values of 0.28 and 0.57 Mev, for the two energy ranges. All the possible sources of error pointed out in the foregoing discussion, however, are in the direction of tending to favor the electrons which have lost the least energy. Therefore it is probable that the real values are somewhat higher than our experimental values, which would make them differ still more from the theoretical values. On the other hand, it has been pointed out to us by Professor H. A. Bethe that the use of the Born approximation for lead may give theoretical values which are as much as 30 percent too low. We believe that the largest uncertainty in our experiment lies in the estimation of the average length of path of the electrons in the absorber. We have had valuable discussions on this point with Dr. M. E. Rose of Cornell University, who is making some calculations[6] on the path length-thickness ratio in this energy region. Dr. Rose suggested that the ratio which would be necessary to bring our results into agreement with theory might be well within the calculated limits.

The large discrepancy between the results reported here and those of Skobeltzyn and Stepanowa, Leprince-Ringuet and of Klarmann and Bothe should not be considered as a direct contradiction, because the methods used are by no means the same. The most important difference in the various experiments lies in the treatment of those electrons which appear to lose their entire energy in the absorbing material. We have excluded tracks of this class on the assumption that the great majority of them do not represent actual energy loss, but scattering out of the field of vision or stopping by a separate process. At the same time, we do not wish to lose sight of the possibility that the theory may be quite wrong in the region in which the electron loses nearly its entire energy in a single event, and that these cases should rightfully be included in the data. If so, the value for the average energy loss will be much higher. There remains also the possibility, as suggested by several authors, that this complete stopping may be a new process which is separate from radiative stopping in the ordinary sense.

Investigations similar to these are now under way on the absorption of the beta-rays from Li^8, which will extend the data up to about 11 Mev. These results will follow shortly in a separate publication.

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[6] M. E. Rose, Abstract, Washington Meeting, 1937.

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Absorption Coefficients for Thermal Neutrons

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A discussion is given of several integrals arising in the interpretation of experiments on the absorption of thermal neutrons in a $1/V$ absorber. In order to evaluate these integrals they have been expressed in the form of convenient series. There is also given a table of numerical values, which should be useful in the interpretation of experimental data on slow neutron absorption.

In studying the absorption of slow (thermal) neutrons one has in general to deal with neutrons having a given angular distribution and approximately a Maxwellian energy distribution, and with absorbers whose coefficient of absorption is a function of the relative energy of the neutrons with respect to the individual absorbing nuclei. If the scattering cross section is of importance, as in the case of neutron-proton collisions, the general problem (of the absorption and back-scattering of the a layer of absorber-scatterer) is complicated by the scattering; but if the scattering cross section is negligible in comparison with the absorption cross section, then the neutrons may be considered as moving in straight lines without deviation until they are finally absorbed. The latter condition is apparently very nearly valid[1] for the capture of neutrons by the heavier nuclei. If the absolute value of the resonance energy for neutron capture is large compared to the average thermal energy, the absorption coefficient may be regarded as inversely proportional to the square root of the relative energy ($1/V$ absorption); but if, as in the case of cadmium, the thermal and resonance regions overlap appreciably, one must consider the general form of the Breit-Wigner formula for neutron capture. Presumably most cases of neutron capture fall in the class of $1/V$ absorption.

As is well known, in the case of $1/V$ absorption the average absorption cross section for neutrons is quite independent of the angular and energy distributions of the absorbing nuclei, and therefore independent of the temperature of the absorber,—and in fact equal to the value obtained by regarding the absorbing nuclei at rest relative to their center of gravity and replacing the relative velocity in the $1/V$ formula by the actual velocity of the neutrons relative to the absorber as a whole.

