before a promising term array can be proven fortuitous. In fact, many fine looking arrays, involving hundreds of lines, were obtained only to be discarded later as entirely fortuitous. As there are many possible intervals between the known high states that one can start with in the search for the lower metastable states, there are many intervals to eliminate. So much can be obtained by chance alone that the real is hidden by the false.

A glance at the theoretical possible terms from the two electron configurations in question will show the cause of the great line density from this electron transition.

There will be 114 states from $4f^6(^7F)5d6s$ including

$$^{9,\,7,\,7,\,5}(HGFDP)$$

and 334 high states from $4f^6(^7F)5d6p$ including

$$^{9,\,7,\,7,\,5}(IHG,\ HGF,\ GFD,\ FDP,\ DPS).$$

The intense lines from the transition $4f^65d6s - 4f^66s6p$ will be in the inaccessible infrared.

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Electromagnetic Waves in Conducting Tubes

LEIGH PAGE AND N. I. ADAMS, JR.
Sloane Physics Laboratory, Yale University, New Haven, Connecticut
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It is shown that the waves in a hollow conducting tube described by Carson and by Barrow may be represented as semi-standing waves due to the superposition of plane waves with normal phase velocity $c/(\kappa\mu)^{\frac{1}{2}}$ reflected back and forth from one side of the tube to the other. It is also shown that certain types of waves can be transmitted without attenuation in tubes of triangular, rectangular and hexagonal cross section.

THE theory of the propagation of electromagnetic waves in cylindrical conducting tubes of circular cross section has been presented by Carson, Mead and Schelkunoff[1] and independently by Barrow,[2] and has been verified experimentally by Southworth.[3] While the mathematical analysis in these papers pursues the most direct and obvious method of attack, it fails to reveal the details of the physical process involved in the transmission of waves in the interior of a hollow tube. Fundamentally the "E" and "H" types of wave described by Carson and by Barrow are the result of the superposition of an infinite number of elementary plane waves traveling at an angle with the axis of the tube and having the normal phase velocity $c/(\kappa\mu)^{\frac{1}{2}}$ characteristic of the permittivity κ and permeability μ of the homogeneous isotropic nonconducting medium filling the interior of the tubular conductor, these waves being reflected back and forth from one side of the tube to the other. The two types of wave differ only in the state of polarization of the component elementary plane waves to whose superposition they are due. The elementary plane wave with normal phase velocity is more fundamental physically than the waves discussed by Carson and by Barrow for the reason that the Poynting flux is everywhere in the direction of wave propagation.

It is the object of this communication to show that the superposition of the specified plane waves yields the waves described by Carson and by Barrow. Incidentally we shall discuss the possibility of the propagation of electromagnetic waves in cylindrical conducting tubes of polygonal cross section. In all cases we shall limit ourselves to tubes which are perfectly conducting, using

[1] J. R. Carson, S. P. Mead and S. A. Schelkunoff, Bell System Tech. J. **15**, 310 (1936).
[2] W. L. Barrow, Proc. I. R. E. **24**, 1298 (1936).
[3] G. C. Southworth, Bell System Tech. J. **15**, 284 (1936).

the symbol v for the normal phase velocity $c/(\kappa\mu)^{\frac{1}{2}}$.

(1) Parallel Planes

First consider waves propagated in the X direction in a homogeneous isotropic medium between two conducting planes $y=0$ and $y=a$. Solutions of the field equations satisfying the boundary conditions are

$$\mathbf{E}=\mathbf{j}A\cos(lx-\omega t),\quad l=\omega/v; \tag{1}$$

$$\mathbf{E}=A\{\mathbf{i}m\sin my\cos(lx-\omega t) -\mathbf{j}l\cos my\sin(lx-\omega t)\}, \tag{2–1}$$

$$\mathbf{E}=B\{\mathbf{k}m\sin my\cos(lx-\omega t)\}, \tag{2–2}$$

$$m=k\pi/a,\quad l^2+m^2=\omega^2/v^2,$$

where k is an integer; and

$$\mathbf{E}=A\{\mathbf{i}(m^2+n^2)\sin my\sin nz\cos(lx-\omega t) -\mathbf{j}lm\cos my\sin nz\sin(lx-\omega t) -\mathbf{k}nl\sin my\cos nz\sin(lx-\omega t)\}, \tag{3–1}$$

$$\mathbf{E}=B\{-\mathbf{j}n\cos my\sin nz\cos(lx-\omega t) +\mathbf{k}m\sin my\cos nz\cos(lx-\omega t)\}, \tag{3–2}$$

$$m=k\pi/a,\quad l^2+m^2+n^2=\omega^2/v^2.$$

As (1) represents a plane wave with normal phase velocity v no special comment is needed.

