Velocity Analysis by Means of the Stern-Gerlach Effect

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(Received June 11, 1937)

A Stern-Gerlach experiment furnishes a convenient method of studying the velocity distribution in a molecular beam. In addition it is possible to obtain some information concerning the behavior of the field gradient as a function of position. This method has been applied to a beam of alkali atoms and shows that with low vapor pressure in the oven the Maxwell distribution law is fulfilled whereas at oven pressures such that the mean free path is much less than the width of the oven slit a departure sets in. It is found that the field gradient is quite uniform over almost half the trough height for deflections inside the trough. Toward the wedge the gradient varies rapidly.

COMPARATIVELY little accurate work has been done on the analysis of the distribution in velocity of the molecules in a molecular beam. Although several methods are available for dispersing the molecules according to velocity there are certain serious difficulties associated with each. Use of rotating slits or sectored disks, the method used by several investigators[1] involves the attainment of very high and constant peripheral speeds of the rotating parts. In addition there is considerable loss of intensity in the beam. Such work has yielded only qualitative agreement with the Maxwell distribution law. Experiments on deflection by inhomogeneous electric fields[2] have yielded a dispersion insufficient for accurate velocity analysis. In order to secure appreciable resolving power with this method it would be necessary to use potentials and gradients considerably greater than those used in the past. The well-known method of deflection by an inhomogeneous magnetic field has been successfully applied to the measurement of atomic and nuclear magnetic moments as well as to secure a beam of single-velocity atoms.[3] However, no accurate extensive study has been made for the purpose of determining the velocity distribution in the beam. Particularly well adapted to this work are the alkali metals for whose detection we have the Taylor-Langmuir gauge[4] by far the most sensitive, accurate, and convenient molecular beam detector available.

A simple and rugged velocity analyzer of fair resolving power, even though limited to the alkalis will enable one to investigate several rather interesting problems. One question which almost every investigator in the molecular beam field must have asked himself is whether the distribution in velocity in the beam is that of Maxwell. It has been assumed that if the oven pressure is low enough so that the mean free path in the oven is large compared to the width of the slit, this will be true. If it were not for the need of greater oven pressures in many experiments this would be a sufficient answer. When high beam intensities are required the experimenter is immediately faced with the question of just how large the mean free path must be in order to preserve, approximately, the Maxwell distribution. This can be answered only by a velocity analysis of the beam.

THEORY OF THE ANALYZER

An essential condition for the satisfactory operation of a velocity analyzer of the magnetic type is that the gradient of the magnetic field be very nearly constant over the height of the beam, otherwise atoms of the same velocity may suffer quite different deflections depending upon how far they are from the center of the beam. Taylor[5] has shown by direct measurement, that in the so-called "Hamburg set-up" in which the beam is sent through the field just at the edge of the trough, this condition is very well realized, provided the beam height is not much more than one quarter of the height of the trough. We find

[1] Stern, Zeits. f. Physik **2**, 49 (1920); Eldridge, Phys. Rev. **30**, 931 (1927).
[2] Estermann, Zeits. f. physik. Chemie **B1**, 161 (1928); Scheffers and Stark, Physik. Zeits. **25**, 452 (1934).
[3] Rabi and Cohen, Phys. Rev. **46**, 707 (1934).
[4] Taylor, Zeits. f. Physik **57**, 242 (1929).
[5] Fraser, *Molecular Rays* (Cambridge, 1931), p. 120.

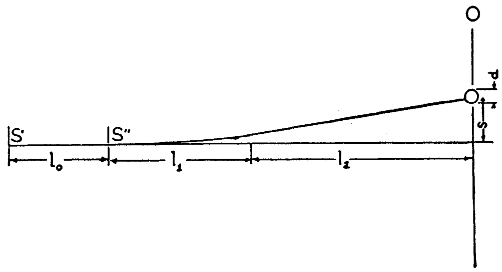


FIG. 1. Arrangement of the analyzer; S', effective source slit; S'', defining slit; l_1 beam path in the field; l_2, beam path after traversing field; d, width of detector wire; s, distance from center of detector wire to center of undeflected beam.

