Theory of the Townsend Method of Measuring Electron Diffusion and Mobility

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The Lorentz method is applied to the combined drift and diffusion of the electron stream in the Townsend experiment. The zeroth-order approximation is found to lead to the equation found by Townsend. It is shown how the next approximation is obtained. The same method is then applied to the drift of electrons in crossed electric and magnetic fields. It is found that the magnetic field changes the energy distribution as well as the direction of drift. The latter is found in close agreement with Townsend, but not with Huxley.

Townsend[1] measures the random motion of electrons in gases under the influence of an electric field E by means of an apparatus sketched in Fig. 1. Electrons liberated from the plate P pass through a circular hole at A and on down to a circular electrode C_2 with a ring electrode C_1C_3 around it. Potentials are applied to the various parts of the apparatus so as to insure a uniform field E from P to C. The ratio of the current to C_1C_3 to that to C_2 determines the ratio of the diffusion constant D to the drift velocity u_E, and hence the temperature T by the relation

$$T = eED/ku_E. \tag{1}$$

The drift velocity u_E is measured in a similar apparatus in which the circular hole is replaced by a slit and the electrodes C_1, C_2, C_3 are sections of a disk. The electron stream is deviated

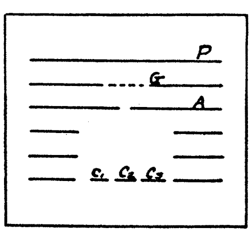

Fig. 1. Arrangement of electrodes used by Townsend for measuring random motion of electrons in gases.

by a magnetic field H directed parallel to the slit, and the angle of deviation is given by

$$\tan\theta = Hu/E. \tag{2}$$

The derivation of these relations is, however, not free from objections. The existence of a "temperature" implies a Maxwell distribution of velocities, yet it is known that the distribution cannot be Maxwellian. Druyvesteyn[2] has derived a distribution similar to the Maxwellian but with the energy squared, instead of to the first power, in the exponent.

The authors have therefore applied to the theory of these experiments the method originated by Lorentz[3] and extended by Morse, Allis and Lamar[4] to give both the drift velocities and the energy distribution. The method is superior to "mean free path" methods in that one does not average over velocities until the very last step, and hence can easily handle any velocity distribution or mean free paths which are a function of the velocity. Townsend[5] has recently tried to avoid the difficulty by considering groups of electrons of the same energy, but this is difficult as the groups do not stay together. The Lorentz method considers the electrons entering and leaving an element of volume fixed in phase space. It therefore resembles the Euler method in hydrodynamics whereas Townsend follows Lagrange.

The as yet unknown distribution f is expanded in surface harmonics in velocity space and only the linear terms are kept. This is well justified, as random motions are always much larger than the drift velocity. The thermal motions of the gas atoms are neglected compared to the motions of the electrons. Davydov[6] has shown how to correct for the thermal motions if it is necessary to do so. Collisions with atoms are considered to be elastic. This imposes an upper limit on the parameter E/p which will be investigated in the following paper by one of

[1] Townsend and Tizard, Proc. Roy. Soc. **88**, 336 (1913); Townsend and Bailey, Phil. Mag. **42**, 873 (1921).

[2] M. J. Druyvesteyn, Physica **10**, 61 (1934).

[3] H. A. Lorentz. *Theory of Electrons*, p. 269.

[4] Morse, Allis and Lamar, Phys. Rev. **48**, 412 (1935).

[5] J. S. E. Townsend, Phil. Mag. **22**, 145 (1936).

[6] D. Davydov, Physik. Zeits. Sowjetunion **8**, 59 (1935).

us. The probability of elastic scattering, $\sigma(v, \Theta)$ is, however, an arbitrary function of angle and velocity. The cross section for momentum transfer is then defined by

$$Q(v)=\int_0^\pi 2\pi(1-\cos\Theta)\sigma(v,\Theta)\sin\Theta d\Theta \quad (3)$$

and the mean free path by

$$\lambda(v, p)=1/NQ. \quad (4)$$

Interactions between electrons are neglected, and hence the theory applies only to electron densities which are very small in comparison to that of the gas.

The two cases of a circular hole and a slit at A are treated quite separately as cylindrical coordinates are appropriate to the first and cartesian coordinates to the second.

