# Relativistic Effects for the Deuteron

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It is seen from Tables I, II, and III than an appreciable spin-spin dependence of the neutron-proton interaction can be expected to arise as a result of using Eqs. (1) and (9). This spin dependence is of the wrong sign to account for the empirically known difference in energies of the singlet and triplet states of the deuteron. Its presence suggests that the spatial dependence of spin dependent forces is not wholly the same as that of the other forces. The possibility of finding a simple equation in which the empirical spin dependence will be a natural consequence is still open. It is shown that the relativistic correction to the kinetic energy is only a small part of the corrections called for by invariance. The relativistic refinements made so far in the combined theory of $H^2$ and $H^3$ are shown to be questionable from this point of view, as well as on account of the possible presence of terms in relative momenta which are not determined by requirements of invariance alone.

## 1. Introduction and Summary

THE effect of relativity on the energy of the deuteron has been considered by Blochinzew,[1] by Margenau,[1] and by Feenberg.[2] The discussions of Blochinzew and Margenau presuppose that the actual two-body problem can be replaced by a suitable one-body problem and that the latter can be treated by the Klein-Gordon equation. These assumptions are incorrect. Feenberg's calculation takes into account only corrections for the kinetic energy of the two particles. In addition to these, however, it is necessary[3] to consider also the terms in the Hamiltonian which correspond to the interaction of the orbits of the two particles and which are formally of the same order of magnitude as the pure kinetic energy effects. In the present note the results of calculations which take these terms into account, together with all terms of the order $v^2/c^2$, where $v$ is the velocity of the particles and $c$ is the velocity of light, are presented. Since the Hamiltonian to the order $v^2/c^2$ can be put into the form of a sum of two relativistic Dirac Hamiltonians and an interaction energy also expressible by means of Dirac's operators[3] the corrections to order $v^2/c^2$ corresponding to these forms are also given here.

When the motion of two particles is described by means of Diracian equations there appears among other terms an interaction between the spins of the two particles which brings about an energy difference between their triplet and singlet conditions. Calculations given below show that this energy difference is of the same order of magnitude although somewhat smaller than that due to the usual spin dependent Heisenberg force. In the equations used the order of the triplet and singlet terms is opposite to the empirical. Therefore, the spin dependence following from the equations used here cannot be claimed to be the explanation of the Heisenberg force. Nevertheless, the results suggest that a suitable equation could be found in which the Heisenberg force could be replaced by a relativistic correction term to the Majorana force. Whether this is so or not, the fact that a spin dependence of the interaction energy of a magnitude comparable with the empirical arises out of relativistic Diracian equations indicates that the spatial dependence of the spin dependent force is probably different from that of the main Majorana force and that further one should consider in addition to energies of type $(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)$ also energies of type $(\mathbf{r}\boldsymbol{\sigma}_1)(\mathbf{r}\boldsymbol{\sigma}_2)$, where $\boldsymbol{\sigma}$ are the Pauli spin operators for particles 1, 2 and $\mathbf{r}$ is the vector from particle 2 to particle 1, $=\mathbf{r}_1-\mathbf{r}_2$. Thus for Eq. (17.6) of reference 3 two spin operators are involved only in the following terms

$$E=2Mc^2-J-\frac{\hbar^2 f}{2M^2c^2}(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)-\frac{\hbar^2}{4M^2c^2}\frac{df}{rdr}[r^2(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)-(\mathbf{r}\boldsymbol{\sigma}_1)(\mathbf{r}\boldsymbol{\sigma}_2)]+\cdots,\quad f=\frac{dJ}{rdr}.$$

[1] D. Blochinzew, Physik. Zeits. Sowjetunion **8**, 270 (1935); H. Margenau, Phys. Rev. **50**, 342 (1936).
[2] E. Feenberg, Phys. Rev. **50**, 674 (1936).
[3] G. Breit, Phys. Rev. **51**, 248 (1937). Eqs. (16.1), (18.1), (17.6), (18.2) are respectively Eqs. (1), (2), (1′), (2′) of present paper.