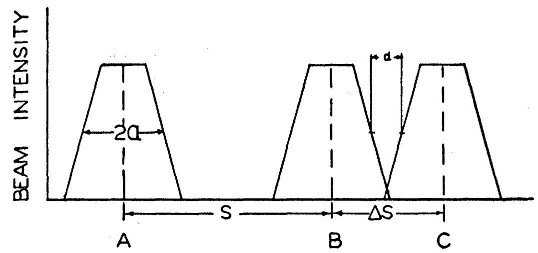


FIG. 2. Distribution of intensity in (A) undeflected beam; (B) deflection pattern of atoms with velocity v; (C) deflection pattern of atoms with velocities $v+\Delta v$.

no observable difference in the form of the intensity curve for beam heights of 1.6 and 3.25 mm. The total trough height is 8 mm. We take this to indicate a constant gradient over this distance.

Let us consider the resolving power which may be realized with a velocity analyzer of this type. Fig. 1 represents the schematic arrangement of the system. The beam is defined by slits S' and S'', separated by a distance l_0, and is detected by a wire of diameter d free to move laterally in the plane 0. In the region l_1 the atoms are accelerated in a direction normal to that of their original motion by an inhomogeneous magnetic field, while the region l_2 is field free. In the absence of an applied field the curve representing the number of atoms crossing the plane 0 per unit time between s and $s+\Delta s$ will if the slits be correctly aligned, be a trapezoid whose width at one-half its maximum height is

$$2a=((l_0+l_1+l_2)/l_0)w'', \qquad (1)$$

where w'' is the width of slit S'' (Fig. 2).

In the presence of a field, atoms having a velocity v will give rise to a precisely similar distribution but shifted bodily by an amount

$$s=(1/2mv^2)\mu(\partial H/\partial S)(l_1^2+2l_1l_2). \qquad (2)$$

Atoms of velocities v and $v+\Delta v$ will then give rise to a distribution represented by the superposition of two such trapezoids, the separation of their centers being

$$\Delta s=(2s/v)\Delta v. \qquad (3)$$

If a detector of width d is now run across this pattern its response will measure the total intensity in the pattern between the limits $s+\frac{1}{2}d$ and $s-\frac{1}{2}d$, where s is the position of the center of the wire. A sufficient condition for resolution, if d is less than $2a$, is that Δs be greater than $(2a+d)$. In this case

$$V/\Delta V=2s/(2a+d). \qquad (4)$$

With the apparatus to be described, the resolving power obtained for a typical curve is about 13 in the region of the most probable velocity of the Maxwell distribution characteristic of 450°K. Clearly the resolution increases as the velocities decrease. As the dispersion increases the intensity decreases correspondingly. We see from (2) that the gradient is the most easily adjusted parameter, so that in order to study any particular velocity range one may choose a value of the gradient to compromise between convenient intensity and sufficient resolving power.

Inspection of (1) and (2) shows that to design the instrument for high resolving power one must use narrow slits, as long a path in the field as is possible, and reduce the constant a by making l_0 comparible to l_1+l_2. Very little will be gained by increasing l_2 much above l_1 since a gain in deflection will be offset by an increase in a.

In order to obtain a calibration curve for the instrument (velocity *vs.* deflection) we have used as a source an oven operating at a vapor pressure so low that the Maxwell velocity distribution is certainly realized. The form of the intensity distribution curve to be expected may be calculated if the displacement of the beam by the field is so small that the gradient of the field may be assumed to be constant. If, by suitable choice of parameters, a curve of the computed form may be fitted to the data, it appears to be a reasonable inference that the gradient is essentially constant.