Mobility and Diffusion Coefficients

As a first approximation we shall neglect the diffusion sideways and assume the energy distribution to be that for the homogeneous case[7]

$$f_0=A\exp\left[-2\mu\int_0^\epsilon \epsilon d\epsilon/\epsilon_0^2\right], \quad (5)$$

where $\mu=3m/M$, $\epsilon_0=\lambda eE$, $\epsilon=\frac{1}{2}mv^2$ and the electron density

$$n=4\pi\int_0^\infty f_0 v^2 dv. \quad (6)$$

Eq. (7) of Morse, Allis and Lamar can then be applied directly, giving

$$f_1=(2\mu\epsilon/\epsilon_0)f_0-(\lambda/n)(dn/dz)f_0, \quad (7)$$

where we are using z instead of x. The drift velocity is then given by

$$u=\frac{1}{n}\int\frac{\dot{z}^2}{v}f_1 d\gamma=\frac{8\pi}{3nm^2}\times\left[\int\frac{2\mu\epsilon^2}{\epsilon_0}f_0 d\epsilon-\frac{1}{n}\frac{dn}{dz}\int\lambda\epsilon f_0 d\epsilon\right]=u_E-(1/n)(dn/dz)D, \quad (8)$$

where u_E is the drift due to the field and D the diffusion constant. If the mean free path is constant, the two integrals are easily evaluated and give

$$u_E=\frac{\mu^{\frac{1}{4}}}{3\Gamma(3/4)}\left(\frac{2\pi\epsilon_0}{m}\right)^{\frac{1}{2}},\qquad D=\frac{\lambda}{3\Gamma(3/4)\mu^{\frac{1}{4}}}\left(\frac{2\epsilon_0}{m}\right)^{\frac{1}{2}}. \quad (9)$$

The cases of greatest interest, however, are those where λ is not constant and the above quantities must be determined by integrating.

Energy Distribution

Let r, θ, z be cylindrical coordinates, $\dot{r}$, $r\dot{\theta}$, $\dot{z}$ the corresponding velocities. Following Lorentz, we have

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\dot{r}\frac{\partial f}{\partial r}+\dot{\theta}\frac{\partial f}{\partial\theta}+\dot{z}\frac{\partial f}{\partial z}+\ddot{r}\frac{\partial f}{\partial\dot{r}}+\ddot{\theta}\frac{\partial f}{\partial\dot{\theta}}+\ddot{z}\frac{\partial f}{\partial\dot{z}}=b-a. \quad (10)$$

The first, third, and sixth term in the above expression vanish because we are considering a stationary state with cylindrical symmetry. In the fifth and seventh terms

$$\ddot{r}=r\dot{\theta}^2 \quad\text{and}\quad \ddot{z}=eE/m, \quad (11)$$

but the nature of these two accelerations is entirely different. The second is due to the applied electric field and takes place independently of the orthogonal velocity components $\dot{r}$ and $r\dot{\theta}$ and these therefore have to be kept constant in the partial derivative. On the other hand the radial acceleration is due to the curvature of the coordinate lines and takes place as the electron moves uniformly along a straight line. The derivative is therefore to be taken with v constant.

[7] Morse, Allis and Lamar, Eq. (14) which is incorrectly written.

$$\frac{df}{dt}=\dot{r}\frac{\partial f}{\partial r}+\dot{z}\frac{\partial f}{\partial z}+r\dot{\theta}^2\left(\frac{\partial f}{\partial \dot{r}}\right)_v+\frac{eE}{m}\left(\frac{\partial f}{\partial \dot{z}}\right)_{\dot{r},\,\dot{\theta}}=b-a. \tag{12}$$

We now substitute

$$f=f_0+(\dot{z}/v)f_1+(\dot{r}/v)f_2 \tag{13}$$

and replace the resulting quadratic terms by their average values

$$\langle\dot{z}^2\rangle_{\mathrm{Av}}=\langle\dot{r}^2\rangle_{\mathrm{Av}}=r^2\langle\dot{\theta}^2\rangle_{\mathrm{Av}}=v^2/3,\quad \langle\dot{z}\dot{r}\rangle_{\mathrm{Av}}=0 \tag{14}$$

$b-a$ can readily be obtained from Morse, Allis and Lamar by an obvious extension of their Eq. (6a). Equating separately the terms in v alone, in $\dot{z}$, and in $\dot{r}$ gives

$$\frac{\partial f_1}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(rf_2)+\frac{eE}{mv^3}\frac{\partial}{\partial r}(v^2 f_1)=\frac{\mu}{v^3}\frac{\partial}{\partial v}\left(\frac{v^4 f_0}{\lambda}\right),\quad \frac{\partial f_0}{\partial z}+\frac{eE}{mv}\frac{\partial f_0}{\partial v}=-\frac{f_1}{\lambda},\quad \frac{\partial f_0}{\partial r}=-\frac{f_2}{\lambda}. \tag{15}$$

It is convenient now to use energy units

$$\zeta=zeE,\quad \rho=reE,$$

and further to substitute as independent variable $\eta=\epsilon-\zeta$ in place of ζ.

