev. **39**, 601 (1932).

The relative uncertainty in the value of Q is

$$\Delta Q/Q=\{[\Delta a/(b-a)]^2+[\Delta b/(b-a)]^2+(\Delta l/l)^2\}^{\frac{1}{2}}.$$

Combining the above results gives

$$Q=8.1375\pm0.10 \text{ percent}.$$

Determination of I

The moment of inertia of the inner cylinder was determined by the customary procedure of measuring the period of the rotational oscillations with and without the addition of a ring whose moment of inertia was calculated from its dimensions. Since the torsion constant of a suspension depends upon the weight it supports, it was necessary to use a compensating weight when the ring was not in place. This weight had the form of a rod and was supported on the axis of the cylinder by three small wires extending to one of the ribs. The period was then measured under four different conditions.

Let T_0 be the period of oscillation of the cylinder alone, T_1 the period of the cylinder and the ring, T_2 the period of the cylinder and the weight, and T_3 the period of the cylinder with both the ring and the weight. Let these be the periods after the correction for the observed damping has been applied so that the square of the corrected period is proportional to the effective moment of inertia divided by the force constant. The correction was made in terms of the observed decrement. Let $\delta=\log_e (X_n/X_{n+1})$, where X_n and X_{n+1} are successive maximum displacements on one side. Then $T=T'/(1+\delta^2/8\pi^2)$, where T' is the observed period. In many cases this correction was entirely negligible.

The suspensions for the determination of I were steel and tungsten wires of various sizes. When first inserted they showed a tendency to change with the time, and the torsion constant always showed a slight change with the temperature. A determination of the moment of inertia thus involved an extended series of observations of each of the four periods until the permanent change was negligible and a curve of period against temperature could be established. The cylinder was mounted in a wooden case on a heavy block of concrete to be shielded from air currents and as much as possible from outside mechanical disturbances. In spite of these precautions it was found that the long period observations were erratic and not reproducible, so that no determinations were included in the final results for which T_0 was greater than 100 sec.

Table III. *Moment of inertia of rings and weights.*

No.	I_R	I_W	M_R	M_W
1	3264.5±1.0	43.5±0.5	116.10	118.16
2	6364.8±1.2	130.5±0.5	226.78	228.83

The oscillations were observed with a telescope and scale and the amplitude was recorded to compute the damping correction. The time between a passage through the central position and an integral minute on the clock was measured by means of a stopwatch. Part of the time the clock used was a chronometer loaned by the Mt. Wilson Observatory, but after the Institute power lines were connected to the Boulder Dam plant, the frequency was sufficiently constant to use an ordinary electric clock. Errors in the clock were treated as negligible. Observations were made of three passages through the center at the beginning of a run and again about 50 minutes later. The nine differences permitted averaging to minimize the end point errors so that the periods are believed to be correct to 0.005 percent.

Since these observations were made in air at atmospheric pressure it is necessary to make a correction for the moment of inertia of the air dragged around with the cylinder. This quantity is approximately proportional to $T^{\frac{1}{2}}$, and its exact form and development are given in the appendix. Let ΔI_0, ΔI_1, ΔI_2, ΔI_3 be the corrections to be added to the true moment of inertia of the cylinder under the four conditions of oscillation. Then

$$\begin{aligned}
T_0^2&=4\pi^2(I+\Delta I_0)/k,\\
T_1^2&=4\pi^2(I+\Delta I_1+I_R)/k(1+\epsilon),\\
T_2^2&=4\pi^2(I+\Delta I_2+I_W)/k(1+\epsilon),\\
T_3^2&=4\pi^2(I+\Delta I_3+I_W+I_R)/k(1+2\epsilon).
\end{aligned}\quad(2)$$

I_R is the calculated moment of inertia of the ring, I_W that of the weight, and k is the torsion constant of the suspension when supporting the weight of the cylinder alone. In these equations it is assumed that the change in the

