when the electron is deflected into the element of solid angle $\sin\vartheta d\vartheta d\varphi$ is given by

$$(\Omega/(2\pi c)^3)\omega_s^2 d\omega_s \sin\vartheta_s d\vartheta_s d\varphi_s \times\sum_\lambda \sum_{n_{s\lambda}=0}^{\infty} n_{s\lambda}\hbar\omega_s U \sin\vartheta d\vartheta d\varphi \quad (30)$$

$$=(1/4\pi^2\hbar^4\Omega)\sin\vartheta d\vartheta d\varphi(m^2 v c/(1-v^2))|F(\mathbf{q})|^2 \cdot(e^2/4\pi^2 c)\left[\left(\frac{\boldsymbol{\mu}}{1-\mu_s}-\frac{\mathbf{v}}{1-\nu_s}\right)^2 -\left(\frac{\mu_s}{1-\mu_s}-\frac{\nu_s}{1-\nu_s}\right)^2\right]d\omega_s \sin\vartheta_s d\vartheta_s d\varphi_s. \quad (31)$$

This has been obtained by making use of (26), (13) and (2) and of the formula

$$\sum_{n=0}^{\infty} ne^{-x}x^n/n!=x.$$

On account of the extra factor ω_s in (30), the mean total energy, unlike the mean total number of quanta, is finite. Formula (31) is just the total probability that the electron be scattered into the element of solid angle $\sin\vartheta d\vartheta d\varphi$ multiplied by the amount of energy which it would radiate classically in such a deflection. The expansion in powers of $e^2/\hbar c$ in the limit of small frequencies, where alone Formula (31) may be supposed to be valid, leads to the same result for the mean energy radiated, though its results are entirely misleading so far as transition probabilities are concerned.[15]

The above considerations can be applied almost literally also to cases such as the theory of β-decay, in which the external perturbation does not act on the electron coordinates, but creates an electron. The only difference, so far as the electromagnetic field is concerned, consists in replacing $\boldsymbol{\mu}c$ by the velocity of the nucleus and $\mathbf{v}c$ by the velocity with which the electron is created. Here again the total probability of β-decay is unaltered by the interaction of the electron with the low frequency radiation and the mean radiated energy is in agreement with that calculated by expanding in powers of $e^2/\hbar c$.

[15] This remark does not affect the cross section for the emission of a high frequency quantum as calculated by Bethe and Heitler, and experimentally verified. However, the cross section so derived has to be interpreted, not as the probability that the high frequency quantum alone be emitted, but as the probability that this happens no matter how many other light quanta are emitted.

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The Low Frequency Radiation of a Scattered Electron

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The radiative scattering of a nonrelativistic electron is treated by an approximate method which neglects the reaction of the radiation field on the motion of the electron. In this approximation the different modes of oscillation of the radiation field are independent of one another, and can therefore be treated individually. For frequencies of the radiation small compared to the inverse impact time of the electron, it is shown that the probability of emitting any finite number of quanta is zero, and that the mean radiated energy depends only on the total change in velocity of the electron, the amount of energy radiated being given by the same formula as in classical theory.

1. INTRODUCTION

IN a previous paper[1] Bloch and the author have treated the interaction of a relativistic electron and the radiation field by a method which consists essentially of an expansion in powers of the parameters $e^2\omega/mc^3$, $\hbar\omega/E$, $\hbar\omega/c\Delta p$ (ω = angular frequency of radiation; E = kinetic energy, Δp = change in momentum, of electron). Applications of the method to actual physical processes were made in an approximate way by regarding external influences on the electron, such as atomic

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[1] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937); subsequently referred to as I.

fields or the interactions responsible for β-ray processes, as small perturbations. It was shown that, to the extent to which these approximations are legitimate, the classical prediction concerning the radiation emitted is valid in the mean, and the motion of the electron is unaffected by the radiation field.