The above equations then become

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho f_2)+\frac{1}{\epsilon}\frac{\partial}{\partial\epsilon}(\epsilon f_1)=\frac{2\mu}{\epsilon}\frac{\partial}{\partial\epsilon}\left(\frac{\epsilon^2 f_0}{\epsilon_0}\right)+2\mu\frac{\epsilon}{\epsilon_0}\frac{\partial f_0}{\partial\eta},\quad \partial f_0/\partial\epsilon=-f_1/\epsilon_0,\quad \partial f_0/\partial\rho=-f_2/\epsilon_0. \tag{16}$$

It is noticed that the variable η which has replaced z no longer appears on the left-hand sides with the diffusion terms, but enters only in the term arising from the energy loss in collisions. If there were no energy losses, η, the total energy, would be a constant of the motion and hence fixed by the initial conditions and have no place in the differential equation.

It is now easy to eliminate f_1 and f_2 and separate the variables

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f_0}{\partial\rho}\right)+\frac{1}{\epsilon_0\epsilon}\frac{\partial}{\partial\epsilon}\left(\epsilon_0\epsilon\frac{\partial f_0}{\partial\epsilon}\right)+\frac{2\mu}{\epsilon_0\epsilon}\frac{\partial}{\partial\epsilon}\left(\frac{\epsilon^2 f_0}{\epsilon_0}\right)+\frac{2\mu\epsilon}{\epsilon_0{}^2}\frac{\partial f_0}{\partial\eta}=0. \tag{17}$$

Set

$$f_0=R(\rho)Z(\eta)E(\epsilon), \tag{18}$$

$$(1/\rho)(\partial/\partial\rho)(\rho\partial R/\partial\rho)=-a^2R,\quad R=J_0(a\rho),\quad \partial Z/\partial\eta=bZ,\quad Z=e^{b\eta}=e^{b(\epsilon-\zeta)}, \tag{19}$$

where a and b are the separation constants to be determined by the boundary conditions and the equation for $E(\epsilon)$ is

$$\frac{1}{\epsilon_0\epsilon}\frac{\partial}{\partial\epsilon}\left(\epsilon_0\epsilon\frac{\partial E}{\partial\epsilon}+\frac{2\mu\epsilon^2}{\epsilon_0}E\right)+\left(\frac{2b\mu\epsilon}{\epsilon_0{}^2}-a^2\right)E=0.$$

Substituting

$$E=\exp\left[-b\epsilon-2\mu\int_0^\epsilon \epsilon d\epsilon/\epsilon_0{}^2\right]G(\epsilon), \tag{20}$$

$$G''+\left(\frac{1}{\epsilon}+\frac{\epsilon_0'}{\epsilon_0}-2b-\frac{2\mu\epsilon}{\epsilon_0{}^2}\right)G'+\left(b^2-a^2-\frac{b}{\epsilon}-\frac{b\epsilon_0'}{\epsilon_0}+\frac{4b\mu\epsilon}{\epsilon_0{}^2}\right)G=0, \tag{21}$$

and finally the solution is expressed as a series

$$f_0=\exp\left[-2\mu\int\epsilon d\epsilon/\epsilon_0{}^2\right]\sum_i A_iJ_0(a_i\rho)\sum_k B_{ik}\exp\left[-b_{ik}\zeta\right]G_{ik}(\epsilon), \tag{22}$$

where the constants are determined by the boundary conditions at the plane $z=0$.

$$\sum B_{ik}G_{ik}(\epsilon)=1, \quad \sum_i A_iJ_0(a_i\rho)=\begin{cases} A & \text{for } \rho<\rho_0 \text{ the radius of the hole,} \\ 0 & \text{for } \rho>\rho_0. \end{cases} \tag{23}$$

The equation for G is, however, not one that has been studied and therefore it would be difficult to determine the coefficients B_{ik} even if the characteristic values b_{ik} were known. It is better therefore to determine a single mean value $\bar{b}_i$ corresponding to each Bessel function $J_0(a_i\rho)$ and then solve Eq. (21) by power series or asymptotic series. The resulting function f_0 will satisfy the equation exactly but will not give quite the correct energy distribution at the hole.