TABLE IV. *Determinations of I.*

SERIES SUSPENSION	1 8 mil steel	2 5 mil steel	3 8 mil W	4 5 mil W	5 5 mil W
Ring No.	2	1	2	1	1
Temp.	20.0	20.5	21.0	22.5	22.2
T_0	35.408	89.496	26.6174	56.637	
T_1	43.910	101.169	33.0030	63.990	63.969
T_2	35.618	89.775	26.7680	56.779	56.763
T_3	44.100	101.436	33.1340	64.103	
ΔI	37.5	58.0	31.5	47.5	
ΔI	41.6	61.3	36.0	50.2	50.2
ΔI_2	37.7	58.1	31.5	47.5	47.5
ΔI_3	41.7	61.3	36.0	50.2	
I'	11857	11839	11839	11824	
I''	11837	11850	11826	11862	
I'''	11835	11830	11834	11844	11847
Mean I	11843	11840	11833	11843	11847
Uncertainty	18	14	9	16	

Mean $I = 11841 \pm 10$

torsion constant is proportional to the weight added, and also that the small difference between the weights of the rings and the corresponding compensating weights is negligible.

From Eq. (2) it follows that

$$\begin{aligned}(1+\epsilon)&=\{1+(I_W+\Delta I_2-\Delta I_0)/\\&\qquad(I+\Delta I_0)\}T_0^2/T_2^2,\\(1+\epsilon)&=\{1+(I_W+\Delta I_3-\Delta I_1)/\\&\qquad(I+I_R+\Delta I_1)\}T_1^2/T_3^2.\end{aligned}\tag{3}$$

An approximate value of I can be inserted in these equations to calculate $(I+\epsilon)$. The agreement of the two values obtained gives some indication of the consistency of the observations. The mean value was then used to determine I.

From Eqs. (2) it also follows that

$$\begin{aligned}I'&=T_0^2(I_R+\Delta I_1-\Delta I_0)/\\&\qquad[(1+\epsilon)T_1^2-T_0^2]-\Delta I_0,\\I''&=T_2^2(I+\Delta I_3-\Delta I_2)/\\&\qquad[(1+\epsilon)T_3^2-T_2^2]-I_W-\Delta I_2,\\I'''&=T_2^2(I_R+\Delta I_1-\Delta I_2-I_W)/\\&\qquad(T_1^2-T_2^2)-I_W-\Delta I_2.\end{aligned}\tag{4}$$

The differences among these three values of I reflect the difference between the two values of $(1+\epsilon)$. The root-mean-square deviation of these three values from their mean gives an estimate of the consistency of the four observed periods and of the reliability of the mean value of I.

Table III gives the calculated moments of inertia of the rings and the weights and Table IV gives the observed periods and the values of I determined from them. In series 5 observations were made of T_1 and T_2 only so that no value of the dispersion can be given. The temperatures indicate those at which the periods were taken, but no correction to the standard temperature of 22.0° was made since the correction is well within the uncertainty.

The uncertainty in I given in Table IV is based on the consistency of the different results only. In addition the uncertainty in I_R and I_W must be included. The fact that two different rings are used with no apparent systematic difference in the results indicates that the difference between the errors in the two rings is within the dispersion of the measurements. However to make some allowance for these uncertainties the value was taken as

$$I=11841\pm 0.10 \text{ percent gram cm}^2.$$

DETERMINATION OF D

The rotating cylinder was driven at constant speed by a reed controlled motor. The reed was enclosed in a thermostated box and its temperature did not vary over 0.2°. The angular velocity of the cylinder, when connected to the motor by gears reducing the speed by a factor of 120, was determined from a run of several hours to be $\Omega_0=0.034967\pm 0.002$ percent radians per second. In making observations several different gear ratios were used and the ratio of the angular velocity used to the one given above is tabulated in column 2 of Table V.