For $S$ terms this is equivalent to having a spin dependent potential of magnitude[3]

$$-(\hbar^2/6M^2c^2)\Delta J(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2),$$

which depends on $r$ differently from the dependence of $J$ on $r$, while for $P$, $D$, $\cdots$ terms still different dependences result. The terms in $(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)$, $(\mathbf{r}\boldsymbol{\sigma}_1)(\mathbf{r}\boldsymbol{\sigma}_2)$ that are discussed here are not required directly by invariance of order $v^2/c^2$. It is nevertheless interesting that they disappear only accidentally when relativistic Diracian forms are used. One may expect that neutrino-electron field theories will lead to Diracian forms that are invariant to order $v^2/c^2$ and it is therefore probable, although not certain, that the $(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)$, $(\mathbf{r}\boldsymbol{\sigma}_1)(\mathbf{r}\boldsymbol{\sigma}_2)$ terms have reality. Wheeler's[4] and Wigner's[5] enumerations of possible rotationally invariant interactions include such terms. The fact that they arise as relativistic corrections and are only somewhat smaller than the spin dependent Heisenberg part of the force indicates that not only are they mathematical possibilities but that they most probably correspond to reality.

## 2. Ordinary Interactions

In reference 3 there were considered two forms of the Hamiltonian for two particles:

$$H=-c(\boldsymbol{\alpha}_1\mathbf{p}_1)-c(\boldsymbol{\alpha}_2\mathbf{p}_2)-(\beta_1+\beta_2)Mc^2-J+\tfrac{1}{2}(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)J-\tfrac{1}{2}(\boldsymbol{\alpha}_1\mathbf{r})(\boldsymbol{\alpha}_2\mathbf{r})(dJ/rdr), \tag{1}$$

$$H=-c(\boldsymbol{\alpha}_1\mathbf{p}_1)-c(\boldsymbol{\alpha}_2\mathbf{p}_2)-(\beta_1+\beta_2)Mc^2-\beta_1\beta_2J-\tfrac{1}{2}(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)J-\tfrac{1}{2}(\boldsymbol{\alpha}_1\mathbf{r})(\boldsymbol{\alpha}_2\mathbf{r})(dJ/rdr), \tag{2}$$

which when reduced to equivalent Pauli forms give, respectively,[3]

$$\begin{aligned}H=2Mc^2-J&+\frac{p_1^2+p_2^2}{2M}-\frac{p_1^4+p_2^4}{8M^3c^2}+\frac{\hbar f}{4M^2c^2}([\mathbf{r}\times(2\mathbf{p}_2-\mathbf{p}_1)]\boldsymbol{\sigma}_1-[\mathbf{r}\times(2\mathbf{p}_1-\mathbf{p}_2)]\boldsymbol{\sigma}_2)+\frac{J}{2M^2c^2}\mathbf{p}_1\mathbf{p}_2\\&-\frac{f}{2M^2c^2}x^ax^bp_1^ap_2^b-\frac{3\hbar f}{4iM^2c^2}\mathbf{r}(\mathbf{p}_2-\mathbf{p}_1)-\frac{\hbar}{4iM^2c^2}\frac{rdf}{dr}\mathbf{r}(\mathbf{p}_2-\mathbf{p}_1)-\frac{\hbar^2 f}{8M^2c^2}(15+4\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\\&-\frac{\hbar^2}{4M^2c^2}\frac{df}{rdr}[5r^2+r^2\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2-(\mathbf{r}\boldsymbol{\sigma}_1)(\mathbf{r}\boldsymbol{\sigma}_2)]-\frac{\hbar^2r^3}{8M^2c^2}\frac{d}{dr}\left(\frac{df}{rdr}\right).\end{aligned} \tag{1'}$$

and

$$\begin{aligned}H=2Mc^2-J&+\sum_{i=1}^{2}\left[\frac{p_i^2}{2M}+\frac{p_i^2J+2\mathbf{p}_iJ\mathbf{p}_i+Jp_i^2}{8M^2c^2}-\frac{p_i^4}{8M^3c^2}+\frac{\hbar}{4M^2c^2}[\nabla_iJ\times\mathbf{p}_i]\boldsymbol{\sigma}_i\right]-\frac{J}{2M^2c^2}\mathbf{p}_1\mathbf{p}_2\\&-\frac{f}{2M^2c^2}x^ax^bp_1^ap_2^b-\frac{\hbar}{4iM^2c^2}\left(5f+\frac{rdf}{dr}\right)\mathbf{r}(\mathbf{p}_2-\mathbf{p}_1)-\frac{\hbar^2}{8M^2c^2}\left[15f+10\frac{rdf}{dr}+\frac{r^3d}{dr}\left(\frac{df}{rdr}\right)\right].\end{aligned} \tag{2'}$$