Several types of absorption problem have arisen in the interpretation of experimental data on slow neutrons:

I. Absorption of a parallel beam of neutrons from a Maxwellian distribution, in a $1/V$ absorber.[2]

II. Absorption of the same beam by a layer of $1/V$ absorber, and subsequent detection by a $1/V$ detector.[3]

III. Absorption of neutrons having an isotropic angular distribution and a Maxwelliam energy distribution.[4]

IV. Absorption of neutrons emerging from the surface of a scatterer such as paraffin or water.[5]

V. Detection of the latter beam by a $1/V$ detector.

In the interpretation of data obtained in connection with the above types of experiment there have appeared in the literature two definite integrals,[2, 3] which apparently cannot be expressed in closed form, and which have been evaluated presumably by numerical integration. It is proposed here to show that these integrals, together with some other useful related integrals, can be expressed in terms of a single integral, which in turn can be evaluated by means of a convenient series,—and to give for the various integrals a table of values, which it is hoped may be found useful in the interpretation of absorption data.

Case I

Parallel beam with Maxwellian distribution

The energy distribution may be expressed as $ye^{-y}dy$, where y is the energy measured in units of kT, and T is the "temperature" of the beam. (This distribution gives the relative numbers of neutrons arising from a Maxwellian distribution and falling on a given area per second, in contradistinction to the ordinary Maxwellian distribution for the number in a given volume.) If the absorption coefficient of the $1/V$ absorber is given by $K=A/E^{\frac{1}{2}}=A/(kTy)^{\frac{1}{2}}$, and the thickness of the absorbing layer is t, then the fraction of neutrons transmitted is given by:

$$\varphi_1=\int_0^\infty ye^{-y}e^{-At/(kTy)^{\frac{1}{2}}}dy\equiv\int_0^\infty ye^{-y-x/y^{\frac{1}{2}}}dy$$

(where $x\equiv At/(kT)^{\frac{1}{2}}$).

Case II

If the latter neutrons are detected by a *thin* layer of a $1/V$ detector, the observed activity will be proportional to:

$$\varphi_{\frac{1}{2}}=\int_0^\infty y^{\frac{1}{2}}e^{-y-x/y^{\frac{1}{2}}}dy.$$

Case III

Neutrons with an isotropic angular distribution and a Maxwellian energy distribution

For an isotropic angular distribution of neutrons of temperature T the fraction trans-

[1] Cf. G. Breit and E. Wigner, Phys. Rev. **49**, 519 (1936).
[2] D. P. Mitchell, J. R. Dunning, E. Segrè, and G. B. Pegram, Phys. Rev. **48**, 774 (1935); D. P. Mitchell, Phys. Rev. **49**, 453 (1936).
[3] H. H. Goldsmith and F. Rasetti, Phys. Rev. **50**, 328 (1936).
[4] E. Amaldi and E. Fermi, Ricerca Scient. II, VII, I, 3 (1936); E. Segrè, Ricerca Scient. II, VII, I, 9–10 (1936); D. S. Bayley, B. R. Curtis, E. R. Gaerttner, and S. Goudsmit, Phys. Rev. **50**, 570 (1936); C. T. Zahn, E. L. Harrington, and S. Goudsmit, Phys. Rev. **50**, 570 (1936).
[5] E. Fermi, Ricerca Scient. II, VII, II, 3 (1936).

mitted through a foil is given by:

$$\varphi_0=\int_{y=0}^{\infty}\int_{\theta=0}^{\pi/2} ye^{-y-x/y^{\frac{1}{2}}\cos\theta}2\sin\theta\cos\theta dyd\theta.$$

By means of the transformation $z=y\cos^2\theta$ it can be shown that:

$$\varphi_0=\int_0^{\infty} e^{-y-x/y^{\frac{1}{2}}}dy.$$

Hence it is seen that all three of the above integrals belong to the class:

$$\varphi_n\equiv\int_0^{\infty} y^n e^{-y-x/y^{\frac{1}{2}}}dy.$$

(The effect of changing from a parallel beam to an isotropic distribution is simply to reduce the exponent n from 1 to 0.)