In this paper the radiative scattering of a non-relativistic electron in a potential field $V(\mathbf{r})$ is considered from the same point of view, but without regarding $V(\mathbf{r})$ as a small perturbation. A treatment of this problem with no restrictions except that the above mentioned parameters shall be small is possible in principle but complicated and dependent on the detailed form of $V(\mathbf{r})$. Since all frequencies except the extremely low ones may be treated adequately by the ordinary expansion in powers of $e^2/\hbar c$, as was shown in I, we shall have offset the additional complication of treating the scattering field exactly by imposing a further restriction on the frequency.

We may again guide our considerations just as was done in I, by reference to the corresponding classical problem. If the potential energy of the electron at a distance $\mathbf{r}$ from the center of the scattering field is $V(\mathbf{r})$ and if $V(\mathbf{r})$ tends to zero faster than $1/r$ as $r\rightarrow\infty$, as we shall for simplicity assume throughout this paper, the classical orbit of the electron has the form $\mathbf{r}=\mathbf{r}(t)$ with

$$\mathbf{r}(t)\sim\mathbf{v}t+\mathbf{a}; \quad t<-T/2,$$
$$\sim\mathbf{w}t+\mathbf{b}; \quad t>T/2,$$

where $\mathbf{v}$ and $\mathbf{w}$ are the initial and final velocities, $\mathbf{a}$ and $\mathbf{b}$ are constants and T is the collision time. For $-T/2<t<T/2$, $\mathbf{r}(t)$ will be a more complicated function, which, however, does not affect the radiation of sufficiently low frequencies. The amount of energy radiated with angular frequency ω in a direction within the element of solid angle $d\Sigma$ is given by

$$(e^2/4\pi^2c^3)\,|\ddot{r}_{\perp,\,\omega}|^2 d\omega d\Sigma, \tag{1}$$

where $\ddot{r}_{\perp,\,\omega}$ is the Fourier component of the component of the acceleration perpendicular to the direction of propagation of the radiation:

$$\ddot{r}_{\perp,\,\omega}=\int_{-\infty}^{\infty}dt\ddot{r}_{\perp}(t)e^{-i\omega t}=-\omega^2\int_{-\infty}^{\infty}dt r_{\perp}(t)e^{-i\omega t}.$$

For frequencies such that $\omega T\ll 1$, this expression depends essentially only on the initial and final velocities $\mathbf{v}$ and $\mathbf{w}$, and may be evaluated without knowledge of the detailed motion of the electron during the collision:

$$\ddot{r}_{\perp,\,\omega}=(v_{\perp}-w_{\perp})(1+O(\omega T)).$$

Hence the amount of low frequency radiation is

$$(e^2/4\pi^2c^3)(v_{\perp}-w_{\perp})^2 d\omega d\Sigma. \tag{2}$$

The simplicity of this result depends physically on the fact that the deflection of the electron takes place suddenly with respect to the periods of the radiation considered, so that the simple methods of sudden impulse dynamics apply.

These considerations indicate that if in the quantum theoretical treatment of the problem we limit ourselves to frequencies small compared to the reciprocal of the impact time, we shall get a correspondingly simple treatment. In fact, when we neglect the additional parameters $\hbar\omega/E$ and $\hbar\omega/c\Delta p$, which are a measure of the effect of the quantum mechanical fluctuations of the radiation field on the electron, we may expect to get exactly the classical formula (2). The motion of the electron itself in the scattering field need not, of course, be describable in classical terms.

2. Radiative Scattering

We shall keep the notation of I except where further symbols are explicitly defined. The Hamilton function is

$$\mathcal{H}=(1/2m)(\mathbf{p}-(e/c)\mathbf{A})^2+V(\mathbf{r})+\tfrac{1}{2}\sum_{s\lambda}\hbar\omega_s(P_{s\lambda}{}^2+Q_{s\lambda}{}^2),$$

where

$$(e/c)\mathbf{A}=\sum_{s\lambda}\mathbf{a}_{s\lambda}[P_{s\lambda}\cos(\mathbf{k}_s,\mathbf{r})+Q_{s\lambda}\sin(\mathbf{k}_s,\mathbf{r})].$$