Waves (2–1) and (2–2) are transmitted only for frequencies greater than $c/2a(\kappa\mu)^{\frac{1}{2}}$. In Carson's notation (2–1) is an "E" wave and (2–2) an "H" wave. The first may be written in the form

$$\mathbf{E}=\tfrac{1}{2}A\{(\mathbf{i}m-\mathbf{j}l)\sin(lx+my-\omega t) -(\mathbf{i}m+\mathbf{j}l)\sin(lx-my-\omega t)\}, \tag{4–1}$$

showing that it is due to the superposition of two plane waves with normal phase velocity v traveling in the XY plane in directions making angles with the X axis whose tangents are $\pm m/l$. Each component elementary wave is polarized with the electric vector in the plane determined by the direction of propagation and the X axis. Similarly (2–2) may be resolved into the two plane waves

$$\mathbf{E}=\tfrac{1}{2}B\{\mathbf{k}m\sin(lx+my-\omega t) -\mathbf{k}m\sin(lx-my-\omega t)\} \tag{4–2}$$

with normal phase velocity v traveling in the XY plane in directions making angles with the X axis whose tangents are $\pm m/l$. In this case the component elementary waves are polarized with the electric vector perpendicular to the plane determined by the direction of propagation and the X axis. The physical process underlying the propagation of these waves, therefore, consists in the back and forth reflection from the two conducting planes of plane waves traveling at an angle with the X axis with the normal phase velocity v. The phase velocity ω/l along the X axis of the resultant wave represents the reciprocal of the component in this direction of the wave slowness of either of the component elementary waves, and for this reason it is greater than v. As the limiting frequency is approached the angles which the directions of propagation of the component waves make with the X axis approach $\pm\pi/2$, and consequently these waves are reflected back and forth between the two conducting planes without progressing in the X direction. Hence the velocity of propagation of the resultant wave becomes infinite.

Waves (3–1) and (3–2) permit the introduction of the conducting planes $z=0$, $z=b$, with $n=k'\pi/b$, in addition to the conducting planes $y=0$, $y=a$, with $m=k\pi/a$. In the first we have an "E" wave and in the second an "H" wave propagated in a conducting tube of rectangular cross section. Excepting the special cases where m or n is equal to zero, these waves are transmitted only for frequencies greater than $(a^2+b^2)^{\frac{1}{2}}c/2ab(\kappa\mu)^{\frac{1}{2}}$. If $n=0$, (3–2) becomes

$$\mathbf{E}=B\{\mathbf{k}m\sin my\cos(lx-\omega t)\},$$

which is a wave identical with (2–2) having the limiting frequency $c/2a(\kappa\mu)^{\frac{1}{2}}$ independent of b, and if $m=0$, we have the analogous wave

$$\mathbf{E}=-B\{\mathbf{j}n\sin nz\cos(lx-\omega t)\}$$

with the limiting frequency $c/2b(\kappa\mu)^{\frac{1}{2}}$ independent of a. The first of these waves satisfies the boundary conditions for any z dimension of the rectangular tube, and the second for any y dimension. In a manner similar to the method employed in the cases of (2–1) and (2–2), it is easily shown that each of the waves (3–1) and (3–2) can be resolved into four elementary plane waves traveling with the normal phase velocity v in directions making equal angles with the X axis. These waves, however, differ from those previously discussed in that they do not travel, in general, in directions lying in planes determined by the normals to the conducting planes and the X axis.