When the apparatus was assembled and the cylinder adjusted to be vertical and coincident with the guard cylinders a series of intermingled observations of P and θ were made. The observations of P were made with the apparatus

TABLE V. *Observations of D.*

P(22.0°)	Ω/Ω_0	TEMP.	θ	$D_T\times 10^5$	$D_{22}\times 10^5$	MEAN
275.42	9/14	21.7	0.032232	1.8912	1.8927	
	6/7	21.3	0.042935	1.8908	1.8943	1.8932
	1/1	21.6	0.050112	1.8905	1.8925	±0.0008
275.35	1/1	21.8	0.050168	1.8924	1.8934	
	4/3	21.8	0.066965	1.8942	1.8952	1.8941
	9/14	21.7	0.032260	1.8920	1.8935	±0.0008
	6/7	21.8	0.043045	1.8933	1.8943	
186.225	6/7	21.7	0.019642	1.8898	1.8913	
	9/14	21.9	0.014750	1.8922	1.8927	1.8909
	1/2	21.9	0.011448	1.8881	1.8886	±0.0017
186.049	1/1	22.0	0.022900	1.8916	1.8916	
	4/3	22.3	0.030563	1.8940	1.8925	1.8921
						±0.0005
186.070	4/3	22.3	0.030640	1.8974	1.8959	
	1/1	21.8	0.022935	1.8948	1.8958	1.8960
	6/7	22.2	0.019696	1.8978	1.8968	+0.0005
	4/3	23.3	0.030731	1.9020	1.8955	

Mean $D_{22}=1.8933\pm 0.0017$

evacuated since the moment of inertia was determined for the cylinder alone. The temperature was measured by means of four thermocouples at various points on the case of the apparatus. The whole was surrounded by an insulating housing to reduce the fluctuations in temperature, and although no great effort was made to control the temperature, it varied so slowly that the change during an observation of an hour was rarely as much as 0.1°C. The values of P were then plotted as a function of the temperature and the suitable value read from the curve for use with each value of the deflection. The value of P at 22.0° for each set of measurements is given in column 1 of Table V.

The air admitted to the apparatus was passed slowly through two tubes, a meter long, filled with calcium chloride and seemed to be adequately dried. To measure θ the motor was run first in one direction and then in the other. The equilibrium position for each direction was determined by observing a series of turning points as the cylinder oscillated through a small amplitude. Two observations of the deflection in each direction gave two values of the double deflection whose mean was then reduced to give the deflection of the cylinder. The scale readings were reduced to angular deflections by the suitable series expansion. The observed values of θ are given in column 4 of Table V and the corresponding temperatures are in the previous column.

The value of θ was then divided by the angular velocity of the outer cylinder and the square of the period at the given temperature to give the quantity D. Although D was observed at a number of temperatures, the temperature range was not sufficient to give a good value of the temperature coefficient. Hence the observed values were corrected to 22.0°C by the usual temperature coefficient of 0.00271 per degree.[6]

The results given in Table V are divided into five groups. The first two groups were made with a watch spring suspension of the type used by Day and Bleakney and the last three were made with a steel wire 0.003 in. in diameter. Each group represents a complete readjustment of the cylinders. The differences between the groups are somewhat larger than the dispersion within them and so are to be attributed to errors in the adjustment of the cylinders. According to Kellström an eccentric adjustment gives too low a value, so that the highest of the values should be considered the best. To make some allowance for this the adopted uncertainty in D was made fifty percent greater than that obtained from the dispersion of the values so that

$$D_{22}=1.8933\times10^{-5}\pm0.14 \text{ percent.}$$

Discussion of the Results

The combination of the adopted values of Q, I, and D given above gives

$$\eta_{22}=1.8243\pm0.25 \text{ percent}\times10^{-4} \text{ c.g.s. units.}$$

The combination of the uncertainties in the three quantities given above leads to 0.19 percent as the uncertainty of the result. To allow for other possibilities this is rounded off to 0.25 percent which is then to be regarded as an estimated limit of error.