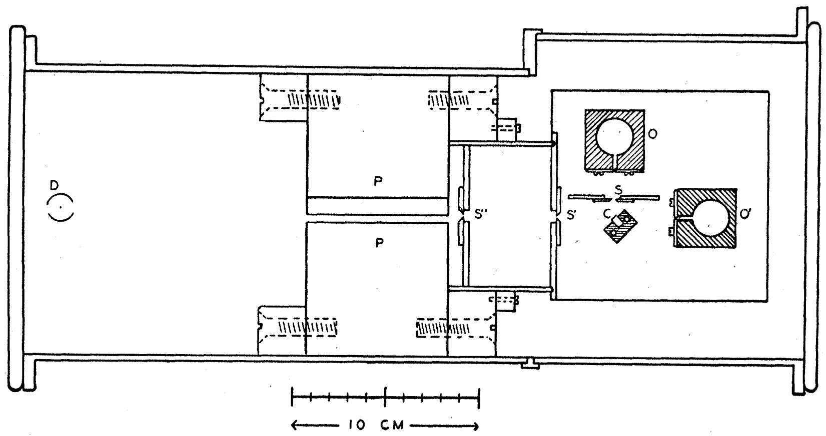


FIG. 3. Plan of apparatus. Oven position O and parts marked S and C are used only in connection with work described in the following paper.

However it seems desirable to see to just what extent a small variation in the gradient may be concealed by a suitable choice of parameters, i.e., of the effective or apparent gradient and of the constants introduced by finite width of slits and detector wire.

Consider the form of the distribution to be expected if we assume, not a constant gradient but merely that atoms of velocity v passing through the field suffer a deflection η and write

$$v=f(\eta). \tag{5}$$

For the number of atoms which, in the absence of field, pass through the plane of the detector within a region of width $d\xi$ at ξ and which have velocities within dv at v let us write

$$N(v)P(\xi)d\xi dv.$$

$P(\xi)$ gives the form of the undeviated beam. For the present we will assume merely that it is symmetrical about $\xi=0$. Both $N(v)$ and $P(\xi)$ are normalized to unity. In the presence of a field these atoms will be deflected a distance η given by (5) and the number of atoms crossing the plane of the detector in ds at s will be given by

$$I'(s)=\int_{\xi=0}^{\xi=+\infty} F(s+\xi)P(\xi)d\xi ds, \tag{6}$$

where $F(s+\xi)=N(v)dv$ in which v has been expressed explicitly in terms $\eta=s+\xi$ by means of (5).

Expanding $F(s+\xi)$ about $\xi=0$ by Taylor's series and remembering that because of the symmetry of $P(\xi)$ all odd moments vanish, we obtain

$$I'(s)=\sum_{n=0}^{n=\infty}\frac{F^{2n}(s)}{(2n)!}M(2n), \tag{7}$$

where
$$M(2n)=\int_{-\infty}^{+\infty}\xi^{2n}p(\xi)d\xi$$

is the $2n$th moment of the field free distribution.

The detector wire of width $2b$ with its center at s measures

$$I''(s)=\int_{s-b}^{s+b} I(s)ds, \tag{8}$$

which may be written

$$I''(s)=\sum_{n=0}^{n=\infty}\sum_{r=0}^{r=\infty}\frac{M(2n)}{(2n)!}\frac{b^{2r+1}}{(2r+1)!}F^{2n+2r}(s), \tag{9}$$

in which the term $bM(0)F(s)$ gives the form of the distribution curve for infinitesimal widths of slit and detector. In any region in which the remaining terms are small we may evaluate them using for the function (5) that form appropriate for the case of a rigorously constant gradient. Now fortunately these terms do become small as s becomes large, and it is only for large s that the question of constancy of gradient becomes serious.

Writing (9) in the form

$$bM(0)F(s)=I''(s)-\sum_{n=1}^{n=\infty}\sum_{r=1}^{r=\infty}\frac{M(2n)}{2n!}\frac{b^{2r+1}}{(2r+1)!}F^{2n+2r}(s) \tag{10}$$

and integrating, we have

$$bM(0)\int_0^s F(s)ds = \int_0^s I''(s)ds - \Delta I'', \quad (11)$$

where the integral on the left-hand side represents the integrated intensity which would be observed in an apparatus with infinitesimal width of slits and detector. Consequently those values of s and v for which

$$\frac{bM(0)\int_0^s F(s)ds}{bM(0)\int_0^\infty F(s)ds} = \frac{\int_\infty^v N(v)dv}{\int_\infty^0 N(v)dv} \quad (12)$$

must satisfy the equation $v = f(s)$.