(2) Cylindrical Tube of Circular Cross Section

We shall now show that the Carson-Barrow waves, like the simpler waves just described, are due to the superposition of plane waves traveling with normal phase velocity $v=c/(\kappa\mu)^{\frac{1}{2}}$ at an angle with the axis of the tube and reflected back and forth one side to the other. As the normal component of the wave slowness of each elementary plane wave is reversed on reflection we shall save labor by taking as primary waves those specified by (2–1) and (2–2) instead of the pairs (4–1) and (4–2) of elementary plane waves into which they may be resolved. We shall need the expansions[4]

$$\sin(mr\cos\theta)=2J_1(mr)\cos\theta-2J_3(mr)\cos 3\theta+2J_5(mr)\cos 5\theta-\cdots, \quad (5\text{–}1)$$

$$\cos(mr\cos\theta)=J_0(mr)-2J_2(mr)\cos 2\theta+2J_4(mr)\cos 4\theta-\cdots, \quad (5\text{–}2)$$

and the recurrence formulas

$$(2n/mr)J_n(mr)=J_{n-1}(mr)+J_{n+1}(mr), \quad (6\text{–}1)$$

$$2J_n'(mr)=J_{n-1}(mr)-J_{n+1}(mr). \quad (6\text{–}2)$$

Take the axis of the tube as X axis and let the circle in Fig. 1 with center at O represent a cross section of the tube. Let the Y axis have the direction of the projection on this cross section of the velocity of propagation of an elementary plane wave. A point P lying on a wave front MN is located by the cylindrical coordinates x, r, ϕ.

First consider the "E" waves. The complete solution of the electromagnetic equations of the form (2–1) is

$$E_x=m[a_1\cos my-a_2\sin my]\cos(lx-\omega t), \quad (7\text{–}1)$$

$$E_y=l[a_1\sin my+a_2\cos my]\sin(lx-\omega t), \quad (7\text{–}2)$$

where a_1 and a_2 are independent arbitrary constants. This solution is obtained immediately from (2–1) by replacing y by $y-\delta$. Now let the number of such primary plane waves propagated in the range of θ between θ and $\theta+d\theta$ be $\cos n(\theta+\phi)d\theta$ where n is an integer. Then, as $y=r\cos\theta$, the X component of the resultant amplitude at P is

$$A_x=ma_1\int_0^{2\pi}\cos(mr\cos\theta)\{\cos n\theta\cos n\phi-\sin n\theta\sin n\phi\}d\theta-ma_2\int_0^{2\pi}\sin(mr\cos\theta)\{\cos n\theta\cos n\phi-\sin n\theta\sin n\phi\}d\theta$$

$$=\left\{\begin{array}{ll}(-1)^{n/2}2\pi ma_1J_n(mr)\cos n\phi, & n\text{ even},\\ (-1)^{(n+1)/2}2\pi ma_2J_n(mr)\cos n\phi, & n\text{ odd},\end{array}\right\} \quad (8\text{–}1)$$

from (5–1) and (5–2). Similarly the component of the resultant amplitude in the direction of increasing ϕ is

$$A_\phi=\tfrac{1}{2}la_1\int_0^{2\pi}\sin(mr\cos\theta)\{[\sin(n+1)\theta-\sin(n-1)\theta]\cos n\phi+[\cos(n+1)\theta-\cos(n-1)\theta]\sin n\phi\}d\theta$$

$$+\tfrac{1}{2}la_2\int_0^{2\pi}\cos(mr\cos\theta)\{[\sin(n+1)\theta-\sin(n-1)\theta]\cos n\phi+[\cos(n+1)\theta-\cos(n-1)\theta]\sin n\phi\}d\theta$$

$$=\left\{\begin{array}{ll}(-1)^{n/2}\pi la_1\{J_{n-1}(mr)+J_{n+1}(mr)\}\sin n\phi, & n\text{ even},\\ (-1)^{(n+1)/2}\pi la_2\{J_{n-1}(mr)+J_{n+1}(mr)\}\sin n\phi, & n\text{ odd},\end{array}\right.$$

$$=\left\{\begin{array}{ll}(-1)^{n/2}2\pi(ln/m)(a_1/r)J_n(mr)\sin n\phi, & n\text{ even},\\ (-1)^{(n+1)/2}2\pi(ln/m)(a_2/r)J_n(mr)\sin n\phi, & n\text{ odd},\end{array}\right\} \quad (8\text{–}2)$$

[4] Gray, Mathews and Macrobert, *Bessel Functions*, p. 16 and 32.

from (5–1), (5–2) and (6–1). In the same manner we obtain for the resultant amplitude in the direction of increasing r

$$A_r=\begin{cases} -(-1)^{n/2}2\pi l a_1 J_n'(mr)\cos n\phi, & n \text{ even,} \\ -(-1)^{(n+1)/2}2\pi l a_2 J_n'(mr)\cos n\phi, & n \text{ odd,} \end{cases} \tag{8–3}$$

from (5–1), (5–2) and (6–2).