In Eqs. (1), (2) the quantities $\boldsymbol{\alpha}$, $\beta$ are the usual Dirac matrices, the momenta are denoted by $p$, and the indices 1, 2 refer to the two particles. In Eqs. (1′), (2′) the notation is similar. The quantities $\boldsymbol{\sigma}$ are Pauli's two row square matrices. Summations are understood for $x^ap_1^a$. Also

$$f=dJ/rdr. \tag{2''}$$

The nonrelativistic approximation to the deuteron equation is

$$\Delta\psi+(M/\hbar^2)(E+J)\psi=0. \tag{3}$$

By using this equation, terms in Eqs. (1′), (2′) which occur in addition to $2Mc^2-J+(p_1^2+p_2^2)/2M$

[4] J. A. Wheeler, Phys. Rev. **50**, 643 (1936).
[5] E. Wigner, Phys. Rev. **51**, 106 (1937).

may be said to be of relativistic origin. Inspection of Eqs. (1′), (2′) shows that there are among them terms explicitly involving momenta. In classical analogy these terms depend on the velocity. In addition there are also present terms which do not depend explicitly on the velocity. Their effects are not easily distinguishable from $J$ because from an empirical point of view they can be considered as part of $J$. The expectation values of the terms are

$$-\frac{1}{8M^3c^2}\int\psi(p_1^4+p_2^4)\psi d\tau=-\frac{1}{4Mc^2}\int(E+J)^2\psi^2 d\tau=-\frac{\alpha a(D+E)(E+D-E/\alpha a)}{4Mc^2(1+\alpha a)}\sim -0.15mc^2,$$

$$\frac{1}{8M^2c^2}\int\psi[(p_1^2+p_2^2)J+J(p_1^2+p_2^2)]\psi d\tau=-\frac{\hbar^2}{2M^2c^2}\int\psi J\Delta\psi d\tau=\frac{(D+E)(D\alpha a-E)}{2Mc^2(1+\alpha a)}\sim 0.31mc^2,$$

$$\frac{1}{2M^2c^2}\int\psi\mathbf{p}_1J\mathbf{p}_1\psi d\tau=\frac{\hbar^2}{2M^2c^2}\int J(\nabla\psi)^2 d\tau=\frac{D(E+D)\alpha a}{2Mc^2(1+\alpha a)}\left[1+\frac{2E}{D\alpha^2a^2}+\frac{E}{D\alpha a}\right]\sim 0.09mc^2,$$

$$-\frac{1}{2M^2c^2}\int\psi J\mathbf{p}_1\mathbf{p}_2\psi d\tau=-\frac{\hbar^2}{2M^2c^2}\int\psi J\Delta\psi d\tau=\frac{(D+E)(D\alpha a-E)}{2Mc^2(1+\alpha a)}\sim 0.31mc^2,$$

$$-\frac{1}{2M^2c^2}\int\psi f x^a x^b p_1^a p_2^b\psi d\tau=-\frac{\hbar^2}{2M^2c^2}\int\psi f r^2\frac{d^2\psi}{dr^2}d\tau=-\frac{2E(D+E)}{Mc^2\alpha a}-\frac{\alpha a(D+E)(E+D/2)}{Mc^2(1+\alpha a)}\sim 0.22mc^2,$$

$$\int\psi\left[-\frac{p_1^4+p_2^4}{8M^3c^2}-\frac{\hbar^2\Delta J}{4M^2c^2}\right]\psi d\tau=-\frac{D+E}{4Mc^2(1+\alpha a)}\left[\alpha a(D+E)+3E+\frac{4E}{\alpha a}\right]\sim 0.08mc^2,$$