If we assume the atoms in the beam to have the Maxwell distribution of velocities

$$N(v)dv = N_0 \exp[-mv^2/2KT]v^3dv = N_1 e^{-x}xdx,$$

where $\qquad x = mv^2/2KT = v^2/v_\alpha^2,$

the integral on the right side of (12) is merely $e^{-x}(x+1)$ while the quantity on the left may be obtained by graphical integration of the observed intensity distribution curves. $\Delta I''$ can be calculated only approximately, however its value proves to be of the order of one-half to two percent of $\int I''(s)ds$ over the useful range of the intensity curve, so that an approximate evaluation is sufficient. The series for $\Delta I''$ converges so rapidly for large s that only the first term is appreciable.

Apparatus

The apparatus is illustrated schematically in Fig. 3. The chamber walls are of brass tubing closed on the ends with brass disks sealed with wax. The system is divided into two parts, an oven chamber and a beam chamber, which are pumped independently. Communication between the two is through slit S'. During runs the pressure in the beam chamber is "flat" as recorded on a McLeod gauge with a constant of 3.6×10^{-7} while in the oven chamber the gauge may read from 2 to 6×10^{-7} mm Hg. The oven is similar in design to that used in other experiments with alkali beams.[3] The oven slit jaws are of rectangular cross section, 2 mm thick.

The beam is effectively defined by slits S' and S'', the oven slit being considerably wider than either of them. By defining the beam in this manner scattering which occurs in the beam itself as well as by foreign gases in the oven chamber does not cause appreciable broadening of the beam with consequent loss in resolving power.

The magnetic field is produced by an electromagnet with Stern-Gerlach type pole pieces made of Armco iron. These pole pieces are held in position inside the vacuum system by a pair of bronze rings to which they are bolted. The slit-holders are carried on a brass tube which slides into one of the bronze rings in such a way that it may be removed for cleaning and be replaced without disturbing the alignment. The oven supports are attached rigidly to the slit unit. In this way the beam always traverses the same region of the magnetic field. Fig. 4 illustrates a side view of the pole piece unit with the slit unit in place.

The slit widths used were 0.15 mm for the oven, 0.022 mm for S', and 0.024 mm for S''. Slits S' and S'' are each formed by a pair of knife edged jaws. A pair of brass stops set into the trough pole piece limits the height of the beam to 3.25 mm. The distance between S' and S'' is 5 cm, the length of pole pieces 7.5 cm, and the path from the end of the pole pieces to the detector 13 cm. The angle of the wedge pole piece is 77°, the trough height, 8 mm; the distance of the vertex of the wedge from the plane of the trough 4 mm.

The detector wire is pure tungsten 0.002″ diameter, maintained at 45 volts above ground potential. The atoms after ionization are collected by a nickel plate connected to the grid of

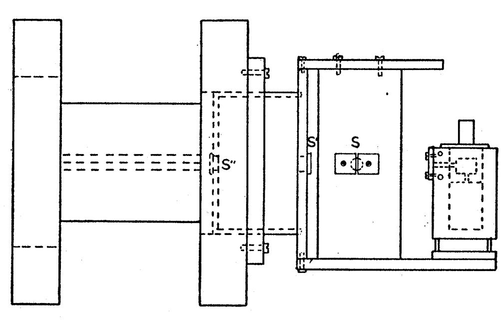
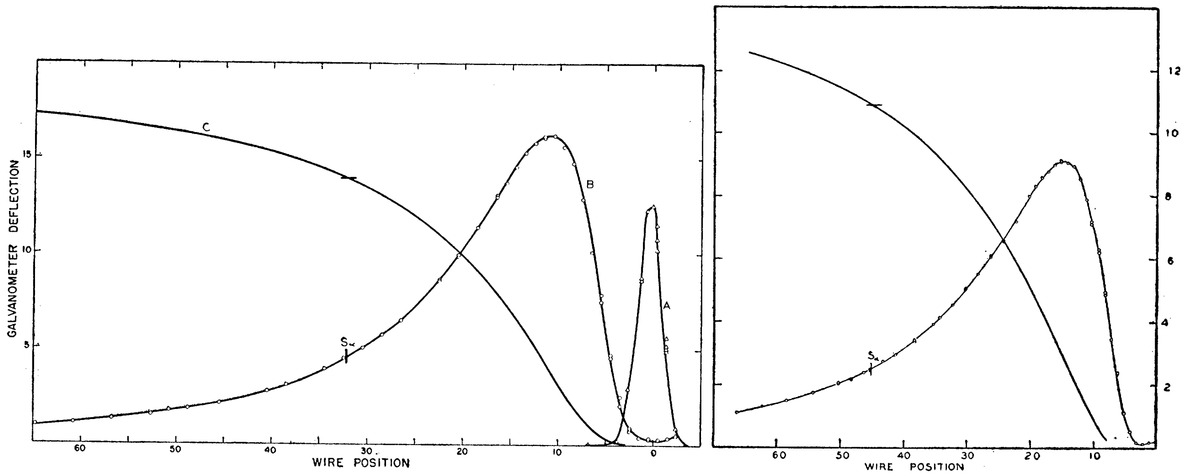