To determine the best value for b one can first assume the energy distribution to be the same everywhere and given by Eq. (5). Townsend's[8] equation, converted to our notations, is then correct

$$\bar{b}_i^2+u\bar{b}_i/eED=a_i^2 \tag{24}$$

and u and D are given by Eq. (8). Solving for $\bar{b}_i$ and substituting in (21) one finds how much the energy distribution differs from that assumed, and substituting in (22) without summing over k one finds how the energy distribution varies from point to point, in first approximation.

Magnetic Deflection

We shall not attempt here the complete theory of the experiment with the magnetic field, but simply that of crossed electric and magnetic fields, leaving out diffusion. Using Cartesian coordinates and taking E and H along the x and z axes respectively, the problem reduces to the $x-y$ plane only. We have

$$\ddot{x}(\partial f/\partial \dot{x})+\ddot{y}(\partial f/\partial \dot{y})=b-a \tag{25}$$

with

$$\ddot{x}=(eE/m)+(eH/m)\dot{y}, \qquad \ddot{y}=-(eH/m)\dot{x}. \tag{26}$$

Setting

$$f=f_0+(\dot{x}/v)f_1+(\dot{y}/v)f_2 \tag{27}$$

and proceeding as before one obtains

$$\frac{eE}{mv^3}\frac{\partial}{\partial v}(v^2f_1)=\frac{\mu}{v^3}\frac{\partial}{\partial v}\left(\frac{v^4f_0}{\lambda}\right), \tag{28}$$

$$\frac{eE}{mv}\frac{\partial f_0}{\partial v}-\frac{eH}{mv}f_2=-\frac{f_1}{\lambda}, \tag{29}$$

$$(eH/mv)f_1=-f_2/\lambda. \tag{30}$$

The last two equations correspond to Huxley's[9] Eq. (24). Huxley has, however, already averaged over all velocities. Defining θ by

$$\tan\theta=f_2/f_1$$

and following Huxley one gets

$$\frac{f_1}{\partial f_0/\partial v}=-(E/2H)\sin 2\theta.$$

Averaging over velocities now does not lead to Huxley's Eq. (25).

Equations (28), (29), (30) are easily integrated and give

$$\log f_0=-\int\frac{2\mu\epsilon}{\epsilon_0^2}d\epsilon-\frac{\mu\epsilon}{m}\frac{H^2}{E^2}, \tag{31}$$

$$f_1=\frac{2\mu\epsilon}{\epsilon_0}f_0, \tag{32}$$

$$f_2=\frac{H\epsilon_0}{Emv}f_1=-\mu\frac{Hv}{E}f_0. \tag{33}$$

Eq. (31) shows that the magnetic field reduces the number of high velocity electrons. Eq. (32) shows that the drift down the electric field of electrons of a given energy is not altered by the magnetic field. This is, however, because of the change in the energy distribution. If f_0 were not allowed to change, (32) would not be true. Eq. (33) shows that the sideways drift increases with v. Comparing (32) and (33) shows that $\tan\theta$ varies inversely as v, so that the high velocity electrons are relatively little affected.

The average drift velocities are given by

$$u_x=\frac{1}{n}\int\frac{\dot{x}^2}{v}f_1d\gamma=\frac{2\pi}{3n}\frac{8\mu}{m^2}\int\frac{\epsilon^2}{\epsilon_0}f_0d\epsilon, \qquad u_y=\frac{1}{n}\int\frac{\dot{y}^2}{v}f_2d\gamma=\frac{4\pi}{3n}\mu\frac{H}{E}\int v^4f_0dv. \tag{34}$$

[8] J. J. Thomson and G. P. Thomson, *Conduction of Electricity through Gases*, p. 82.

[9] L. G. H. Huxley, Phil. Mag. 23, 226 (1937).

These integrals are easily evaluated as power series in $\sqrt{\mu}(Hv_0/E)^2$, where $\frac{1}{2}mv_0^2=\lambda eE$, and give, in terms of the drift velocity without the magnetic field u_E

$$u_x=u_E[1-0.195(\mu)^{\frac{1}{2}}(Hv_0/E)^2+\cdots], \quad (35)$$

$$\frac{u_y}{u_x}=\frac{3}{2(2)^{\frac{1}{2}}}\frac{Hu_E}{E}=1.06\frac{Hu_E}{E}. \quad (36)$$

It is this last formula which Townsend used to measure u_E and it is seen that the numerical factor differs insignificantly from his value 1.