In order to evaluate these integrals in series it is convenient first to evaluate φ_0, and then to derive the others from φ_0 by integration. Integrating by parts it is easily seen that:

$$\varphi_n=n\varphi_{n-1}+\tfrac{1}{2}x\varphi_{n-1\frac{1}{2}}$$

and since $d^m\varphi_n/dx^m=(-1)^m\varphi_{n-\frac{1}{2}m}$, it follows that φ_0 must satisfy the differential equation:

$$d^3\varphi_0/dx^3=-2\varphi_0/x.$$

The latter equation can be solved in the form:

$$\varphi_0=\sum_{n=0}^{\infty}(a_n\log x+b_n)x^n,$$

where
$$a_n=\frac{-2a_{n-2}}{n(n-1)(n-2)};$$

$$b_n=\frac{-2b_{n-2}-(3n^2-6n+2)a_{n-2}}{n(n-1)(n-2)}$$

$$a_0=a_1=0\quad(a_2=-b_0)$$

and b_0, b_1, and b_2 are taken as the three necessary arbitrary constants. The boundary conditions give immediately that $b_0=1$ and $b_1=-\pi^{\frac{1}{2}}$ (by direct integration of the expressions for φ_0 and $d\varphi_0/dx$ after setting $x=0$). The evaluation of b_2 requires special consideration, which for the sake of brevity is not included here. It may be shown that $b_2=\frac{3}{2}(1-C)$, where C is Euler's constant. This value of b_2 was checked by numerical integration, for $x=1$, of the expression

$$\varphi_0=\int_0^1 e^{-x/\log^{\frac{1}{2}}(1/z)}dz.$$

In this way one obtains:

$$\begin{aligned}\varphi_0=1&-\pi^{\frac{1}{2}}x+0.6342x^2+0.5908x^3-0.1431x^4\\&-0.01968x^5+0.00324x^6+0.000188x^7\cdots\\&-x^2\log x(1-0.08333x^2+0.001389x^4\\&\qquad-0.0000083x^6\cdots)\end{aligned}$$

and by direct integration and appropriate choice of integration constants:

$$\begin{aligned}\varphi_{\frac{1}{2}}=&\frac{\pi^{\frac{1}{2}}}{2}-x+\frac{\pi^{\frac{1}{2}}}{2}x^2-0.3225x^3-0.1477x^4\\&+0.03195x^5+0.00328x^6\\&-0.000491x^7-0.0000235x^8\cdots\\&\quad+x^3\log x(\tfrac{1}{3}-0.01667x^2+0.000198x^4\cdots)\end{aligned}$$

and

$$\begin{aligned}\varphi_1=1&-\frac{\pi^{\frac{1}{2}}}{2}x+\frac{x^2}{2}-0.2954x^3+0.1014x^4\\&+0.02954x^5-0.00578x^6\\&-0.00047x^7+0.000064x^8\cdots\\&-x^4\log x(0.0833-0.00278x^2+0.000025x^4\cdots).\end{aligned}$$

Case IV

The case of neutrons emerging from the surface of a block of paraffin requires special consideration. For the angular distribution of such neutrons Fermi[5] has given the approximate expression

$$\frac{2\sqrt{3}}{2+\sqrt{3}}(\cos\theta+\sqrt{3}\cos^2\theta)\sin\theta d\theta.$$

If these neutrons also may be assumed to have approximately a Maxwellian energy distribution, the transmission coefficient for a layer of $1/V$ absorber is given by:

$$\psi=\frac{2\sqrt{3}}{2+\sqrt{3}}\int_{y=0}^{\infty}\int_{\theta=0}^{\pi/2} ye^{-y-x/y^{\frac{1}{2}}\cos\theta}\times(\cos\theta+\sqrt{3}\cos^2\theta)\sin\theta dyd\theta,$$