Let $\psi(\mathbf{r})$ describe the motion of the electron without interaction with the radiation:

$$\{p^2/2m+V(\mathbf{r})-\tfrac{1}{2}mv^2\}\psi(\mathbf{r})=0$$

and suppose ψ to have the asymptotic form for large r appropriate to a scattering experiment:

$$\psi(\mathbf{r})\sim\exp(im(\mathbf{v},\mathbf{r})/\hbar)+f(\theta,\varphi)/r\cdot\exp(imvr/\hbar). \tag{3}$$

If we now write for the solution $\Psi(\mathbf{r},Q_{s\lambda})$ of the equation $\{\mathcal{H}-E\}\Psi=0$,

$$\Psi(\mathbf{r}, Q_{s\lambda})=\psi(\mathbf{r})U(\mathbf{r}, Q_{s\lambda}) \quad (4)$$

we get for U the equation

$$(\psi/2m)(\mathbf{p}-(e/c)\mathbf{A})^2U+(\mathbf{p}\psi/m)(\mathbf{p}-(e/c)\mathbf{A})U +(\psi/2)\sum_{s\lambda}\hbar\omega_s(P_{s\lambda}^2+Q_{s\lambda}^2)U -\psi(E-\tfrac{1}{2}mv^2)U=0. \quad (5)$$

The first term in this equation represents the reaction of the radiation on the electron; its order of magnitude relative to the remaining terms is given by the ratio of the mean recoil momentum to the momentum mv of the electron. In accordance with our program we neglect this term. Such a procedure is quite analogous to what is done in the theory of molecules, where the reaction of the fluctuating motion of the electrons on the nuclei is similarly left out of account. The resulting equation is separable with respect to the various modes of vibration of the radiation field. Thus U is approximately equal to $\prod_{s\lambda}u_{s\lambda}(\mathbf{r}, Q_{s\lambda})$, where

$$(\mathbf{p}\psi/m)(\mathbf{p}-\mathbf{a}_{s\lambda}[P_{s\lambda}\cos(\mathbf{k}_s, \mathbf{r}) +Q_{s\lambda}\sin(\mathbf{k}_s, \mathbf{r})])u_{s\lambda}+(\psi/2)\hbar\omega_s(P_{s\lambda}^2 +Q_{s\lambda}^2)u_{s\lambda}-\psi E_{s\lambda}u_{s\lambda}=0. \quad (6)$$

This approximate separability is an expression of the fact that the behaviors of different components of the radiation field, in particular any changes in the numbers of light quanta present in them, are almost independent. The coupling between different components exists only in virtue of the intermediary action of the electron and is therefore small whenever the reaction of the radiation on the electron is small.

We proceed to the solution of Eq. (6), which is to be carried out in the approximation $\omega\ll 1/T$. Let R be the "radius" of the scattering field, or more precisely, the smallest radius for which $\psi(\mathbf{r})$ is well represented by (3). For $r>R$ we have then, dropping the subscripts s, λ temporarily,

$$\exp(im(\mathbf{v}, \mathbf{r})/\hbar)\{(\mathbf{v}, \mathbf{p}-\mathbf{a}[P\cos(\mathbf{k}, \mathbf{r}) +Q\sin(\mathbf{k}, \mathbf{r})])+\tfrac{1}{2}\hbar\omega(P^2+Q^2)-E\}u +f(\vartheta, \varphi)/r\cdot\exp(imvr/\hbar)\{(\mathbf{w}, \mathbf{p}-\mathbf{a}[P\cos(\mathbf{k}, \mathbf{r}) +Q\sin(\mathbf{k}, \mathbf{r})]+\tfrac{1}{2}\hbar\omega(P^2+Q^2)-E\}\cong 0, \quad (7)$$

where $\mathbf{w}$ is the final velocity of the electron, i.e., $\mathbf{w}$ has the magnitude of $\mathbf{v}$ and the direction of $\mathbf{r}$.

In the derivation of (7) we have neglected terms of order $1/r^2$ coming from the differentiation of $f(\vartheta, \varphi)/r$.