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Electron Temperatures and Mobilities in the Rare Gases

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The results of the theory developed by Allis and Allen are applied to the computation of electron "temperatures" and drift velocities in the rare gases; helium, neon and argon. Curves are given showing the effect of the variable cross sections on the distribution function in the three cases. A distribution function is derived to account for energy lost by inelastic impact at higher values of E/p. This distribution function depends on an adjustable parameter, ϵ_1 which has a value between the first resonance potential and the ionization potential, since it is assumed that the number of electrons with energies above ϵ_1 is negligible. Curves are given comparing the computed values of electron "temperatures" and drift velocities with experimental values. In most cases the check is good.

1. Elastic Collisions Only

The rare gases helium, neon, and argon offer an excellent opportunity to check the results of the theory developed in the previous paper by Allis and Allen, of the diffusion and drift velocity of electrons in these gases. The angular distribution of electrons scattered from these gases has been measured down to very low velocities by Ramsauer and Kollath[1] and by Normand[2] and these measurements allow the computation of cross sections for momentum transfer Q. It is found that helium has a falling curve, neon one which is practically flat, while argon has one which rises sharply (Fig. 1). The energy distribution in these three gases will consequently vary greatly on this account as well as because of the greatly differing masses. The argon curve cuts off more sharply on the high energy side. Furthermore the drift velocity u_E and the diffusion coefficient D correspond to the averages of $(\mu/\lambda E)v^3$ and λv, respectively; $\mu=3m/M$, three times the ratio of the masses of the electron and the gas molecule, λ is the mean free path, $1/NQ$. The quantity which is given by Townsend[3] as a result of his measurements is

$$T=eED/ku_E \quad (1)$$

and would be the temperature if the electrons had a Maxwell distribution. In the above expression e is the electron charge, E the field strength and k the Boltzmann constant. Formulae (9) of Allis and Allen show that for a constant cross section T should be proportional to $E\lambda/\mu^{\frac{1}{2}}$ but when the cross section varies it is

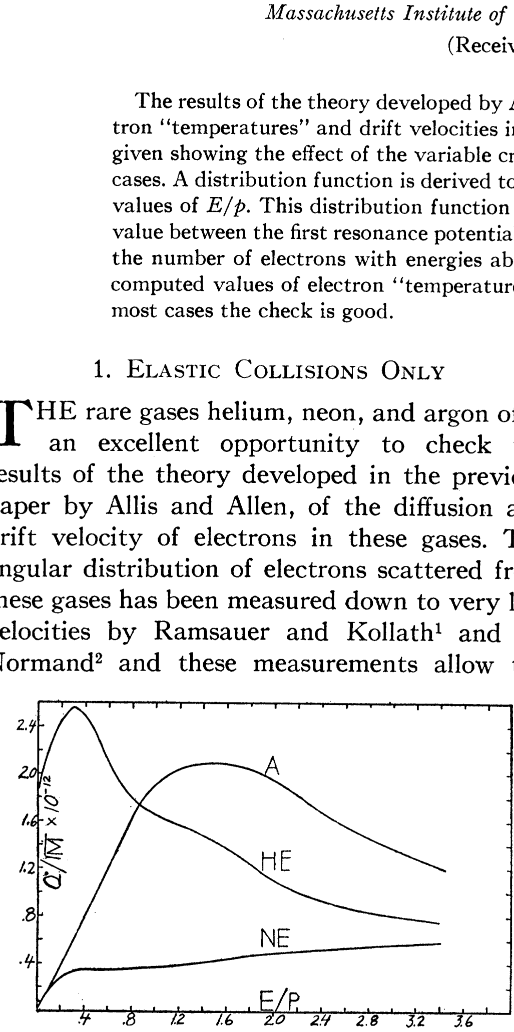

FIG. 1. Cross section in sq. A divided by the square root of the mass. This is the quantity which determines the mean energy $\bar{\epsilon}$ of the electrons.

[1] C. Ramsauer and R. Kollath, Ann. d. Physik (5) **12**, 529 (1932).

[2] C. E. Normand, Phys. Rev. **35**, 1217 (1930).

[3] J. S. Townsend, *Electricity in Gases* (Clarendon Press, Oxford, 1915).