TABLE I. *Values of five integrals φ_0, $2/\pi^{\frac{1}{2}}\varphi_{\frac{1}{2}}$, φ_1, x, and $-\psi'/1.5350$ normalized to unity for $x=0$.*

x	φ_0	$2\varphi_{\frac{1}{2}}/\pi^{\frac{1}{2}}$	φ_1	ψ	$-\psi'/1.5350$
0.00	1.0000	1.0000	1.0000	1.0000	1.0000
.01	.9829	.9888	.9912	.9850	.9612
.02	.9664	.9778	.9825	.9705	.9314
.03	.9506	.9670	.9739	.9564	.9050
.04	.9353	.9564	.9653	.9427	.8810
.05	.9205	.9459	.9569	.9293	.8536
.1	.8527	.8959	.9161	.8671	.7652
.2	.7395	.8063	.8408	.7608	.6282
.3	.6476	.7282	.7729	.6724	.5283
.4	.5711	.6611	.7114	.5975	.4507
.5	.5063	.5989	.6557	.5333	.3867
.6	.4510	.5497	.6051	.4768	.3372
.7	.4031	.4968	.5586	.4289	.2947
.8	.3615	.4538	.5169	.3874	.2588
.9	.3252	.4153	.4784	.3508	.2281
1.0	.2932	.3802	.4431	.3166	.2018

the first part of which is proportional to φ_0; and the second part to

$$\eta=\int_{y=0}^{\infty}\int_{\theta=0}^{\pi/2} ye^{-y-x/y^{\frac{1}{2}}\cos\theta}\cos^2\theta\sin\theta dyd\theta$$

$$\equiv x^3\int_x^{\infty}\frac{\varphi_1(z)}{z^4}dz\equiv x^3\left(B-\int^x\frac{\varphi_1(z)}{z^4}dz\right).$$

It may be shown that B, the value of the upper limit of the latter auxiliary integral, is equal to:

$$\frac{\pi^{\frac{1}{2}}}{6}\left(\frac{9C-11}{6}+\log 2\right)\equiv -0.08078.$$

It then follows that

$$\eta=\frac{1}{3}-\frac{\pi^{\frac{1}{2}}}{4}x+\frac{x^2}{2}-0.08078x^3-0.1847x^4$$
$$-0.01477x^5+0.002236x^6-0.000118x^7$$
$$-0.0000138x^8\cdots+x^3\log x(0.2954+0.0833x$$
$$-0.000927x^3+0.000005x^5\cdots)$$

and

$$\psi=\frac{1}{2+\sqrt{3}}(\sqrt{3}\varphi_0+6\eta)\equiv 1-1.5350x$$
$$+1.0982x^2+0.1443x^3-0.3634x^4-0.03288x^5$$
$$-0.005098x^6+0.00277x^7-0.000022x^8\cdots$$
$$-x^2\log x(0.4641-0.4749x-0.1726x^2$$
$$+0.002135x^4-0.0000119x^6\cdots).$$

CASE V

If the latter neutrons are detected by a thin foil of a $1/V$ absorber, the activity will be proportional to $-d\psi/dx\equiv-\psi'$. Then

$$-\psi'/1.5350=1-1.1285x-0.5915x^2$$
$$+0.8344x^3+0.1071x^4-0.01853x^5-0.001263x^6$$
$$+0.0001079x^7\cdots+x\log x(0.6047-0.9282x$$
$$-0.4497x^2+0.008345x^4-0.0000620x^6\cdots).$$

In Table I are given some values of the five integrals φ_0, $2\varphi_{\frac{1}{2}}/\pi^{\frac{1}{2}}$, φ_1, ψ, and $-\psi'/1.5350$, all normalized so as to have the value unity for $x=0$.