Therefore the components of the electric intensity at P due to the superposition of the group of elementary plane waves reflected back and forth from one side of the tube to the other, for n either even or odd, are

$$E_x=Am^2J_n(mr)\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}\cos(lx-\omega t), \tag{9–1}$$

$$E_r=-AlmJ_n'(mr)\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}\sin(lx-\omega t), \tag{9–2}$$

$$E_\phi=Aln\frac{1}{r}J_n(mr)\begin{Bmatrix}\sin n\phi\\ -\cos n\phi\end{Bmatrix}\sin(lx-\omega t), \tag{9–3}$$

the lower factors in the braces having been obtained from the upper by replacing $n\phi$ by $n\phi-\pi/2$, which is equivalent to making the number of primary waves in the range $d\theta$ proportional to $\sin n(\theta+\phi)$ instead of to $\cos n(\theta+\phi)$. These are, however, just the components of **E** obtained for the "E" type wave by Carson and by Barrow, who have discussed the circumstances under which the boundary conditions are satisfied.

The "H" waves are due to the superposition of primary plane waves of the other state of

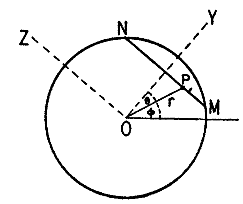

FIG. 1. Cross section of circular tube.

polarization. From (2–2) we have in the primary plane wave

$$E_z=m[b_1\sin my+b_2\cos my]\sin(lx-\omega t), \tag{10}$$

and, if we take the number of such primary waves propagated in the angle $d\theta$ to be $\cos n(\theta+\phi)d\theta$, we get

$$\begin{aligned} B_\phi=&\tfrac{1}{2}mb_1\int_0^{2\pi}\sin(mr\cos\theta)\{[\cos(n+1)\theta+\cos(n-1)\theta]\cos n\phi \\ &\qquad -[\sin(n+1)\theta+\sin(n-1)\theta]\sin n\phi\}d\theta \\ &+\tfrac{1}{2}mb_2\int_0^{2\pi}\cos(mr\cos\theta)\{[\cos(n+1)\theta+\cos(n-1)\theta]\cos n\phi \\ &\qquad -[\sin(n+1)\theta+\sin(n-1)\theta]\sin n\phi\}d\theta \\ =&\begin{cases} -(-1)^{n/2}\pi m b_1\{J_{n-1}(mr)-J_{n+1}(mr)\}\cos n\phi, & n\text{ even,}\\ -(-1)^{(n+1)/2}\pi m b_2\{J_{n-1}(mr)-J_{n+1}(mr)\}\cos n\phi, & n\text{ odd,}\end{cases} \\ =&\begin{cases} -(-1)^{n/2}2\pi m b_1 J_n'(mr)\cos n\phi, & n\text{ even,}\\ -(-1)^{(n+1)/2}2\pi m b_2 J_n'(mr)\cos n\phi, & n\text{ odd,}\end{cases} \end{aligned} \tag{11–1}$$

from (5–1), (5–2) and (6–2). Similarly

$$B_r=\begin{cases} -(-1)^{n/2}2\pi n(b_1/r)J_n(mr)\sin n\phi, & n\text{ even,}\\ -(-1)^{(n+1)/2}2\pi n(b_2/r)J_n(mr)\sin n\phi, & n\text{ odd,}\end{cases} \tag{11–2}$$

from (5–1), (5–2) and (6–1).

Therefore the components of electric intensity in the "H" wave, for n either even or odd, are

$$E_r = Bn\frac{1}{r}J_n(mr)\left\{\begin{matrix}\sin n\phi \\ -\cos n\phi\end{matrix}\right\}\sin(lx-\omega t), \quad (12\text{–}1)$$

$$E_\phi = BmJ_n'(mr)\left\{\begin{matrix}\cos n\phi \\ \sin n\phi\end{matrix}\right\}\sin(lx-\omega t), \quad (12\text{–}2)$$

in agreement with Carson and with Barrow.