With the usual temperature coefficient

$$\eta_{23}=1.8292\pm0.0045\times10^{-4} \text{ c.g.s. units}$$

which gives for the electronic charge as given by the oil drop experiments

$$e=4.796\times10^{-10} \text{ e.s.u.}$$

This value of η_{23} is a little lower than that given by Kellström although the difference is less than the sum of the two limits of error. It is not clear from Kellström's paper whether or not he applied the correction for the length of the cylinder. If this is applied to his result it becomes somewhat lower. Kellström also did not discuss the corrections to the moment of inertia due to the presence of air, but it appears from his description of the method used that this should be rather small and would tend to raise his value a little.

In conclusion I wish to acknowledge the assistance of Mr. F. C. Bennett who helped in making many of these observations.

[6] R. A. Millikan, Ann. d. Physik **41**, 759 (1913).

Appendix

Computation of end correction

The use of guard cylinders at each end of the suspended cylinder very much reduces the end effects and makes the end correction very small. Nevertheless a correction of the order of magnitude of the spacing between the inner cylinder and the guard cylinders must be added to the measured length. To determine the amount of this correction it is necessary to compute the excess of the torque on the inner cylinder over the torque computed in the simple theory.

Since the radius of the cylinders is large compared with the difference in radius it is sufficient to regard them as planes and to treat the problem in two dimensions. In the figure let AB represent a part of the inner surface of the outer cylinder, CD the end of the suspended cylinder and $EFGH$ the section of one guard cylinder. Since FG is about 3 mm and DF about 0.5 mm it is a sufficient approximation to prolong FG to FGJ. The thickness of the sheet CD is about 0.4 mm so it is treated as a thin sheet. The problem is now to find the motion of the fluid in the space surrounding these surfaces when CD and EFJ are fixed and AB is moving into the page with a constant velocity. When the motion of the fluid is known the force on the various surfaces can be found.

Under these circumstances all of the velocity is perpendicular to the page, and if v is the magnitude of the velocity at any point (x, y),

$$\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 = 0. \qquad (1')$$

This equation and the accompanying boundary conditions are just those for the distribution of electrical potential between conducting sheets when AB is at one potential and the other surfaces are at another.

When the distribution of v is known the force per unit area on one of the surfaces is the normal component of $(-\eta \operatorname{grad} v)$ and the total force is the surface integral of this quantity. To determine the correct value of l for use in Eq. (1) it is only necessary to find the point P such that the lines of "flux" (grad v) from BP go to EFJ and those from AP go to CD. If P were joined to J by a surface normal to the figure and coincident with the grad v it follows from Gauss' theorem that the force on BP is equal and opposite to that on EFJ, and the force on AP is equal and opposite to that on CD. If the guard cylinder and the suspended cylinder were identical in section it would follow from symmetry that P would lie just midway between D and F. However, since this symmetry does not exist the problem was solved by means of a conformal transformation.[7] This shows that the fraction of the spacing to be added depends upon the ratio of the distance DF to the distance DP. As DF/DP approaches zero the correction approached $DF/2$, and for the dimensions used in this work 0.47 DF must be added to each end of the swinging cylinder. Since the thickness of CD is really almost as great as DF the correct result is between 0.47 and 0.50.

[7] I am indebted to Professor W. R. Smythe for assistance in working out this transformation. The method is described in his book, *Static and Dynamic Electricity* published by Edwards Brothers.

Since the wall of the cylinder is very thin the whole force is applied effectively at the surface, so the excess torque is simply given by this addition to the length. In other cases allowance would have to be made for the point on the end of the cylincer at which the force was applied.

Effect of the air on the period of the cylinder

When the cylinder is rotating about its axis in air a certain amount of the air, both inside and outside, rotates

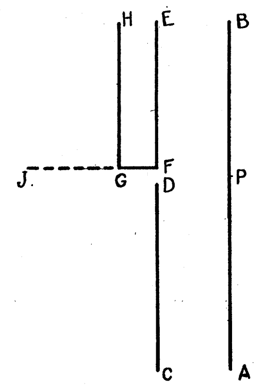

FIG. 1. Section at end of cylinder.

with it and contributes to the effective moment of inertia of the system. The correction necessary on account of this effect was computed by considering an infinite hollow cylinder. This is in error due both to the finite length of the cylinder and to the presence of the inside ribs but there seems to be no simple way to take these into account. Of course the moment of inertia could actually have been determined in vacuum but this presents considerable complication.