Fig. 4. Side view of slit and pole piece units.



(a) (b)

FIG. 5. (a) A. Intensity distribution in typical undeviated beam. B. Intensity distribution in deflection pattern, magnet current 6 amp. C. Graphical integral of B as a function of s. (b) Same as (a) but for magnet current 9 amp.

an F. P. 54 vacuum tube amplifier. As beam intensities vary over a factor of 500, the amplifier sensitivity is varied by using several grid leaks whose resistances range from 10^8 to $2\times10^{10}\Omega$ so that in any single run only potentials less than 0.05 volt are applied to the grid. In this region the response of the amplifier is quite linear. The detector wire is carried eccentrically on a conical spindle bearing which can be rotated externally by means of a ground glass joint. A telemicroscope with an ocular micrometer magnifying 80 times is used to measure the position of the detector wire. The observed galvanometer deflections are corrected for amplifier drift and residual ion emission of the wire by cutting off the beam after every few readings with a magnetically operated shutter.

Calibration of the Analyzer

Figures 5(a) and (b) represent two typical experimental curves taken with magnet currents of 6 and 9 amperes. The small triangles are points of the undeviated beam, the circles those of the velocity spectra, while the solid curves marked C represent the results of graphical integration. If corresponding to any value of x we calculate the value of $e^{-x}(x+1)$ we may pick the corresponding value of s from the curve. This procedure may be used to obtain values of s as a function of x. The short horizontal dashes represent the ordinates for $x=1$ namely, 0.7359. The corresponding values of s are the abscissas of the intersections, or the s_α's. For a constant gradient (5) takes the form (2), from which it is evident that the product sx should be constant. A plot of sx against s is given in Fig. 6. Curves A and A' are for the beam defined vertically by stops 3.25 mm high and deflected toward the trough. These curves show the gradient to be constant to within one percent. Curve A'' represents sx under the same conditions (oven pressure 0.004 mm Hg, 9 amperes magnet current) but deflected toward the wedge. The field gradient is evidently not constant on this side. As constancy of the gradient greatly simplifies the interpretation of the results we have used the beam deflected toward the trough exclusively. Curve D shows the result of a run made with the beam limited to a height of 6 mm. Clearly the gradient at the upper and lower extremes varies rapidly, the average being roughly represented by curve D. Curves B and C show the effect of increasing the oven vapor pressure until the mean free path becomes smaller than the width of the oven slit. B was taken with an oven pressure of 0.13 mm Hg, (m.f.p. $\sim$0.04 mm) while C was taken at 1.4 mm Hg (m.f.p. $\sim$0.003 mm) while in both the beam stops were the same as in case A. The difference is no doubt due to a departure from the Maxwell distribution. This point will be discussed in more detail later.