$$-\frac{\hbar}{4iM^2c^2}\int\psi r\frac{df}{dr}\mathbf{r}(\mathbf{p}_2-\mathbf{p}_1)\psi d\tau=\frac{2\pi\hbar^2}{M^2c^2}\int f\frac{d}{dr}\left(r^4\psi\frac{d\psi}{dr}\right)dr$$

$$=-\frac{E(1-\alpha a)(D+E)}{Mc^2\alpha a}+\frac{\alpha a(D+E)(E+D/2)}{Mc^2(1+\alpha a)}\sim 0.34mc^2,$$

$$-\frac{5\hbar}{4iM^2c^2}\int\psi f\mathbf{r}(\mathbf{p}_2-\mathbf{p}_1)\psi d\tau=-\frac{5\hbar^2}{2M^2c^2}\int\psi f\frac{d\psi}{dr}d\tau=\frac{5E(D+E)}{Mc^2\alpha a}\sim -1.14mc^2.$$

The numerical values are for $a=2.3\times10^{-13}$ cm, $E=-4.3mc^2$. In these formulas the second expression in each case refers to Eq. (3) while the last refers to the case of a constant $J=D$ through $0<r<a$ and $J=0$ for $r>a$ (square well). The quantity $\alpha$ is the reciprocal length

$$\alpha=(-ME/\hbar^2)^{\frac{1}{2}}. \tag{4′}$$

Either by means of Eqs. (4) and Eqs. (17.3), (17.4) of reference 3 or by direct calculation using the spherical symmetry of $S$ states one obtains the expectation values

$$(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)J=\frac{\hbar^2}{2M^2c^2}\int\{\psi J\Delta\psi-J(\nabla\psi)^2-(2/3)(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)[\psi J\Delta\psi+J(\nabla\psi)^2]\}d\tau,$$
$$(\boldsymbol{\alpha}_1\mathbf{r})(\boldsymbol{\alpha}_2\mathbf{r})f=\frac{\hbar^2}{2M^2c^2}\int\frac{dJ}{dr}\left\{r\psi\Delta\psi-2\psi\frac{d\psi}{dr}-r(\nabla\psi)^2-(2/3)(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\psi\frac{d\psi}{dr}\right\}d\tau. \tag{5}$$

Here $(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)$ stands for the expectation value of the corresponding operator. For a square well

$$(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)J=-\frac{(E+D)(D\alpha a+E/\alpha a)}{Mc^2(1+\alpha a)}-\frac{2}{3}\frac{E(E+D)}{Mc^2\alpha a}(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2),$$

$$(\boldsymbol{\alpha}_1\mathbf{r})(\boldsymbol{\alpha}_2\mathbf{r})f=\frac{E(D+E)}{Mc^2(1+\alpha a)\alpha a}[1+(\alpha a)^2D/2E+\tfrac{2}{3}(1+\alpha a)(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)]. \tag{5'}$$

According to Eq. (17′) of reference 3, the sum of corrections of order $v^2/c^2$ that correspond to Eq. (1) is

$$(\Delta E)_1=-\frac{p_1^4+p_2^4}{8M^3c^2}-\frac{\hbar^2\Delta J}{4M^2c^2}+\tfrac{1}{2}(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)J-\tfrac{1}{2}(\boldsymbol{\alpha}_1\mathbf{r})(\boldsymbol{\alpha}_2\mathbf{r})f, \tag{6.1}$$

and according to Eq. (18) of reference 3, the sum of similar corrections that correspond to Eq. (2) is

$$(\Delta E)_2=\sum_{i=1,\,2}\left[-\frac{p_i^4}{8M^3c^2}+\frac{p_i^2J+2\mathbf{p}_iJ\mathbf{p}_i+Jp_i^2}{8M^2c^2}\right]-\tfrac{1}{2}(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)J-\tfrac{1}{2}(\boldsymbol{\alpha}_1\mathbf{r})(\boldsymbol{\alpha}_2\mathbf{r})f. \tag{6.2}$$