The treatment of problems such as this apparently goes back to Stokes[8] who gave a simple solution for an oscillating plane and worked out a number of more complicated cases. For a large enough cylinder and a short enough period of oscillation the approximation of a plane can be used. However, for the cylinder and the periods used in this measurement the error in such an approximation is considerable. I have been unable to find the cylindrical case treated in any of the standard works which I have consulted, and since it seems one which might have a number of applications I shall outline it here.

For the slow motion of a viscous fluid subject to no external volume forces, and in which the differences in pressure and density are negligible, the equation of motion is

$$\rho(\partial \mathbf{v}/\partial t) = \nabla^2 \mathbf{v}, \qquad (2')$$

where ρ is the density, η the viscosity, and $\mathbf{v}$ is the vector velocity of the fluid. For the problem in hand all the motion takes place in cylindrical laminae so that v is always perpendicular to the axis and to the radius. Expressed in cylindrical coordinates under these restrictions the equation is

$$\partial v/\partial t = (\eta/\rho)\{\partial^2 v/\partial r^2 + (1/r)\partial v/\partial r - v/r^2\}, \qquad (3')$$

[8] G. G. Stokes, Math. and Phys. Papers, vol. III.

where v is now the scalar velocity. The velocity at any distance from the axis will vary harmonically with the time so let

$$v = A(r)\sin(2\pi t/T) + B(r)\cos(2\pi t/T). \quad (4')$$

Then
$$\begin{aligned} d^2A/dr^2 + (1/r)dA/dr - A/r^2 + \lambda B &= 0,\\ d^2B/dr^2 + (1/r)dB/dr - B/r^2 - \lambda A &= 0, \end{aligned} \quad (5')$$

where $\lambda = 2\pi\rho/\eta T$.

The general solution of these equations is

$$\begin{aligned} A &= a\, ber_1(x) - b\, bei_1(x) + c\, ker_1(x) - d\, kei_1(x),\\ B &= a\, ber_1(x) + b\, bei_1(x) + c\, ker_1(x) + d\, kei_1(x), \end{aligned} \quad (6')$$

where $x = \lambda^{\frac{1}{2}} r$. For numerical computation it is a little more convenient to have these expressed in terms of the functions with the index zero, so by using the recursion formulas one obtained

$$\begin{aligned} A &= \alpha\, ber'(x) - \beta\, bei'(x) + \gamma\, ker'(x) - \delta\, kei'(x),\\ B &= \beta\, ber'(x) + \alpha\, bei'(x) + \delta\, ker'(x) + \gamma\, kei'(x). \end{aligned} \quad (7')$$

The torque on length l of the cylinder due to the fluid is

$$L = \pm 2\pi l R^2 \eta \{(dA/dr - A/r)\sin(2\pi t/T) + (dB/dr - B/r)\cos(2\pi t/T)\}. \quad (8')$$

The upper sign is to be used for the fluid outside the cylinder and the lower sign for that inside it. If θ is the angular displacement of the cylinder, let

$$\ddot{\theta} = (2\pi/RT)A_0\cos(2\pi t/T), \qquad \dot{\theta} = (A_0/R)\sin(2\pi t/T),$$

where R is the radius of the cylinder. The equation of motion of the cylinder is

$$\{I \mp (2\pi\rho l R^4/\lambda^{\frac{1}{2}} R A_0)(dB/dx - B/x)_{x=\lambda^{\frac{1}{2}}R}\}\ddot{\theta} + \{p \mp (2\pi l R^3 \eta \lambda^{\frac{1}{2}}/A_0)(dA/dx - A/x)_{x=\lambda^{\frac{1}{2}}R}\}\dot{\theta} + k\theta = 0, \quad (9')$$

in which p represents the damping constant due to sources other than the fluid in which the cylinder is immersed.