The solution of (7) which we require is defined by the condition that the incident electron be accompanied by no free light quanta. Hence at $r=\infty$, u must be the function appropriate to the motion of a free electron with velocity $\mathbf{v}$, accompanied only by its proper field. This function $u^{(0)}$ is a solution of $\{F-E\}u^{(0)}=0$, where $\{F-E\}$ is the first bracket in (7), and is, according to I,

$$u^{(0)}=\exp\{i\sigma\cos(\mathbf{k}, \mathbf{r})[Q-\tfrac{1}{2}\sigma \times\sin(\mathbf{k}, \mathbf{r})]\}\cdot h_0(Q-\sigma\sin(\mathbf{k}, \mathbf{r})), \quad (8)$$

$$E=\tfrac{1}{2}\hbar\omega$$

where $\sigma=(\mathbf{v}, \mathbf{a})/\hbar(\omega-(\mathbf{v}, \mathbf{k}))$, and where we have neglected a term of order v^2/c^2 in E.

Now since the scattering of the electron occurs within a small fraction of a period of the radiation, we should expect the wave function of the field to remain unaltered by the scattering. Hence $u^{(0)}$ should be a solution of the complete Eq. (6) in our approximation. To show that this is true, we must prove first that $\{G-E\}u^{(0)}\sim 0$ where $\{G-E\}$ is the second bracket in (7); and second, that the values of $u^{(0)}$ at different points on the sphere $r=R$ join on to each other correctly according to Eq. (6).

That $u^{(0)}$ is an approximate solution of $\{G-E\}u^{(0)}=0$, and thus of the whole Eq. (7), is best shown by expanding $u^{(0)}$ in terms of the proper functions of G. We have according to I

$$u^{(0)}=\sum_{n=0}^{\infty}\alpha_n^{(0)}f_n(Q) \quad (9)$$

where

$$f_n(Q)=\exp\{i\tau\cos(\mathbf{k}, \mathbf{r})[Q-\tfrac{1}{2}\tau\sin(\mathbf{k}, \mathbf{r})]\} \cdot h_n(Q-\tau\sin(\mathbf{k}, \mathbf{r})),$$

$$\alpha_n^{(0)}=\exp\{-in(\mathbf{k}, \mathbf{r})\}\cdot K(\sigma, 0; \tau, n),$$

the quantity K being given by formula (20) in I. Hence

$$\{G-E\}u^{(0)}=\sum_n\alpha_n^{(0)}n\hbar(\omega+(\mathbf{w}, \mathbf{k}))f_n(Q).$$

By an alteration of $u^{(0)}$ which is negligible for $\hbar\omega\ll mv^2$ we may bring $\{G-E\}u^{(0)}$ to vanish: we need evidently only replace $\alpha_n^{(0)}$ by

$$\alpha_n^{(0)} \exp\{-in(\omega+(\mathbf{w},\mathbf{k}))r/v\}.$$

This involves an alteration of $u^{(0)}$ consisting of inserting factors of the kind $\exp\{-i(mvr/\hbar)\cdot(\hbar\omega/mv^2)\}$. On account of the smallness of $\hbar\omega/mv^2$, such factors vary much less rapidly with $\mathbf{r}$ than ψ, the wave function of the electron, and so may be replaced by unity.[2] To be consistent we also replace $\cos(\mathbf{k},\mathbf{r})$ by one and $\sin(\mathbf{k},\mathbf{r})$ by zero, since k, the wave number of the radiation, is likewise small compared to $mv/\hbar$. The solution of (7) is then

$$u^{(0)} \sim e^{i\sigma Q} h_0(Q). \tag{10}$$

It remains to be shown that the values of $u^{(0)}$ at various points on the surface of the sphere $r=R$ are correctly related to each other in virtue of Eq. (6). The argument by which this may be proved is essentially that used for the behavior of a quantum mechanical system under the influence of a sudden external perturbation.[3] Let us write Eq. (6) in the form

$$(\hbar/i)(\mathbf{v}(\mathbf{r}),\text{grad})u = \{(\mathbf{v}(\mathbf{r}),\mathbf{a})P-\tfrac{1}{2}\hbar\omega(P^2+Q^2)+E\}u,$$