From these formulas and the Eqs. (5) the corrections can be computed. One finds

$$(\Delta E)_1=\frac{\hbar^2}{4M^2c^2}\int\left\{E\psi\Delta\psi-3J(\nabla\psi)^2-\psi^2\Delta J-\frac{rdJ}{dr}[\psi\Delta\psi-(\nabla\psi)^2]-(2/3)(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\psi^2\Delta J\right\}d\tau, \tag{7.1}$$

$$(\Delta E)_2=\frac{\hbar^2}{4M^2c^2}\int\left\{(E-2J)\psi\Delta\psi+3J(\nabla\psi)^2-\psi^2\Delta J-\frac{rdJ}{dr}[\psi\Delta\psi-(\nabla\psi)^2]\right\}d\tau. \tag{7.2}$$

For square wells these become

$$(\Delta E)_1=-\frac{(D+E)[4D\alpha^2a^2+E(\alpha^2a^2+3\alpha a+8)]}{4Mc^2(1+\alpha a)\alpha a}-\frac{2E(E+D)}{3Mc^2\alpha a}(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2), \tag{8.1}$$

$$(\Delta E)_2=\frac{D+E}{Mc^2(1+\alpha a)}\left[D\alpha a+E\left(\frac{1}{4}-\frac{\alpha a}{4}+\frac{1}{\alpha a}\right)\right]. \tag{8.2}$$

By using these formulas the numerical values in Table I were obtained. The values for $^3S$ have a minimum in the range shown and go through zero at $\alpha a\sim 0.29$. The deuteron energy was taken to be $-4.35mc^2$ in this and in Table II.

TABLE I. *Relativistic corrections for square wells. Energies in units of $mc^2$.*

| $amc^2/e^2=$ | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 2.00 |
|---|---|---|---|---|---|---|
| $(\Delta E)_1$ $^3S$ | −0.023 | −0.034 | −0.026 | −0.020 | −0.016 | −0.012 |
| $^1S$ | −2.45 | −0.82 | −0.39 | −0.22 | −0.14 | −0.067 |
| $E_1(^3S)-E_1(^1S)$ | 2.43 | 0.79 | 0.36 | 0.20 | 0.12 | 0.06 |
| $(\Delta E)_2$ | 1.47 | 0.50 | 0.24 | 0.14 | 0.09 | 0.05 |

TABLE II. *Relativistic corrections for $J=Ae^{-\alpha r^2}$ computed by using numerically integrated wave functions, for $\alpha^{-\frac{1}{2}}\sim 2.3\times 10^{-13}$ cm and $A=72.3$ $mc^2$.*

| $(\Delta E)_1$ $^3S$ | $(\Delta E)_1$ $^1S$ | $E_1(^3S)-E_1(^1S)$ | $(\Delta E)_2$ |
|---|---|---|---|
| −0.025 | −0.34 | 0.32 | 0.26 |

Collecting terms of order $v^2/c^2$ that contain the $\mathbf{p}_i$ explicitly in Eqs. (1), (2) one obtains for a square well

$$(\Delta E)_1'=\frac{D+E}{Mc^2(1+\alpha a)}\left[E\left(\frac{7}{4}+\frac{3\alpha a}{4}\right)-\frac{3}{4}D\alpha a\right]=\frac{(D+E)E}{Mc^2(1+\alpha a)}\left[\frac{3\pi^2}{16\alpha a}+\frac{13}{4}+\left(\frac{3}{2}-\frac{3}{\pi^2}\right)\alpha a+\cdots\right], \tag{8.3}$$

$$(\Delta E)_2'=\frac{D+E}{Mc^2(1+\alpha a)}\left[E\left(\frac{11}{4}+\frac{3}{\alpha a}+\frac{3\alpha a}{4}\right)+\frac{5}{4}D\alpha a\right]$$
$$=\frac{(D+E)E}{Mc^2(1+\alpha a)}\left[\left(3-\frac{5\pi^2}{16}\right)\frac{1}{\alpha a}+\frac{1}{4}+\left(\frac{5}{\pi^2}-\frac{1}{2}\right)\alpha a+\cdots\right], \tag{8.4}$$

where the first three terms of the series

$$\frac{D}{E}=-\frac{\pi^2}{4\alpha^2a^2}-\frac{2}{\alpha a}-\left(1-\frac{4}{\pi^2}\right)-\left(\frac{32}{\pi^4}-\frac{8}{3\pi^2}\right)\alpha a\cdots \tag{8.5}$$

have been used. It is curious that the coefficients in the square bracket of Eq. (8.4) are small and that the first three terms of the series in the square bracket partly cancel each other for $a=2.3\times10^{-13}$ cm. For this range of force $(\Delta E)_2'$ is practically negligible. For the same range $(\Delta E)_1'=-0.58mc^2$.