It has frequently been assumed that the intensity distribution in the observed deflection pattern may be represented by the well-known formula of Stern.[6]

$$I_0''(s)=\frac{I_0}{2a'}\Big[e^{-y}(y+1)\Big]_{s_\alpha/(s-a')}^{s_\alpha/(s+a')}. \qquad (14)$$

One may readily show that for s large enough so that only the first term in $\Delta I''$ is appreciable this equation should be valid. It gives the density of atoms incident upon the plane of the detector if the intensity in the undeviated beam is constant over a breadth $2a'$ and zero outside this region. It is merely (6) with $P(\xi)=\frac{1}{2}a'$ for $|\xi|<a'$ and $P(\xi)=0$ for $|\xi|>a'$. In this case the integration may be carried out and gives (14). Instead of integrating, we may expand as we did in the general case and obtain the Stern formula in a form more convenient for comparison with (9). This gives

$$I_0''(s)=\sum_{n=0}^{n=\infty}\frac{M(2n)}{2n!}F^{2n}(s). \qquad (15)$$

Eq. (9) may be written for comparison

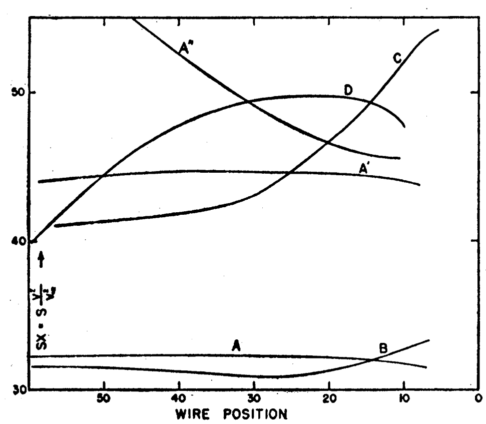


FIG. 6. Product sx as a function of s:

- A Beam height 3.25 mm, oven pressure 0.004 mm, magnet current 6 amp.
- A' Beam height 3.25 mm, oven pressure 0.004 mm, magnet current 9 amp.
- A'' Beam height 3.25 mm, oven pressure 0.004 mm, magnet current 9 amp. (wedge).
- B Beam height 3.25 mm, oven pressure 0.13 mm, magnet current 6 amp.
- C Beam height 3.25 mm, oven pressure 1.4 mm, magnet current 9 amp.
- D Beam height 6 mm, oven pressure 0.004 mm, magnet current 9 amp.

$$I''(s)=\sum_{\nu=0}^{\nu=n}\left\{\sum_{n=0}^{n=\infty}\frac{M(2n)}{2n!}\frac{b^{2(\nu-n)+1}}{[2(\nu-n)+1]!}F^{2\nu}(s)\right\}. \quad (16)$$

If the undeviated beam is a rectangle of width $2a$ this becomes

$$I''(s)=b\left\{F(s)+\frac{(b^2+a^2)}{3!}F^2(s)+\left[\frac{a^4+b^4+10a^2b^2}{5!}\right]F^4(s)+\cdots\right\}$$

expanding (15)

$$I_0''(s)=F(s)+\frac{a'^2F^2}{3!}(s)+\frac{a'^4}{5!}F^4(s)+\cdots. \qquad (17)$$

If $a'^2=a^2+b^2$ these will evidently not differ until the third term becomes of importance. It should perhaps be pointed out that under these circumstances the breadth of the *observed* undeviated beam at half-maximum intensity is a, not $(a^2+b^2)^{\frac{1}{2}}$. For any other form of $P(\xi)$ the higher moments will be larger than for the rectangle, so that the comparison is for the most favorable case.

The solid curve of Fig. 7 shows the calculated intensity while the circles show the observed points for the velocity spectrum of Fig. 5(*a*). For the evaluation of the intensity (14) was used in the region of the maximum while the first two terms of (17) were used for larger s. The higher

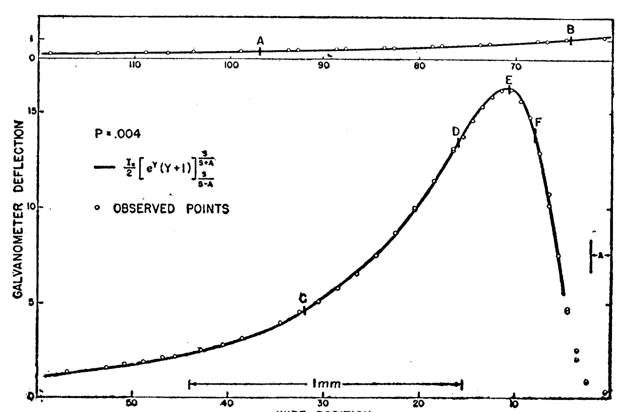


FIG. 7. Comparison between calculated and observed intensities for spectrum of Fig. 5(a). Points labeled A, B, C, E and F represent deflections of $(3)^{-\frac{1}{2}}$, $(2)^{-\frac{1}{2}}$, 1, $(2)^{\frac{1}{2}}$, $(3)^{\frac{1}{2}}$ and 2 times v_α.