Obviously the same methods can be applied to the dielectric rod of circular cross section, the analysis being complicated, however, by the fact that the elementary plane waves are partially reflected and partially transmitted at the surface of the dielectric.

(3) Cylindrical Tube of Polygonal Cross Section

Last we shall investigate the propagation of a wave inside a cylindrical tube whose cross section is a regular polygon. The square tube is a special case of the rectangular tube already considered, and we have seen that both an "E" wave and an "H" wave may be transmitted by it. For other cross sections it seems that no "E" wave can be transmitted, and an "H" wave is propagated only in the cases of a triangular or hexagonal section, the transmission in these types of tube being due to the circumstance that cos 60° is the reciprocal of an integer.

Let Fig. 2 represent a cross section of the tube, O being on the axis and the heavy lines being the traces of the faces of the tube. Primary plane waves of the "H" type (2–2) with directions of propagation in planes containing the X axis and the normals to the faces of the tube give rise to a resultant amplitude of the electric intensity at a point $P(y, z)$ equal to

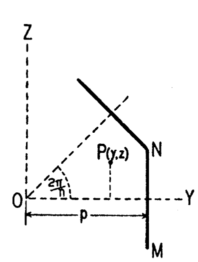

Fig. 2. Cross section of hexagonal tube. O is on the axis.

$$\begin{aligned} A_z &= a\sum_{k=0}^{n-1}\cos k(2\pi/n)\sin m[y\cos k(2\pi/n) \\ &\qquad + z\sin k(2\pi/n)] \\ &= a\sum_{k=0}^{n-1}\cos k(2\pi/n)\sin[my\cos k(2\pi/n)] \\ &\qquad \times\cos[mz\sin k(2\pi/n)] \end{aligned} \quad (13)$$

no matter whether the number n of faces is odd or even.

The boundary condition at the face MN is that A_z should vanish for all values of z. But this condition can be satisfied only if every $\cos(2\pi k/n)$ is an integral multiple of the same number. Such a situation exists only for $n=3$, 4, 6. Consequently we conclude that this type of wave can be transmitted only by triangular, square, and hexagonal tubes.

For the triangular tube $2\pi/n=120°$ and $\cos 2\pi k/n$ assumes the values 1, $-1/2$, $-1/2$. Therefore

$$\begin{aligned} A_z &= a\left\{\sin my + \sin(my/2)\cos\frac{(3)^{\frac{1}{2}}}{2}mz\right\} \\ &= a\sin\tfrac{1}{2}my\left\{2\cos(my/2)+\cos\frac{(3)^{\frac{1}{2}}}{2}mz\right\}, \end{aligned} \quad (14)$$

which vanishes for all values of z if y is an integral multiple of $2\pi/m$.

Hence the minimum frequency for transmission of this wave is $\nu = c/p(\kappa\mu)^{\frac{1}{2}}$, where p is the perpendicular distance from the axis to one of the faces.

For the hexagonal tube $2\pi/n=60°$. Here $\cos 2\pi k/n$ has the values 1, 1/2, $-1/2$, -1, $-1/2$, 1/2 and $\sin 2\pi k/n$ the values 0, $\sqrt{3}/2$, $\sqrt{3}/2$, 0, $-\sqrt{3}/2$, $-\sqrt{3}/2$. Hence

$$\begin{aligned} A_z &= a\left\{2\sin my + 4\sin(my/2)\cos\frac{(3)^{\frac{1}{2}}}{2}mz\right\} \\ &= 2a\sin(my/2)\left\{2\cos(my/2)+\cos\frac{(3)^{\frac{1}{2}}}{2}mz\right\}, \end{aligned} \quad (15)$$

which is just double A_z for the triangular tube.

When we pass to the tube of circular cross section by making $n\to\infty$ we see that (13) gives us A_ϕ for the "H_0" wave. No further cases of transmission are obtained by directing the elementary waves toward the corners N of the polygonal tube instead of directing them at right angles to the faces.