It may be noted that all of these integrals are functions only of x. Hence x plays the role of a characteristic absorption function. $x=At/(kT)^{\frac{1}{2}}$, where A is the constant in the expression for the absorption coefficient for homogeneous neutrons of relative energy E, and t is the thickness of the absorber. $A/(kT)^{\frac{1}{2}}$ is then the value of K for $E=kT$ (that is, for an energy equal to the most probable energy, or to one-half the average energy of the incident neutrons). It is of interest to calculate the apparent absorption exponent, $K't$, for φ_0, defined by:

$$\varphi_0(x)\equiv e^{-K't}\quad\text{or}\quad K't=-\log\varphi_0(x).$$

TABLE II. *Values of the fraction p for the five functions of Table I.*

x	$\left(\frac{x}{\log\varphi_0}\right)^2$	$\left(\frac{x}{\log(2\varphi_{\frac{1}{2}}/\pi^{\frac{1}{2}})}\right)^2$	$\left(\frac{x}{\log\varphi_1}\right)^2$	$\left(\frac{x}{\log\psi}\right)^2$	$\left(\frac{x}{\log(-x'/1.5350)}\right)^2$
0.00	0.318	0.785	1.273	0.425	0.000
.01	.336	.786	1.277	.437	.064
.03	.350	.799	1.286	.453	.090
.05	.364	.808	1.286	.465	.100
.1	.394	.826	1.303	.492	.140
.3	.476	.892	1.355	.571	.221
.5	.540	.950	1.403	.632	.277
.7	.594	1.001	1.446	.684	.328
.9	.642	1.049	1.490	.737	.371
1.0	.664	1.069	1.510	.755	.391

One may then calculate the fraction $y=p$ of the energy kT to which this value of $K't$ corresponds:

$$p=(x/\log \varphi_0)^2.$$

In Table II are given some values of the fraction p for the five different functions, and in Fig. 1 are shown the graphs of the functions given in Table I, together with the exponential function e^{-x}, for the purpose of comparison. The graphs show, for example, that the functions e^{-x} and $2\varphi_{\frac{1}{2}}/\pi^{\frac{1}{2}}$ lie close to each other and actually coincide for a value of x around 0.6, or for about 45 percent absorption.[3] In the range from 45 percent to 55 percent absorption the latter functions practically coincide.

For thin absorbers one may obtain an idea of the importance of the present considerations by comparing the slopes of the various curves of Fig. 1 for $x=0$. The general behavior of the various functions is the same for all except $-\psi'/1.5350$, which has an infinite slope for $x=0$. This is due to the presence of a term as low as $x \log x$ in the logarithmic part of the expansion for this function. The function $\varphi_{-\frac{1}{2}}$ would show a similar behavior. For these cases the apparent absorption for a thin absorber would not be proportional to the thickness of the absorber.

The expansions given here for the functions $\varphi_n(x)$ are sufficient for the determination of fairly accurate values in the interval $0<x<1$. By adding a few terms to the series it would be easy to extend Table I for values of the argument considerably greater than unity, since the coefficients a_n and b_n themselves converge fairly rapidly. For large values of x, Professor O. Laporte has derived asymptotic expansions. (See a note immediately following.) The two sets of expansions will give the general behavior of the functions in the entire interval $0<x<\infty$.

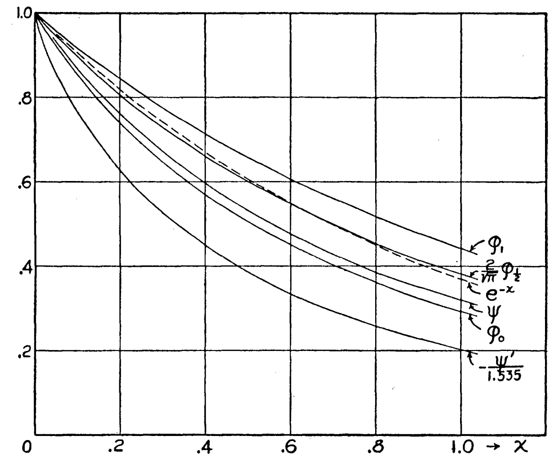

FIG. 1. Graphs of the functions given in Table I. Dashed curve is e^{-x}.

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