[6] Stern, Zeits. f. Physik **41**, 563 (1927).

terms in (17) are negligible when $s > s_\alpha/3$. s_α was evaluated by the use of (12).

Unless $s_\alpha > 20a$ the error introduced by assuming a rectangular form for $P(\xi)$ becomes significant in the region of the maximum. The agreement between the observed and calculated values is well within the experimental error for $s > s_\alpha/3$. The difference is slightly higher for smaller s but this is probably due to the error introduced by the use of the Stern formula in this region. A trapezoidal distribution improved the fit in the one case for which the necessary calculations were carried out. Its principal effect was to shift the peak of the curve about 1/10 division farther from the center. With a higher gradient and consequent increased resolution ($s_\alpha = 45$) the fit is satisfactory even in the region $s = s_\alpha/3$.

The uncertainty in the observed points due to fluctuations in oven pressure and to electrical disturbances affecting the amplifier ranges from about one percent near the maximum to about ten percent beyond $2s_\alpha$. In runs made with several values of the field gradient at oven pressures of 0.01 mm (potassium) and 0.04 mm (sodium) there is no evidence of systematic deviation from the Maxwell distribution, the calculated curves fitting the observations in substantially the same way as in Fig. 7. At the higher pressure (0.04 mm Hg) the mean free path in the oven is approximately equal to the width of the oven slit. This slit has the form of a canal 0.15 mm wide by 2 mm long so that one might very well expect forward scattering to produce some deviation from the Maxwell distribution.

Effect of Oven Pressure on Velocity Distribution

At higher pressures the effect of forward scattering becomes clearly evident. A distinct departure from the Maxwell distribution is shown by the velocity spectrum in Fig. 8. Here we have a curve taken with an oven pressure of 1.4 mm. At this pressure the mean free path is

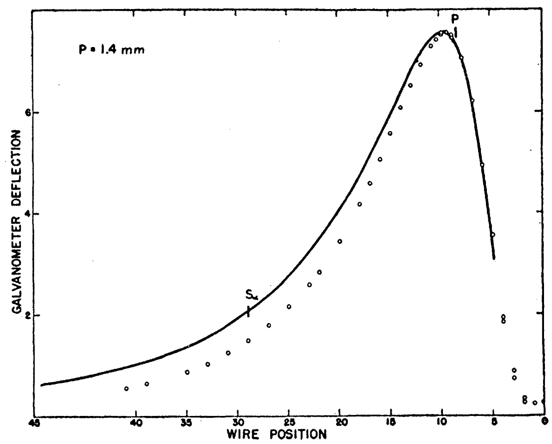


Fig. 8. Comparison of calculated and observed distributions for oven pressure of 1.4 mm Hg.

about 0.003 mm. The solid line represents the distribution as calculated for the appropriate temperature, assuming a Maxwell distribution. Clearly there is a deficiency of slow atoms. One might, at first glance, suppose that a better fit might be produced by assuming a smaller value for s_α. However, if one were to fit a theoretical curve in the region of s_α, the peak would occur at P which is clearly outside the limits of error.[7] At oven pressures of 0.13 and 0.5 mm deviations from the Maxwell distribution are observed but to a less extent than at the higher pressure. If one were to attempt to measure the atomic magnetic moment from the observed intensity distribution of Fig. (8) using the method of equal intensities,[6] the error introduced would range from 2 percent for points chosen near the maximum, to about 7 percent for points taken at three-tenths of the peak height.

From the above results one can draw no certain inference concerning the distribution from a slit of different form. If it were desired to preserve the Maxwell distribution at high beam pressures one would expect that the use of narrow knife-edged slits would tend to eliminate forward scattering.

[7] This would also imply a change in calibration, amounting to 13 percent. A recheck of the calibration failed to reveal any such error.