where $\mathbf{v}(\mathbf{r})=(\mathbf{p}\psi)/m\psi$. Integrating each side of this equation roughly from a point $\mathbf{r}_1$ on the sphere $r=R$ through the sphere to the point $\mathbf{r}_2$ on the sphere, we find

$$(\hbar R/T)(u(\mathbf{r}_2)-u(\mathbf{r}_1)) \sim \hbar\omega R\bar{u},$$

where $\bar{u}$ is a certain average value of u in the region $r<R$. Hence

$$u(\mathbf{r}_2)-u(\mathbf{r}_1) \sim (\omega T)\bar{u} \sim 0.$$

Thus (10) is the solution of (6) for frequencies such that $\omega T \ll 1$, $\hbar\omega/mv^2 \ll 1$ and $\hbar\omega/mvc \ll 1$.

The whole wave function Ψ may now be written down. For purposes of interpretation we use the expansion (9), which expresses the function $u^{(0)}$ appropriate to the initial motion $\mathbf{v}$ of the electron in terms of the functions $f_n(Q)$ appropriate to the final motion $\mathbf{w}$, in writing the coefficient of the scattered wave:

$$\Psi \sim \exp(im(\mathbf{v},\mathbf{r})/\hbar)\prod_{s\lambda} u_{s\lambda}^{(0)}(Q_{s\lambda}) + f(\vartheta,\varphi)/r\cdot\exp(imvr/\hbar)\prod_{s\lambda}\{\sum_{n_{s\lambda}} K(\sigma_{s\lambda},0;\tau_{s\lambda},n_{s\lambda})f_{s\lambda,\,n_{s\lambda}}(Q_{s\lambda})\}. \tag{12}$$

From this it follows that the cross section for scattering with the emission of $n_{s\lambda}$ light quanta of the kind s, λ is

$$|f(\vartheta,\varphi)|^2 \sin\vartheta d\vartheta d\varphi \prod_{s\lambda} |K(\sigma_{s\lambda},0;\tau_{s\lambda},n_{s\lambda})|^2. \tag{13}$$

For the interpretation of this fomula the reader is referred to I, where the equivalent formula (25) is discussed. In particular, it is easily to be seen by using the value of K given in I, that the total probability of scattering of the electron irrespective of what radiation is emitted is just $|f(\vartheta,\varphi)|^2\sin\vartheta d\vartheta d\varphi$; that the probability of emission of any finite number of quanta is zero; and that the mean number of quanta emitted with definite frequency and direction when the electron is deflected into the solid angle $\sin\vartheta d\vartheta d\varphi$ is

$$(e^2/4\pi^2c^3)(v_\perp-w_\perp)^2(d\omega_s/\hbar\omega_s)d\Sigma, \tag{14}$$

which corresponds exactly to the classical formula (2).

As was shown in I, the method of expanding in powers of the interaction between electron and radiation leads to the same result for the mean radiated energy. This may be understood on the basis that the various modes of vibration of the radiation field behave independently, as we have already mentioned in connection with Eq. (6). Because of this independence, the method of expanding in powers of $e^2/\hbar c$ gives correctly the probability that a definite amount of radiation be emitted in one mode irrespective of the behavior of all other modes. One may therefore calculate the mean radiated energy in each mode separately and then add them together, and this is just the procedure used in the method of expanding in powers of $e^2/\hbar c$.

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[2] The rigorous justification for neglecting the variation of such factors lies in the possibility of forming a wave packet describing the motion of the electron, with dimensions so large that the energy of the electron is well defined and yet small compared to $(v/c)\lambda$, where λ is the wavelength of the radiation. The factors concerned are approximate constants over the region occupied by such a wave packet. Our method is thus valid provided only that an electron with relative uncertainty in energy $\delta(mv^2)/mv^2 \sim \hbar\omega/mv^2$ is scattered in sensibly the same way as one of definite energy.

[3] See e.g. W. Pauli, *Handbuch der Physik*, Vol. 24, pp. 163–4.