From Eq. (9′) it is evident that the effect of the fluid is to add an additional moment of inertia to that of the cylinder and to increase the damping constant. Since only the correction to the moment of inertia is of immediate interest, the final expressions will be given for it only. The damping can be worked out in the various cases by similar methods.

In applying the boundary conditions it is convenient to consider three cases.

Case I. The fluid is inside the cylinder only.—In this case the velocity must vanish at the origin and so both A and B must vanish. Since $ker'(x) \rightarrow \infty$ as $x \rightarrow 0$, γ and δ must both be zero in this case.

Let $X = \lambda^{\frac{1}{2}} R$ where R is the radius of the inside of the cylinder. At $r = R$, $B(X) = 0$, and $A(X) = A_0$. Hence

$$\alpha = \frac{A_0\, ber'(X)}{ber'^2(X) + bei'^2(X)}, \qquad \beta = -\frac{A_0\, bei'(X)}{ber'^2(X) + bei'^2(X)}. \quad (10')$$

From the equations which they satisfy it follows that

$$(d/dx)\, ber'(x) - ber'(x)/x = bei(x),$$
$$(d/dx)\, bei'(x) - bei'(x)/x = -ber(x).$$

Using these expressions in Eq. (9′) it follows that

$$\Delta I = (\pi\rho l R^4/2) F(X)/X, \quad (11')$$

where
$$F(X) = 4\frac{ber'(X)\, ber(X) + bei'(X)\, bei(X)}{ber'^2(X) + bei'^2(X)}.$$

The coefficient of $F(X)/X$ in Eq. (11′) is just the moment of inertia of the fluid inside the cylinder if it were moving as a solid. This is its motion in the limit in which $X \rightarrow 0$. This limit can be approached by a low density, a high viscosity, a long period, or a small radius of the cylinder. The other limit in which $X \rightarrow \infty$ is that in which the approximation of the cylinder by a plane is justified. In this limit $F(X) \rightarrow 8^{\frac{1}{2}}$.

Case II. The fluid is outside the cylinder and extends to infinity.—In this case both A and B must vanish at infinity and B vanishes at $x = X$. Since $ber'(x)$ and $bei'(x)$ both go to infinity, α and β are both zero. These conditions give

$$\gamma = \frac{A_0 ker'(X)}{ker'^2(X) + kei'^2(X)}, \qquad \delta = \frac{-A_0 kei'(X)}{ker'^2(X) + kei'^2(X)}, \quad (12')$$

$$\Delta I = (\pi\rho l R^4/2) G(X)/X,$$

$$G(X) = -4\frac{ker'(X)\, ker(X) + kei'(X)\, kei(X)}{ker'(X) + kei'(X)}. \quad (13')$$

A few of the values of $F(X)$ and $G(X)$ which lie in the range needed for the moment of inertia determinations are given in Table VI.

TABLE VI. *Values of F(X) and G(X).*

X	$F(X)$	$G(X)$
0.0	X	
1.0	0.997	2.463
2.0	1.922	2.684
3.0	2.536	2.752
4.0	2.745	2.781
5.0	2.780	2.794
6.0	2.791	2.805
7.0	2.802	2.811
∞	2.828	2.828

Case III. Fluid enclosed between concentric cylinders.—In this case all four coefficients will be different from zero and must be determined by the solution of four simultaneous equations. Although laborious this solution presents no difficulties.

Since when the moment of inertia of the cylinder was determined the cylinder was enclosed in a house, this third case might have been of importance. However for the periods used it differed only negligibly from case II. With the dimensions of the cylinder used and approximate values for the density and viscosity of air $X = 34.0/T^{\frac{1}{2}}$, and $\Delta I = 152.5\{F(X) + G(X)\}/X$.