It should be noted that the relativistic correction to the energy of the deuteron or any other nucleus with atomic weight $>1$ is not unique. Thus $(\Delta E)_1'$, $(\Delta E)_2'$ are as justifiable corrections as $(\Delta E)_1$, $(\Delta E)_2$ so far as invariance of the equations is concerned. However the $(\Delta E)'$ do not have a simple obvious counterpart in a Diracian form and may from this point of view be regarded as less probable. It should also be noted that terms in relative velocities such as $-(3\hbar f/4iM^2c^2)(\mathbf{r}(\mathbf{p}_2-\mathbf{p}_1))$ may be added to the Hamiltonian without destroying its invariance. These terms are not negligible. Thus for $a=2.3\times10^{-13}$ cm

$$-\frac{5\hbar}{4iM^2c^2}f(\mathbf{r}(\mathbf{p}_2-\mathbf{p}_1))=-1.14mc^2,\qquad -\frac{\hbar}{4iM^2c^2}\frac{rdf}{dr}(\mathbf{r}(\mathbf{p}_2-\mathbf{p}_1))=0.34mc^2.$$

Since these values are of the same order of magnitude as $(\Delta E)_1$ and $(\Delta E)_2$ and are larger than the absolute value of the correction to the kinetic energy $-(p_1^4+p_2^4)/8M^3c^2=-0.15mc^2$ one must regard the relativistic corrections made so far in the combined theory of $H^2$, $H^3$ as uncertain[6] not only because they do not take into account terms like $J\mathbf{p}_1\mathbf{p}_2/2M^2c^2$ that are essential for invariance but also because the requirement of invariance does not suffice to fix all the parts of the Hamiltonian that contain the velocity explicitly.

## 3. Exchange Interaction

It can be shown that

$$\begin{aligned}H=&-c(\boldsymbol{\alpha}_1\mathbf{p}_1)-c(\boldsymbol{\alpha}_2\mathbf{p}_2)-(\beta_1+\beta_2)Mc^2-JP^M+\tfrac{1}{2}(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)JP^M-\tfrac{1}{2}(\boldsymbol{\alpha}_1\mathbf{r})(\boldsymbol{\alpha}_2\mathbf{r})fP^M\\&-\frac{i}{4Mc}\{([(\mathbf{p}_2-\mathbf{p}_1)\times\boldsymbol{\sigma}_1]\cdot\mathbf{r})(\boldsymbol{\alpha}_2\mathbf{r})f+f(\boldsymbol{\alpha}_1\mathbf{r})(\mathbf{r}\cdot[(\mathbf{p}_1-\mathbf{p}_2)\times\boldsymbol{\sigma}_2]+([(\mathbf{p}_2-\mathbf{p}_1)\times\boldsymbol{\sigma}_1]\cdot\boldsymbol{\alpha}_2)J\\&+J(\boldsymbol{\alpha}_1\cdot[(\mathbf{p}_1-\mathbf{p}_2)\times\boldsymbol{\sigma}_2])\}P^M\end{aligned} \tag{9}$$

is invariant to order $v^2/c^2$. Here $P^M$ is the Majorana exchange operator which exchanges in this equation only the space coordinates. There are other equations but the above was used because it is analogous to Eq. (1). The proof of the invariance of this equation will be given in another paper. One finds for the expectation values

$$\begin{aligned}(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)JP^M&=\frac{\hbar^2}{2M^2c^2}\int\{\psi J\Delta\psi+J(\nabla\psi)^2-(2/3)(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)[\psi J\Delta\psi-J(\nabla\psi)^2]\}d\tau\\&=\frac{(D+E)E}{Mc^2\alpha a}+\frac{2}{3}(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\frac{(D+E)(D\alpha a+E/\alpha a)}{Mc^2(1+\alpha a)},\end{aligned} \tag{9.1}$$

$$\begin{aligned}(\boldsymbol{\alpha}_1\mathbf{r})(\boldsymbol{\alpha}_2\mathbf{r})fP^M&=\frac{\hbar^2}{2M^2c^2}\int\frac{dJ}{dr}\left\{-2\psi\frac{d\psi}{dr}+r\psi\Delta\psi+r(\nabla\psi)^2-\frac{2}{3}(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\psi\frac{d\psi}{dr}\right\}d\tau\\&=\frac{(D+E)E}{Mc^2\alpha a(1+\alpha a)}\left\{\frac{D}{2E}\alpha^2a^2+3+4\alpha a+2\alpha^2a^2+\frac{2}{3}(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)(1+\alpha a)\right\},\end{aligned} \tag{9.2}$$

[6] W. Rarita and R. D. Present, Phys. Rev. **51**, 788 (1937).

$$-\frac{i}{4Mc}\{([(\mathbf{p}_2-\mathbf{p}_1)\times\boldsymbol{\sigma}_1]\cdot\mathbf{r})(\boldsymbol{\alpha}_2\mathbf{r})f+f(\boldsymbol{\alpha}_1\mathbf{r})(\mathbf{r}\cdot[(\mathbf{p}_1-\mathbf{p}_2)\times\boldsymbol{\sigma}_2]+([(\mathbf{p}_2-\mathbf{p}_1)\times\boldsymbol{\sigma}_1]\cdot\boldsymbol{\alpha}_2)J$$

$$+J(\boldsymbol{\alpha}_1\cdot[(\mathbf{p}_1-\mathbf{p}_2)\times\boldsymbol{\sigma}_2])\}P^M=\frac{\hbar^2}{3M^2c^2}(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\int\left\{[-\psi\Delta\psi+(\nabla\psi)^2]J-\frac{dJ}{dr}\psi\frac{d\psi}{dr}\right\}d\tau$$

$$=\frac{2}{3}(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\frac{D+E}{Mc^2}\left\{\frac{D\alpha a+E/\alpha a}{1+\alpha a}+\frac{E}{\alpha a}\right\},\quad (9.3)$$

and the whole correction is obtained by a formula similar to Eq. (6.1). For a square well this is

$$(\Delta E)_{\text{exc}}=-\frac{(D+E)[2D\alpha a+E(8/\alpha a+9+5\alpha a)]}{4Mc^2(1+\alpha a)}+(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\frac{D+E}{Mc^2}\left[\frac{D\alpha a+E/\alpha a}{1+\alpha a}+\frac{E}{3\alpha a}\right].\quad (10)$$

For $E=-4.3mc^2$, $a=2.3\times10^{-13}$ cm, $\alpha a=0.533$, $D=56.1mc^2$ one obtains from Eq. (10) $(\Delta E)_{\text{exc}}=0.58mc^2$ for the $^3S$ state and $-0.73mc^2$ for the $^1S$ state. Other values are given in Table III.

The spin dependence of the interaction following from Eq. (9) is greater than that corresponding to Eq. (1). By using a range of force only somewhat smaller than the accepted range it is possible to have the spin dependence equal in magnitude but opposite in sign to the empirical value $\sim 5mc^2$. Eq. (9) is only one of many possible forms of invariant equations. It is therefore possible, although not certain, that the empirical spin dependence can be explained as a consequence of a simple interaction between particles obeying Dirac's equation. Such an explanation would not be an ultimate one but would be useful in indicating the sort of field theory that should be used for explaining the interaction between heavy particles.

TABLE III. *Relativistic exchange corrections for Eq. (9).*

| $amc^2/e^2$ | | 0.50 | 0.82 | 1.00 |
|---|---|---|---|---|
| $(E)_{\text{exc}}$ | $^3S$ | 2.08 | 0.58 | 0.36 |
| | $^1S$ | −2.76 | −0.73 | −0.45 |
| $E_{\text{exc}}(^3S)-E_{\text{exc}}(^1S)$ | | 4.84 | 1.31 | 0.81 |

The relatively large magnitude of the spin-spin interactions obtained here indicates that: (a) The spatial dependence of spin dependent forces is likely to be different from the forces averaged over all spin directions. (b) The determination of the magnitude of the Heisenberg force from the empirically known spin dependence of the deuteron energy is questionable.

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